CHAPTER 3
ON A GENERAL STORAGE MODEL

The theory of Queue is concerned with the development of mathematical models to predict the behaviour of systems that provide services for randomly arising demands. The main problems in queueing theory are with the waiting time and queue length estimation.

We present basic results of Queue in section 1, Markov equation representation and integral equation of queueing systems are given in section 2. Applications of queue in Dam models and general storage models are discussed in section 3. The numerical solutions[51] of the integral equations through Gram-Charlier representation and other orthogonal polynomials are reviewed in section 4. We close this chapter by our general storage model for fish feeding[52].

3.1 FUNDAMENTALS OF QUlES

In order to describe a given queueing system, it is necessary to specify the following components of the system: [11]

Definition 3.1.1 The input process expresses the probability law governing the arrival of "customers" at the "counter", where service is provided. Suppose customers arrive at the counter at times $t_1, t_2, \ldots, t_n$ ($0 < t_1 < t_2 < \ldots < t_n < \infty$) and $T_n = t_{n+1} - t_n$ denote the difference between the time of arrival of the $(n+1)^{th}$ and $n^{th}$ customers.

The input process is given by the probability law governing the sequence of arrival times $\{t_n\}$ and sequence of inter arrival times $\{T_n\}$.

Definition 3.1.2 The Queue discipline is the rule or moral code determining
the manner in which customers form a queue and the manner in which they behave while waiting. Here we will assume that the queue discipline can be expressed as "first come, first served".

**Definition 3.1.3**  The service mechanism can be described as follows: Let the random variable \( \xi_n \) denote the time required to serve the \( n \)th customer; hence, the probability law governing the sequence of service times \( \{\xi_n\} \) expresses the service mechanism of the queueing system. It is natural to assume that the successive service times, \( \xi_1, \xi_2, \ldots, \xi_n, \ldots \), are statistically independent of one another and of the sequence of interarrival times \( \{T_n\} \) and that they have the same distribution function \( B(\xi), 0 < \xi < \infty \).

**Definition 3.1.4**  A Levy process is a process \( \{X(t): t \geq 0\} \) independent increments which satisfy the conditions. (i) \( X(t) \) is continuous in probability. That is, for each \( \varepsilon > 0 \), \( p\{|X(t)| > \varepsilon\} \rightarrow 0 \) as \( t \rightarrow 0^+ \). (ii) There exist left and right limits \( X(t-) \) and \( X(t+) \) and we assume that \( x(t) \) is right continuous; that is \( x(t+) = X(t) \).

**Definition 3.1.5**  A First passage Time [2]. Let \( Z(t) \) be the storage level at time \( t \). For the storage model we define the random variable \( T \) as follows:

\[
T = \inf \{ t: Z(t) = 0 \}, Z(0) > 0.
\]

Then \( T \) has the distribution \( T(x) = \inf \{ t: y(t) \leq -x \}, x > 0 \). Then \( T(x) \) is called the first passage time of the Levy process \( y(t) \) into the set \( (-\infty, -x] \). For completeness we define \( T(0) = 0 \).

**Definition 3.1.6**  Customers arrive in a poisson process with parameter \( \lambda \) and join the queue with probability 1 if the server is free, and with probability \( p(<1) \) otherwise, as long as the system is busy, the number of effective arrivals forms a poisson process with parameter \( \lambda p \). This is called **Balking**.

**Definition 3.1.7**  Customers arrive in a poisson process of random size having
the distribution \( \{ c_n : n = 1, 2, 3, \ldots \} \). The service time of each customer has the d.f \( B(x) \). Clearly, the input in this case is the compound poisson process. This is called **Batch arrivals**.

**Definition 3.1.8 Modified service Rule.**

Suppose that in the M|G|1 system the customer who initiates a busy period has a service time with d.f \( B_0(x) \), while all others have service times with d.f \( B(x) \). The busy period \( T_i \) initiated by a single customer can be obtained from \( B_0 \).

**Definition 3.1.9 Compound Poisson Process \( \{ T(x), x \geq 0 \} \)**

Let us consider a continuous infinity of M|G|1 systems, and observe for each system the duration of its first busy period. Our observations will yield a process \( \{ T(x), x \geq 0 \} \). We have

\[
T(x) = x + T_{A(x)}, \ldots
\]

where \( A(x) \) is the number of arrivals during \((0,x]\). Therefore

\[
P\{ T(x) \leq t \} = \sum_{n=0}^{\infty} e^{\lambda x} (\lambda x)^n / (n!) \cdot G_n(t-x) \quad (t \geq x \geq 0)
\]

where \( G_n(x) \) is the d.f of \( T_n \). The equation (1) shows that \( T(x) \) is a compound poisson process with unit drift, in which jumps occur at a rate \( \lambda \) and jump sizes have the same distribution as \( T_i \).

### 3.2 REPRESENTATION OF QUEUEING PROCESSES.

#### 3.2.1 Markov chain Representation. The method of the imbedded Markov chain.

Let the state of the queueing system at time \( t \) be denoted by the random variable \( y(t) \), so that, in any realization of the process \( \{ y(t), t \geq 0 \} \), the history of the system can be represented as a function \( y(\cdot) \) of time with domain \( (-\infty, \infty) \). Let the set \( \Omega \) denote the collection of functions with domain \( (-\infty, t] \) and having the
same range as \( y(\cdot) \). For each \( t \in (-\infty, \infty) \) let \( \theta_t \) be a specified subset of \( \Omega_t \) and corresponding to any actual realization of the process. Let \( L \) be the set of those values of \( t \in (-\infty, \infty) \) for which \( \theta_t \) contains as an element the contraction of \( y(\cdot) \) to the reduced domain \((-\infty, t]\).

We define the random variable \( X(t) = f_t\{ Y(T) : T \leq t, \ t \in T \} \) where \( f_t \) is some specified functional with domain \( \theta_t \). If we can select a domain \( \theta_t \) and a functional \( f_t \) for \( t \in (-\infty, \infty) \), such that

1. The set \( T \) almost certainly has no finite point of accumulation. (Hence the elements \( t \) can be ordered as follows: \( \ldots, t_1, t_2, t_3, \ldots \)).

2. If \( X_n = X(t_n) \) for each \( t_n \in T \), then

\[ P\{X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots\} = P\{X_n = x_{n+1} \mid X_n = x_n\} \text{ for all } n. \]

The process \( \{X_n, n = 0, 1, \ldots\} \) will then be said to be an imbedded Markov chain.

Remark: In applications of the above method three conditions must be satisfied if it to be of any value. First, the queueing system must be simple enough to permit a mathematical formulation of the above procedure. Second, for the random variable \( X(t) \) to be useful in describing the state of the system, the functional \( f_t \) must be sufficiently and suitably sensitive to variations in its argument \( Y(T) \). Third, the stochastic mechanism governing the transition from one instant in \( T \) to the next must be such that the transition probabilities

\[ p_{ij} = P(X_{n+1} = j \mid X_n = i) \] of the imbedded chain can be calculated.

We state some general results which provide criteria for determining whether the queueing system, as described by the Markov chain, is ergodic, transient or recurrent.
**Proposition 3.2.1** The system is ergodic if there exists a non-negative solution of the inequalities

\[ \sum_{j=0}^{\infty} p_{ij} x_j \leq x_{i-1}, \quad i \neq 0 \]

such that

\[ \sum_{j=0}^{\infty} p_{ij} x_j < \infty \]

**Proposition 3.2.2** If the system is ergodic then the finite mean first passage times \( d_i \) from the \( i^{th} \) state to the zero state satisfy the equations

\[ \sum_{j=0}^{\infty} p_{ij} d_j = d_{i-1}, \quad i \neq 0 \]

and

\[ \sum_{j=0}^{\infty} p_{ij} d_j < \infty \]

**Proposition 3.2.3** The system is transient if and only if there exists a bounded non-constant solution of the equations

\[ \sum_{j=0}^{\infty} p_{ij} x_j = x_i, \quad i \neq 0 \]

**Proposition 3.2.4** The system is recurrent if there exists a solution \( x_i \) of the inequalities

\[ \sum_{j=0}^{\infty} p_{ij} x_j \leq x_i, \quad i \neq 0 \]

such that \( x_i \rightarrow \infty \) as \( i \rightarrow \infty \).

**3.2.2. Markov process Representation. The Kolmogorov equations.**

Let \( p(t) = \{ p_{ij}(t) \} \) denote the matrix of transition probabilities associated with the process \( \{ x(t) \} \). Then \( p(t) \) satisfies the system of Kolmogorov equations

\[ \frac{dp(t)}{dt} = p(t)A(t), \quad \frac{dp(t)}{dt} = A(t)p(t) \]

with \( p(0) = 1 \) the identity matrix.
In (6) \( A(t) = (a_{ij}(t)) \) is the matrix of infinitesimal transition probabilities. Therefore, in terms of the matrix elements, the above equations become:

\[
\frac{dp_{ij}(t)}{dt} = \sum_{k=0}^{\infty} p_{ik}(t) a_{kj}(t) \\
\frac{dp_{j}(t)}{dt} = \sum_{k=0}^{\infty} a_{ik}(t) p_{kj}(t) \\
\text{for } i,j = 0,1,\ldots
\]

with \( p_{ij}(0) = \delta_{ij} = 0 \) for \( i \neq j \) \( = 1 \) for \( i = j \)

### 3.2.3 Integral Equation Representation. The theory of Lindly.

We made the following assumptions:

1. The processes \( \{ T_n \} \) is a recurrent process, and the distribution function \( A(t) \) is arbitrary.

2. The service times \( \xi_1, \xi_2, \ldots \) are equidistributed mutually independent random variables with common distribution function \( B(\xi) \).

3. There is only one server at the counter.

In view of the above assumptions, the queueing system we consider is of the type GI/G/1.

Let the random variable \( w(t) \) denote the waiting time of the customer arriving at the service at time \( t \). In general, the process \( \{ w(t) : t \geq 0 \} \) is non-Markovian.

Now let \( w(t_n,0) = w_n \) denote the waiting time of the \( n^{th} \) customer to arrive at the counter. Hence, the process \( \{ w_n, n=0,1,2,\ldots \} \) is the imbedded Markov chains associated with the queueing system. If at time \( t=0 \) the server is free, then \( w_0 = 0 \) however if the server is not free \( w_0 \) denotes the time that elapses before the server is free.
If \( w_0 \) is known, the random variables \( w_n \) may be determined successively from the following equation:

\[
  w_{n+1} = w_n + \xi_n - T_{n+1} \quad \text{if} \quad T_{n+1} - \xi_n < w_n \\
  = 0 \quad \text{if} \quad T_{n+1} - \xi_n \geq w_n
\]

The interpretation of this equation should be clear.

Let \( F_n(t) = P(w_n \leq t) \) denote the distribution function of \( w_n \): that is, \( F_n(t) \) is the distribution function of the waiting time of the \( n \)th customer to arrive at the service. Since the random variable \( w_n \) are non-negative \( F_n(t) = 0 \) for \( t < 0 \), and that \( F_n(0) \) is the probability that the \( n \)th customer will not have to wait.

If we start with \( F_0(t) \), the sequence of distribution functions \( \{ F_n(t) \} \) may be determined by the recurrence formula

\[
  F_{n+1}(t) = \int_0^t k(t,u) dF_n(u) \quad n=0,1,2, \ldots
\]

where the kernel function \( k(t,u) \) expresses the transition probabilities of the imbedded Markov chain \( \{ w_n \} \); that is

\[
  k(t,u) = P\{ w_{n+1} \leq t | w_n = u \} = \int_0^t [1 - A(w+u-t)] dB(w)
\]

The computations indicated may be carried out by utilizing the recurrence formulas

\[
  k_n(t) = \int_0^t B(t-u) dF_n(u)
\]

and

\[
  F_{n+1} = \int_0^t [1 - A(u - t)] dk_n(u)
\]

It is of interest to determine under what conditions an equilibrium distribution of waiting times exists; i.e., when does the unity distribution function
Proposition 3.2.5 A necessary and sufficient condition $\lim_{n \to \infty} F_n(t) = F(t)$ exist is that either $\xi_n(t_n) \leq \xi(T_n)$ or $T_n - \xi_n = 0$. If $\xi_n(t_n) \geq \xi(T_n)$ and $T_n - \xi_n \neq 0$ then $\lim_{n \to \infty} F_n(t) = 0$ for every $i \geq 0$.

The limiting distribution $F(t)$, if it exists, is independent of the initial distribution $F_0(t)$ and is the unique solution of the integral equation.

$$F(t) = \int_0^\infty k(t,u)dF(u).$$

3.3 QUEUE MODELS

3.3.1 Application to Dam Models[41].

Theorem 3.3.1 If $X(t)$ has an absolutely continuous distribution with density $k(x;t)$ then the random variable $T(x)$ has also an absolutely continuous distribution with density $g(t,x)$, where

$$g(t,x) = \begin{cases} k(t-x,u) & \text{for } t-x>0, \\ t \quad & \text{otherwise} \end{cases}$$ (1)

Lemma 3.3.1 For any Levy process $X(t)$ we have

$$\{M(t), M(t)-X(t)\} \sim \{X(t)-m(t), -m(t)\}$$

Here $M(t)$ & $m(t)$ are the supremum and infimum functionals of the Levy process $Y(t)$.

Proof: Using this lemma we have

$$r_+ (s,w) = s \int_0^\infty e^{w t} E[e^{w m(t)}]dt = (s/\eta) ((\eta +iw)/(s+\varphi(w)))$$ (2)

$$r_- (s,w) = s \int_0^\infty e^{w t} E[e^{w m(t)}]dt = \eta / (\eta + iw)$$ (3)

when $M(t)$ and $m(t)$ are the supremum and infimum of the net input process $Y(t)$ and $E(e^{w Y(t)}) = e^{t \varphi(w)}$ we therefore have
The identity is Wiener - Hopf factorization for the Levy process $Y(t)$ in the following sense. We have

\[
s/(s+\varphi(w)) = \exp\left\{ \log \left( \frac{s}{s + \varphi(w)} \right) \right\}
= \exp \left\{ \int_{0}^{\infty} e^{-st} E[e^{wY(t)} - 1] dt \right\}
= \exp \left\{ \int_{-\infty}^{\infty} e^{s'x} - 1 \right\} V_s(dx),
\]

when $V_s(0) = 0$, $V_s(dx) = \int_{0}^{\infty} e^{-st} t^{-1} k(t+x,t) dt dx \quad (x \neq 0)
\]

For fixed $s > 0$, $V_s$ is a Levy measure and the expression on the left side of (4) is an infinitely divisible c.f. For fixed $s > 0$, $r_+(s,w)$ and $r_-(s,w)$ are also infinitely divisible c.f's of distributions concentrated on $(0, \infty)$ and $(-\infty, 0)$ respectively. Thus (4) is a Wiener - Hopf factorization of $s[s+\varphi(w)]^{-1}$, and this factorization is unique (up to a factor $e^{aw}$, where $a$ is a real function of $s$) if restricted to infinitely divisible c.f's on the right side of (4).

Now fixed $s > 0$, $\eta (\eta + i\omega)^{-1}$ is the c.f of the exponential density on $(-\infty, 0)$ which is infinitely divisible and has Levy measure with density $e^{\omega x}(-x)^{-1} \quad (x<0)$. On account of the uniqueness of the factorization (4) we therefore have from (5)

\[
e^{\omega t}(-x) = \int_{0}^{\infty} e^{-st} t^{-1} k(t+x,t) dt \quad (x<0)
\]

or

\[
e^{\omega t} = \int_{0}^{\infty} e^{-st} g(t,x) dt \quad (x>0)
\]

since $e^{\omega t}$ is the L.T of the random variable $T(x)$ it follows that $g(t,x)$ is the required density.
Theorem 3.3.2  Let $T(x)$ be the random variable defined by the first passage time and assume that $T(x) \rightarrow 0$ as $x \rightarrow 0^+$. Then we have the following:

(i) $E[e^{sT(x)}] = e^{-\eta x}$ \quad (s > 0);

(ii) $\{T(x) < \infty\} = 1$ if $q \leq 1$, and $= e^{-\eta_0 x}$ if $q > 1$;

(iii) If $q < 1$, $E[T(x)] = x(1-q)^{-1}$, $\text{Var}[T(x)] = x \sigma^2(1-q)^{-3}$;

(iv) If $q = 1$, $E[T(x)] = \infty$;

Where $\eta = \eta(s)$ and $\eta_0 = \eta(0^+)$ are given by the following Lemma.

Lemma 3.3.2  The functional equation $\eta = s + \varphi(\eta)$ \quad (s > 0)\quad has a unique continuous solution $\eta = \eta(s)$ with $\eta(\infty) = \infty$ further more:

(i) as $s \rightarrow 0^+$, $\eta(s) \rightarrow \eta_0$ where $\eta_0$ is the largest positive root of the equation $\eta_0 = \varphi(\eta_0)$, and $\eta_0 > 0$ iff $q \geq 1$;

(ii) $\eta(0^+) = (1-q)^{-1}$ if $q < 1$ and $= \infty$ if $q = 1$

Proof:  If (i) is proved then (ii) - (iv) follow in the usual manner from the Laplace Transform (L.T) of the distribution of $T(x)$, given by (i).

The proof of (i) rests on the following properties:

(1) $T(x+y) - T(x)$ is independent of $T(x)$ and has the same distribution as $T(y)$;

(2) $T(x) \sim x + T(X(x))$.

(1) is a consequence of the fact that $Y(t)$ has drift $-1$ and a passage from $0$ to $-y < 0$ can occur only after a passage to $-x < 0$. Independence of $T(x)$ and $T(x+y) - T(x)$ is a consequence of the strong Markov property of the Levy process $Y(t)$. Property (2) follows from the definition.
It follows from (1) that \( f(s;x) = \mathbb{E}[e^{sX(x)}] \) satisfies the property \( f(s;x+y) = f(s;x) f(s;y) \). Using the fact that \( f(s;x) \to 1 \) as \( x \to 0^+ \) we obtain the result that \( f(s;x) = e^{\eta x} \). The property (2) now yields
\[
e^{\eta x} = e^{sx} \mathbb{E}[e^{\eta X(x)}] = e^{sx} \cdot \varphi(\eta x).
\]
Since this is true for all \( x > 0 \) it follows that the \( \eta \) in (i) is indeed the solution of the equation \( \eta = s + \varphi(\eta) \). The proof is therefore completed.

**Application to M/G/1 and Related systems**

The Busy period in M/G/1. We consider the single-server queueing system M/G/1 in which customers arrive in a poisson process with parameter \( \lambda (0 < \lambda < \infty) \) and are served on a first-come, first served basis. From the L.T of the Levy process \( X(t) \) we have \( \varphi(0) = \lambda - \lambda \psi(0) \) where \( \psi(0) \) is the L.T of the service time d.f \( B(x) \). It follows from the theorem 3.3.2(1) that L.T of the busy period \( T(x) \) is given by \( e^{\eta x} \), where \( \eta = \eta(s) \) is the unique continuous solution of the equation \( \eta = s + \lambda - \lambda \psi(\eta) \), with \( \eta(\infty) = \infty \).

At time \( T = 0 \), let us suppose that there are \( n (\geq 1) \) customers in the system and service is due to commence on the first of them. Let \( T \) be the busy period that follows. Clearly \( T_n = T(v_1 + v_2 + \ldots + v_n) \) where \( v_1, v_2, \ldots, v_n \) are the service times of these \( n \) customers. Therefore
\[
\mathbb{E}(e^{sT_n}) = \mathbb{E} \left[ e^{sT(v_1 + v_2 + \ldots + v_n)} \right]
\]
\[
= \mathbb{E}[e^{s(\psi(\eta) + 1 + \psi(\eta) + \ldots + \psi(\eta))}]
\]
\[
= [\psi(\eta)]^n.
\]
In particular $T^*$ is the conventionally defined busy period.

**Illustration 1:**

Let us consider a dam model with input process $X(t)$ having the stable density

$$k(x,t) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} \quad (x > 0, \ t > 0)$$

Then for $t > x > 0$

$$g(t,x) = \frac{1}{\sqrt{2\pi t}} \frac{x}{(t-x)^{3/2}} e^{\frac{(t-x)^2}{2(t-x)}}$$

$$= e^{2\alpha} n(t-x,x)$$

where $n(t',x)$ is the inverse Gaussian density,

$$n(t',x) = \frac{1}{\sqrt{2\pi t'}} \frac{x}{t'^{3/2}} e^{\frac{-(t'-x)^2}{2t'}} \quad (t' > 0, \ x > 0)$$

Note that in this case $T(x) < \infty$ with probability $e^{2\alpha}$.

**Illustration 2.** Let $X(t)$ have the gamma density $X^{t+1}$

$$k(x,t) = e^{-\lambda t} \frac{x^{t-1}}{\Gamma(t)} (x > 0, \ t > 0)$$

Then for $t > x > 0$

$$g(t,x) = e^{-\lambda t} \frac{x^{t-1}}{t^{t-1}} \frac{(t-x)^{t-1}}{\Gamma(t)}$$

$$= x e^{-(\lambda t)^{t-1}} \frac{(1-(x/t))^{t-1}}{\Gamma(t+1)}$$

Using the results $\sqrt{2\pi t^{t+1/2}} e^{t}$ (striling approximation) and $\lim_{t\to\infty} \frac{(1-(x/t))^{t-1}}{\Gamma(t+1)} = e^{-x}$ we find that

$$g(t,x) \sim \frac{e^{-\lambda t} x^{t-1}}{\sqrt{2\pi t^t}}$$

if $\lambda \neq 1$
If $Q = 1$ we also have
\[ p\{T(x) > t\} \sim \sqrt{\frac{2}{\pi}} t^{1/2} \quad (t \to \infty) \]
However, this result is true in the general case $Q = 1$, $\sigma^2 < \infty$.

Illustration 3  The process $\{T(x), x \geq 0\}$ and Behaviour of $T(x)$ as $x \to \infty$.

From the proof of theorem 3.3.2 it is clear that $\{T(x), x \geq 0\}$ is a Levy process with $E(e^{xT(0)}) = e^{x\eta(0)}$. Further into the M/G/1 system, we know that $T(x)$ is a compound poisson process.

From the relation $\eta = s + q(\theta)$ with $q(\theta) = \int_0^\infty (1 - e^{\theta x})x^2 M(dx)$ ($\theta > 0$)
where $M^*(0) \leq \infty$, we obtain
\[ \eta = s + \int_0^\infty (1 - e^{\eta x})x^2 M(dx) \]
\[ = s + \int_0^\infty x^2 M(dx) \{ \int x(1 - e^{\eta x}) G(dt,x) + (1 - e^{-\eta x}) \} \]
\[ = s + \eta_0 + \int_0^\infty x^2 M(dx) \int x(1 - e^{\eta x}) G(dt,x), \]
where $G(t,x)$ is the d.f of $T(x)$ and $\eta_0 \geq 0$. Writing
\[ N(dt) = t^2 \int_0^t x^2 M(dx) G(dt,x) \quad \text{(6)} \]
we find that
\[ \eta = \eta_0 + s + \int_0^\infty (1 - e^{\eta x})t^2 N(dt) \quad \text{(7)} \]
which is of the form (6). The presence of the constant $\eta_0$ in (7) indicates the possibility that the event $\{T(x) = \infty\}$ may have a non-zero probability.

If $X_0 = \inf \{ x: T(x) = \infty \}$ then we define $T(x) = \infty$ for all $x \geq X_0$. Thus the
random variable $X_0$ is the life time of the process $T(x)$, and $X_0 < 0$ with probability one iff $Q > 1$.

Behaviour of $T(x)$ as $x \to \infty$, assuming that the input $X(t)$ has finite mean and variance. Since $T(x)$ is a Levy process the limit distribution (if it exists) belongs to the stable family. In particular the central limit theorem behaviour (following theorem 3.3.3) is an obvious consequence of the fact that when $Q < 1$, $T(x)$ has finite mean and variance when $Q = 1$, $T(x)$ has infinite mean and we find the limit distribution to be the stable distribution with exponent 1/2 (following theorem 3.3.4).

**Theorem 3.3.3** If $Q < 1$ and $\sigma^2 < \infty$ then

$$\lim_{x \to \infty} p \left\{ \frac{x (1 - Q)^{-1}}{\sigma \sqrt{\frac{Q}{(1 - Q)^{3/2}}} \leq t} \right\} = N(t)$$

**Lemma 3.3.3** If $Q = 1$ and $\sigma^2 < \infty$ then

$$\eta(s) = \frac{1}{\sqrt{2\pi \sigma}}$$

**Proof:** Under our assumptions $\phi(0) = Q = 1$ and $\phi'(0) = -\sigma^2$

$$\eta = s + \phi(0) + \eta \phi'(0) + (1/2) \eta^2 \phi''(0) + o(\eta^3)$$

$$= s + (1/2) \sigma^2 \eta^2 + o(\eta^3)$$

This gives

$$\eta^2 = (2s/\sigma^2) + o(\eta^3)$$

$$= (2s/\sigma^2)[1 + o(1)]$$

which leads to the desired result.

**Theorem 3.3.4:** If $Q = 1$ and $\sigma^2 < \infty$ then

$$\sigma^2 T(x)$$
Proof: We have \( E \{ e^{sT(z/t^2)} \} = e^{-\eta(s) (z^2/2)} \)

Using Lemma 3.3.3 for each fixed \( s > 0 \)

\[
x \eta \left[ (x \sigma^2)/(x^2) \right] = x \left[ (\sqrt{2s}/x) + o(1/x) \right]
= \sqrt{2s} + o(1) \quad (x \to \infty)
\]

Therefore

\[
E e^{sT(z/t^2)} \to e^{\sqrt{2s}} \quad (x \to \infty)
\]

and the desired result follows since \( e^{\sqrt{2s}} \) is the L.T of the d.f. \( G_{1/2}(t) \)

### 3.3.2 Storage Model with Random output

**Theorem 3.3.5** For the storage process \( \{ z(t); t \geq 0 \} \) defined by

\[
z(t) = \max \{ 0, x + y(t) \} \quad \text{for } 0 \leq t \leq T_i
= \max \{ 0, y(t) - y(T_{i-1}) \} \quad \text{for } t \geq T_i
\]

where \( N = N(t) = \max \{ K: T_k \leq t \} \). Here \( T_i > 0 \),

\( T_k - T_{k-1} > 0 \) \( (k \geq 2) \) a.s. so that \( T_k \to \infty \) as \( k \to \infty \). Then

\[
z(t) = \max \{ y(t) - m(t), x + y(t) \}
\]

where \( m(t) \) is the infimum of the process \( Y \).

**Theorem 3.3.6** For \( I_m(w) \geq 0, I_m(w_0) < 0 \) we have

\[
\int_0^\infty \int_0^\infty e^{iw} e^{iw_0} e^{sE[w|z(0) = x]} \, dt \, dx
= u*(s, w) + \frac{w}{s u*(s, 0) w_i w} u*(s, w) v*(s, w_i)
\]

**Theorem 3.3.7** Let \( z(0) \) have d.f.B. Then for \( s > 0 \).

\[
E \left[ e^{sT_1} \right] = 1 - \left( s/(s+\lambda) \right) \exp \left\{ e^{sT_1} \right\} \left\{ t^{e^{sT_1}} \right\} \, dt
\]
Proof: Let \( y(t) = x + y(t) - J(t) \), where \( J(t) \) is the last jump in \( X_j \), before \( t \). Then \( Y(t) = x + Y(t) - J(t) \) with probability one. If \( x \) has d.f. \( B \), then \( Y(t) \) and \( Y(t) \) have the same finite dimensional distributions. Therefore the \( N_k \) have the same distribution as the (descending) ladder variable of the random walk \( \{ Y(t), n \geq 0 \} \). Recall that \( T_i = tN_i \). For the random walk \( \{ (t, Y(t)) \} \) we have already obtained the distribution of the ascending ladder epoch \( T_i \). The Wiener-Hopf factorization then gives

\[
1 - E(e^{sT}) = \left[ 1 - E(e^{sT_i}) \right] \left[ 1 - E(e^{sT}) \right]
\]

where \( E(e^{sT_i}) = \lambda / (s + \lambda) \)

and (by theorem 3.3.1)

\[
1 - E(e^{sT}) = \exp\left\{- \int_0^\infty e^{st} p\{Y(t) > 0\} dt\right\}
\]

Then the desired result follows.

3.4. NUMERICAL SOLUTIONS

3.4.1 Queue Model and its Numerical Solutions[47,51]

We consider a stochastic process \( X(t) \), \( 0 < t < \infty \), defined on \( [0, \infty) \), which represents the content of a dam or store or the virtual waiting time or workload in a queue etc. Inputs at times \( t_1 < t_2 < \ldots (t_i > t_i = 0) \) occur in a renewal process with time \( T_i = t_{i+1} - t_i \), being identically and independently distributed random variables with a common distribution function \( p(T_i \leq x) = A(x) \), \( 0 \leq x < \infty \), \( A(0+) = 0 \).

The sizes \( S_n \) of the inputs, \( n=1,2,\ldots \) are independently and identically distributed random variables, independent of \( T_i \), with \( p(S_n \leq x) = B(x) \), \( 0 \leq x < \infty \), \( B(0+) = 0 \).

The outflow from the store is determined by a general release rule \( r(\cdot) \) such that, for \( t \neq \) some \( t_i \),
\[
\frac{dX(t)}{dt} = -r(x(t)), \quad (1)
\]

with \( r(0-) = 0 \), \( 0 < r(x) \leq \infty \) for \( 0 < x < \infty \).

The model GI/G/r(x), subsumes not only the work on the single-server queue but also works on general release-rate dams. It will be convenient, although not strictly necessary, to assume that \( A(x) \) and \( B(x) \) are absolutely continuous on \([0, \infty)\) with \( a(x) = A'(x) \) and \( b(x) = B'(x) \) being proper density functions. To apply our approximation procedure below we require that the moments \( \mu_i = E(T') \) and \( \nu_i = E(S') \) exist and are finite at least up to a finite order \( M \).

For some \( \epsilon \geq 0 \) we define
\[
\chi_{\epsilon}(x) = \int_{y \in e} \frac{dy}{r(y)}, \quad 0 < x < \infty \quad (2)
\]

If at time \( T \) the content is \( X(T) = x > \epsilon \), then \( \chi_{\epsilon}(x) \) is the time to run down to content \( \epsilon \) if no inputs occur in the interval \([T, T^*] \). If \( \lim_{\epsilon \to 0} \chi_{\epsilon}(x) < \infty \), then the store can empty in a finite time, and the zero state may be recurrent and will be so if the system is recurrent, while if \( \lim_{\epsilon \to 0} \chi_{\epsilon}(x) = \infty \), then the store can not empty, in a finite time from any positive content and the zero state is not recurrent.

We consider the transformed process
\[
Y(t) = \chi_{\epsilon}(x(t)) \quad (3)
\]
with \( \epsilon = 0 \) if \( \chi_{\epsilon}(x) < \infty \), and call this the 'extinction time' of the store from level \( X(t) \), while if \( \chi_{\epsilon}(x) \) is divergent we choose a suitable, \( \epsilon > 0 \), and call \( Y(t) \) the 'pseudo extinction time' of the store from level \( X(t) \). If \( X(t) < \epsilon \) this can be negative, so \( -\chi_{\epsilon}(X(t)) \) is the pseudo extinction time from \( \epsilon \) to \( X(t) \) in this case.
The relationship (1) is a mapping from $x \in [0, \infty)$ to $y \in [0, \infty)$ if the store can empty, where as if it cannot then (3) is a mapping of $x \in [0, \infty)$ to (possibly a subset of) $y \in (-\infty, \infty)$. It is necessary to complete the frist case; the line $y = \gamma(x)$ defined for non-negative $x$ and $y$ adjoin the half-line $\{x = 0, y \in (-\infty, 0]\}$ We write $x = \bar{\chi}(y)$ as the unique (as $r(x) > 0$ for $x > 0$) inverse mapping of $y = \gamma(x)$. If the store can empty we define $\bar{\chi}(y) = 0$ for $y < 0$. In this case a negative extinction time indicates an empty store, and its magnitude $|y|$ is the time elapsed since it last became empty.

If we consider the imbedded Markov process $(z_n^+, y_n^-) = (\text{content at time } t_n^+, \text{extinction time at time } t_n^-)$ in the $G1/G/1$ queue for $n = 1, 2, \ldots$, we have

$$z_{n+1}^+ = (z_n^+ + s_n - T_n^-)^+ = \chi(\chi(z_n^+ + s_n^+ - T_n^-), (4)$$

and this is the horizontal resolute of a discrete time, continuous-state-space random walk along the line $y = \chi(x)$ in $(x, y)$ space, where in the increment $s$ are applied to the right, i.e in content, and $T$ are applied down wards, ie in elapsed time.

Similarly, the second Lindly or (pseudo extinction time recurrence is

$$Y_{n+1}^+ = Y_n^+ + s_n - T_n^- = \chi(\chi(Y_n^+ + s_n^+) - T_n^-), (5)$$

which is the vertical resolute of the same random walk along the line $y = \chi(x)$. In this instance the negative values are times since the end of the last busy period.
Polynomial approximation.

The distribution function $G_n(y)$ and the density function $g_n(y)$ and its limit $g(y)$ if these exist, may be approximated for $y \in (-\infty, \infty)$ by means of weighted sums of orthogonal polynomials. The question that now arises in whether or not for a given continuous function $f(x)$ one can find polynomial which approximates $f(x)$ to a given accuracy [47]. The answer to this was given by Weistrass in a famous theorem. The Weistrass theorem establishes that a given continuous function $f(x)$ can be approximated to a polynomial of certain degree with a certain degree of accuracy. The statement of the theorem is:

Let $f(x)$ be continuous for $a \leq x \leq b$ and let $\varepsilon > 0$. Then there is a polynomial $p(x)$ for which $|f(x) - p(x)| \leq \varepsilon$, $a \leq x \leq b$. Some of the standard procedures for this approximations are (i) minimax method and (2) least square method.

The least square method implies that of all polynomials $\nu(x)$ of degree $\leq n$ that approximate to a continuous function $f(x)$, $a \leq x \leq b$, we choose the one that

minimises $\int_a^b w(x) \left[ f(y) - \nu(x) \right]^2 dx$ \hspace{1cm} (6)

where $w(x)$ is the weight function. If the various polynomials $\nu_n(x)$ are such that

$\int_a^b w(x) \nu_n(x) \nu_m(x) dx = 0, \quad n \neq m$ \hspace{1cm} (7)

then the polynomials are said to be orthogonal with respect to the weight function $w(x)$. Two of the well known orthogonal polynomials are Hermite and Laguerre polynomial.
Hermite polynomial

For the Hermite polynomial the interval is \((-\infty, \infty)\) and \(w(x) = e^{-\alpha^2 x^2}\)  

Taking \(\alpha^2 = 1\), the Hermite polynomial \(H_r(x)\) of degree \(r\) is defined as  

\[ e^{x^2} H_r(x) = (-1)^r \frac{d^r}{dx^r} (e^{x^2}) \quad (9) \]

The Hermite polynomials possess the property that  

\[ \int_{-\infty}^{\infty} e^{x^2} H_r(\alpha x) H_s(\alpha x) dx = 0 \quad \text{for} \quad r \neq s \quad (10) \]

and  

\[ \int_{-\infty}^{\infty} e^{x^2} H_m(\alpha x) H_n(\alpha x) dx = m! \delta_{mn} \quad \text{if} \quad m = n \quad (11) \]

with \(\alpha^2 = 1\), the first six polynomials are

\[
\begin{align*}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12 \\
H_5(x) &= 32x^5 - 160x^3 + 120x
\end{align*}
\]

The polynomials of higher degree can be found by the recurrence formula

\[
H_{r+1}(x) = 2xH_r(x) - 2rH_{r-1}(x)
\]

Laguerre polynomial

For this polynomial the interval is \([0, \infty)\) and the weight function is \(w(x) = e^{-x}\)

The Laguerre polynomial of degree \(n\) is \(L_n(x)\) defined by

\[
L_n(x) = e^x \frac{d^n}{dx^n} \{x^n e^{-x}\} = \sum_{m=0}^{n} (-1)^m \frac{n!}{m!} x^n/m! \quad (12)
\]

The first few polynomials are

\[
\begin{align*}
L_0(x) &= 1 \\
L_1(x) &= 1 - x \\
L_2(x) &= x^2 - 4x + 2 \\
L_3(x) &= 6 - 18x + 9x^2 - x^3 \\
L_4(x) &= x^4 - 18x^3 + 72x^2 - 96x + 24
\end{align*}
\]
Higher degree Laguerre polynomials can be found from the relation

\[ L_{rs+1}(x) = (1 + 2r - x)L_r(x) - r^2L_{r-1}(x) \quad \text{[28]} \]

The distribution function \( G_n(y) \), density function \( g_n(n) \) and its limit \( g(y) \), if they exist, may thus be approximated for \( y \in (-\infty, +\infty) \) with a generalised Fourier series.

The approximate functions are Hermite polynomials.

Generalised Gram–Charlier (GGC) representation

\[ g(y) = \sum_{j=0}^{\infty} b_j \psi H_j(y) \{ \alpha(y) \}^p \quad \text{[13]} \]

where \( p \in (0, 1) \), and typically \( p = 0.5 \).

We now formally describe the principle of the GC representation. Assume for the present that the following representations are permissible.

\[ g_n(y) = \sum_{j=0}^{\infty} g_{nj} H_j(y) \alpha(y) \quad \text{[14]} \]

\[ l(y, w) = \sum_{m=0}^{\infty} l_m(w) H_m(y) \alpha(y) \quad \text{[15]} \]

\[ l_m(w) = \sum_{r=0}^{\infty} l_{mr} H_r(w) \quad \text{[16]} \]

we apply these to (3.3) and obtain

\[ g_{n+1}(y) = \int l(y, w) H_m(y) \alpha(y) g_{nj} H_j(w) \alpha(w) dw \quad \text{[17]} \]

Assuming order reversal is valued we find

\[ g_{n+1}(y) = \sum_{m=0}^{\infty} H_m(y) \alpha(y) \sum_{r=0}^{\infty} l_{mr} H_r(w) \quad \text{[18]} \]

by orthogonality (6). Hence by comparison with the representation of the form (9) of \( g_{n+1}(y) \) we obtain

\[ g_{n+1,m} = \sum_{r=0}^{\infty} r! l_{mr} g_{nj} \quad \text{[19]} \]
or alternatively in terms of (infinite) vectors $g_{n+1}, g_n$ and matrix

$$L = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = (r!)_n^m \quad \text{that}$$

$$g_{n+1} = Lg_n$$

(20)

If an initial vector $g_0$ is suitably defined, then

$$g_n = L^n g_0$$

(21)

defines the $n$th iterate of the imbedded Markov process in the sense that

$$(g_0, g_1, \ldots)$$

are the GC spectral harmonic weights associated with the GC representations of the density function $g_n(y)$.

As $Y_n = \chi(X(t_n))$ it follows that the distribution function of the content just before the $n$th input is $H_n(x) = p(X(t_n) \leq x) = G_n(\chi^{-1}(x)), \quad 0 < x < \infty$, so we can always recover the content distribution from the extinction-time distribution.

In order to apply (21) we must determine the matrix elements $(L_{qv}, q, v = 0, 1, \ldots M)$.

Consider first for fixed finite $w$

$$I_q(w) = \int_{-\infty}^{\infty} H_q(y) l(y, w) dy,$$

which exists, since $H_q(y)$ is a polynomial of order $q$ in $y$ and $l(y, w)$ is a density function in $y$ for fixed finite $w$, with moments $M_i$ and $v_i$ of $a(.)$ and $b(.)$ existing at least up to order $M$. Using truncated form of (15) to order $M$ and reversing the order of sum and integral, we find

$$I_q(w) = \sum_{m=0}^{M} l_m(w) \int_{-\infty}^{\infty} H_q(y) H_m(y) \alpha(y) dy$$

$$= q! l_q(w).$$

Repeating this kind of argument with

$$I_{qv} = \int_{-\infty}^{\infty} H_q(w) l_q(w) \alpha(w) dw$$

$$= q! l_{qv}(w).$$
with (16) (truncated) we find $I_{qv} = v! = L_{qv}$, so that

$$L_{qv} = \frac{1}{q!} \int_{-\infty}^{+\infty} H_q(w) \alpha (w) \int_{-\infty}^{+\infty} H_q(y) l(y,w) dy dw \quad \text{for } q = v = 0,1,\ldots, M.$$  

(22)

It remains to evaluate $(L_{qv})$. As $l(y,w)$ is a conditional density function in $y$ and $H_0(y) = 1$ for all real finite $y$, it follows that $L_{00} = 1, L_{qv} = 0, v = 1,2,\ldots, M$.

Using the standard form

$$H_q(x) = \sum_{i=0}^{[q/2]} h_{q,i} x^{q-2i} = q! \sum_{j=0}^{[q/2]} \frac{(-1)^j x^{q-2j}}{j! 2^j (q-2j)!}$$

(23)

of the Hermite polynomials it follows that

$$L_{qv} = \sum_{j=0}^{[q/2]} \sum_{i=0}^{q-2j} \frac{(-1)^j x^{q-2j}}{j! 2^j (q-2j)!} \mu_i \mu_{q-2j,i,v}$$

(24)

where $\mu_i^* = \mu_i/i!$, $i = 0,1,\ldots, M$ and

$$J_{sv} = \frac{1}{s!} \int_{-\infty}^{+\infty} H_s(x) \alpha(x) \int_{u}^{+\infty} \chi(u) b(u, \chi^{-}(x)) du dx$$

$$= \frac{1}{s!} \int_{-\infty}^{+\infty} H_s(x) \alpha(x) \int_{-\infty}^{+\infty} \chi(y + \chi(x)) b(y) dy dx$$

(25)

We can approximate $b(x)$, the density function of the size distribution by Hermite polynomials and this can be used in the evaluation of $l(y,w)$ or we can use Laguerre polynomial. This approximation needs the existence of moments of finite order $M$. Similarly the finite summation for $b_x$ in the expansion of $b(x)$
distribution from content distribution. In the case of non-independent random inputs we can choose tractable dependent nature to obtain the content distribution of the input (as in the case of covariance stationary processes).

3.5 FISH FEEDING AS A GENERAL STORAGE MODEL

3.5.1 Finite Store

Suppose that the store has finite capacity k, and that any excess input that would otherwise take the content above k is instantaneously lost by overflow. We have effectively that \( r(x) = \infty \) for \( x \geq k \), and \( \chi(x) = \chi(k) \) for \( x \geq k \). Except for very special cases the finiteness of the store causes a substantial increase in the difficulty of purely analytical methods. However, for our approximating procedure there are only trivial modifications. Thus

\[
J_{sv} = \frac{1}{s!} \int_{-\infty}^{\chi(k)} H_{sv}(x) \alpha(x) \int_{0}^{\infty} \{ \min(\chi(k), \chi(y + \chi^{-1}(x))) \} b(y) dy dx
\]

where the weight \( h_j \) are taken to be 0 for \( \varphi_j > \chi(k) \).

We consider some illustrations of the finite store.

Illustration 3.5.1 \( r(x) = k \sqrt{x} \), where \( 0 < K < \infty \), such as for a square-sided sink.

Then

\[
y = \chi(x) = \frac{2}{k} \sqrt{x}
\]

\[
x = \chi^{-1}(y) = \frac{k^2}{4} y^2
\]

for all \( x \) and \( y \in [0, \infty) \), as \( \chi(x) < \infty \).

Adjoin to this the half line \( \{ x = 0, y \in (-\infty, 0) \} \).
Illustration 3.5.2. \( r(x) = 1 \). This is the single server queue GI/G/1. (Section 3.2.3). The store can clearly empty in finite time. We have \( \chi(x) = x = \chi^-(x) \) for \( x \in [0, \infty) \). Adjoin to this the half-line \( \{ x = 0, y \in (-\infty,0] \} \). The inverse mapping is \( \{ x = 0, y \in (-\infty,0] \} \). The inverse mapping is \( \chi^-(y) = y^* \), where \( y^* = \max(0,y) \), for \( y \in (-\infty,\infty) \).

Illustration 3.5.3. M/M/\( x \). The time dependent as well as the limiting content distribution is known. The limiting content density function is

\[
\nu_x e^{\lambda x} \quad \text{for} \quad 0 < x < \infty
\]

with zero probability of emptiness, and \( f(x) \) is discontinuous at 0 if \( \lambda \leq 1 \).

By transformation \( Y = \frac{1}{\nu} x \), we find

\[
g(y) = \frac{\lambda^\nu \nu_x e^{\lambda y}}{\nu} \quad \text{for} \quad -\infty < y < \infty
\]

\[
G(y) = F(e^y) = \frac{1}{\nu} \int_{e^y}^{\infty} \frac{\lambda^\nu e^{\lambda w}}{\lambda} \, dw
\]

\[
E(Y) = \psi(\lambda) - \frac{1}{\nu}, \quad V(Y) = \psi(\lambda)
\]

Skewness = \( \frac{\psi^{11}(\lambda)}{(\psi^{3/2}(\lambda))^3} \), Kurtosis = \( \frac{\psi^{(3)}(\lambda)}{(\psi^{3/2}(\lambda))^2} \)

\[
\varphi_1(\theta) = E(e^{\psi y}) = \frac{((\lambda - \theta)^\nu)}{((\lambda)^\nu)} \quad \text{for} \quad \theta < \lambda
\]

where \( \psi(\lambda) = (d/d\lambda) \frac{1}{\nu} \left( \frac{1}{\lambda} \right) \) is the digamma function. Provided that \( \lambda > 1 \), we have

\[
g_1 = \sum_{j=0}^{\frac{1}{2}} \frac{(-1)^j \lambda^j}{(j-2i)!} \int d \frac{\lambda^\nu}{e^{\lambda x}} \left( \frac{\lambda}{\nu} \right)
\]

and \( \sum_{j=0}^{\infty} g_j = (\lambda/\nu)e^{-\nu/2} \).
The approximating GC representation has been found to be excellent provided \( \lambda \) is rather larger than unity and a suitable choice of \( \beta \) and \( \delta \) is made, for example

\[
\beta = \mu \quad \text{and} \quad \delta = 1.3\sigma \quad (\delta \text{ rather larger than } \sigma)
\]

due to the effectively exponential left hand tail in \( g(y) \) and \( G(Y) \). We find the mean variance, skewness and kurtosis exactly and for a GC representation with various \( M \) values for \( \nu = 1 \), \( \lambda = 2, 4, 6, 8 \) and the exact and approximate normalized GC (\( M = 11 \)) density function and distribution for \( M/M/x, \lambda = 4, \nu = 1 \).

3.5.2 Food level in the stomach of a fish.

It has been possible to formulate a GI/G/r(x) storage model as an integral equation which can be manipulated albeit approximately, via matrix technique. This solution process provides a unified approach to a wide class of queueing and storage problems.

Despite their historical overtones, GC series have proven surprisingly successful for the representation of the distribution function of the extinction or pseudoextinction times and even better for the low order moments. Of particular interest are models of queues or stores with non-independent inter arrival or input-size processes (this has been applied to the food level in the stomach of a fish), inputs of continuous type or of Markov or auto regressive form. Furthermore, time-dependent results, at input time points, can also be investigated.
Experiment

The above storage problem can be applied to discuss the stomach content of a fish from the known arrival rate of food into the fish tank. An experiment was designed with a fish-pond and a square fish tank in which the fish carp (Catla catla) were cultured. The carp is an omnivorous (eats both plant and animal) feeder, feeding continuously by keeping mouth open and by gulping water with food content. From the fish tank the water along with plankton (food) was allowed to flow into the square fish tank at a known rate \( k \sqrt{x} \) where \( k = 0.37975 \) and \( x \) denotes the volume of water released in 4 minutes flow. This rate of flow was ascertained by repeated analysis of water flow.

Ten healthy fish of the same size and weight were introduced after starving into the square fish tank. The water was allowed to flow out from the square fish tank in the same constant rate. At an interval of 1 hour/1/2 hour, a fish was taken out and the stomach content (wet weight) was estimated after sacrificing the fish. This was repeated until the stomach content of the last fish (10\(^{th}\) fish) was noted. The above experiment was repeated ten times with the fish of the same size and weight for ascertaining the consistency of the results.
DATA POINTS

We obtained the following data from the experiment.

**TABLE 1**

1. **Input**: 6 litres of water with plankton in 4 minutes flow from the fish-pond

2. **Output**: 6 litres of water with plankton in 4 minutes flow from the fish tank.

3. First fish is taken out from the fish tank after 1 hour.

4. Second fish is .. .. .. .. .. 1 1/2 hours.

5. Third fish is .. .. .. .. .. 2 hours.

6. Fourth fish is .. .. .. .. .. 3 hours.

7. Fifth fish is .. .. .. .. .. 4 hours.

8. Sixth fish is .. .. .. .. .. 5 hours.

9. Seventh fish is .. .. .. .. .. 6 hours.

10. Eighth fish is .. .. .. .. .. 6 1/2 hours.

11. Ninth fish is .. .. .. .. .. 7 hours.

12. Tenth fish is .. .. .. .. .. 8 hours.

**TABLE 2**

The content of food is 1.47 grams in 6 litres of water in 4 minutes flow from fish-pond (input). The content of food when the fishes were introduced is at the rate of $0.37975\sqrt{x}$ where $x$ is the volume of the water.

1. Gut content after 1 hour = 1.33578 gram (wet weight)

2. Gut .. 1 1/2 hours = 1.44438 gram

3. Gut .. 2 hours = 1.52228 gram.

4. Gut .. 3 hours = 1.63238 gram.

5. Gut .. 4 hours = 1.73038 gram.

6. Gut .. 5 hours = 1.72148 gram.

7. Gut .. 6 hours = 1.71338 gram.

8. Gut .. 6 1/2 hours = 1.53438 gram.

9. Gut .. 7 hours = 1.32858 gram.

10. Gut .. 8 hours = 1.87918 gram.

**MODEL**

We use the model in illustration 3.5.1.
3.5.3 DISCUSSION AND CONCLUSION

We approximate the general arrival and general service rate as Markovian with parameters \( \lambda = 2 \) and \( \mu = 4 \) respectively and the release rate \( r(x) \) is taken as

\[
0.37975\sqrt{x} = 29.4156 \text{ gram/minutes}
\]

We apply the approximation procedure suggested by Smith and Yeo[47] and obtained the approximate distribution function for the stomach content of the fish, as having mean = 0.6767, standard deviation = 1.28, skewness = -1.72 and kurtosis 2.257. From the actual distribution of Gut content of the experiment, we find that the mean is 0.71 gram with the standard deviation 1.2.

Thus we conclude that the fish Catla can be cultured in a fish tank in which the food released at the rate of 29.4156 gram/minute into a square tank and the fish is to be allowed for 8 hours to get its stomach to be filled completely.