CHAPTER 6

SOME TEST PROCEDURES BASED ON TRIMMED SAMPLE SPACE

6.1 Introduction

A classical theory of Hypothesis Testing, in which tests are derived as solutions of clearly stated optimum problems, was first developed by Neyman and Pearson in the 1930s and since then the research work has been considerably extended.

Let \((x_1, \ldots, x_n)\) be an \(n\)-dimensional random variable having distribution function \(F(x_1, x_2, \ldots, x_n; \mu)\) where \(\mu\) is an \(r\)-dimensional parameter and its space is generally denoted by \(\Omega\) which will be an open region in the Euclidean space \(\mathbb{R}^r\).

If \((x_1, \ldots, x_n)\) is an independent random sample, then

\[
F(x_1, x_2, \ldots, x_n; \mu) = \prod_{i=1}^{n} F(x_i; \mu) \quad \ldots \ldots \quad (6.1.1)
\]

Let \(\omega\) be a subset of \(\Omega\). If \(\mu_0\) is the true value of \(\mu\) in the population, the first step in classical testing theory is to set up a statistical hypothesis \(H_0: \mu_0 \in \omega\) against the alternative hypothesis \(H_1: \mu_0 \in \Omega - \omega = \omega^1\). It is customary to say that \(H_0\) is true if \(\mu_0 \in \omega\) and \(H_1\) is true (or) \(H_0\) is false if \(\mu_0 \in \omega^1\). The hypothesis \(H_0\) is generally called the Null hypothesis and \(H_1\) is generally called the Alternative hypothesis. If \(\omega\) contains only one point \(\mu\), then the hypothesis is called simple otherwise composite and \(\omega\) is called a critical region.

6.2 Type I and Type II Errors

Definition:

Rejecting the Null hypothesis when it is true is called Type I error and its probability is denoted by \(\alpha\).
i.e., \( P [ x \in \omega' \mid H_0 ] = \alpha. \) \hspace{1cm} (6.2.1)

A Type II error is an error in the opposite direction and its probability is denoted by \( \beta. \)

i.e., \( P [ x \in \omega' \mid H_1 ] = \beta \) \hspace{1cm} (6.2.2)

The aim is to test the hypothesis \( H_0 : \mu = \mu_0. \) Suppose that besides the tested value \( \mu_0 \) of the parameter \( \mu \), some other values, say \( \mu_1, \mu_2, \ldots \) of the parameter \( \mu \), may be true. Then the hypothesis \( H_0: \mu = \mu_0; H_1: \mu = \mu_1; H_2: \mu = \mu_2; \ldots \) are called the admissible hypothesis. To distinguish it from the other hypothesis, the hypothesis \( H_0: \mu = \mu_0 \) is called the Null Hypothesis. Every admissible hypothesis is called an Alternative Hypothesis. Considered as a function of all admissible values of \( \mu \), the probability of rejecting the Hypothesis \( H_0: \mu = \mu_0 \) and accepting the alternative hypothesis \( H_1: \mu = \mu_1 \) is called the power function of the test and is denoted by \( \beta(\mu) \).

Thus \( \beta(\mu) = P [ x \in \omega \mid H ]. \)

Hence

\[
\beta(\mu_0) \bigg|_{H_0} = P [ X \in \omega \mid H_0 ] = \alpha \hspace{1cm} (6.2.3)
\]

and \( \beta(\mu_0) \bigg|_{H_1} = P [ X \in \omega \mid H_1 ] = 1 - \beta \) \hspace{1cm} (6.2.4)

Thus the value of \( \beta(\mu) \) under Null Hypothesis is called the size of the test and the values of \( \beta(\mu) \) under Alternative Hypothesis is called power of the test.

The function which is closely connected to the power function of the test is called 'operating characteristic function', denoted by \( \alpha(\mu) \), where

Where \( \alpha(\mu) = 1 - \beta(\mu). \)

Thus \( \alpha(\mu_0) = 1 - \alpha \) and \( \alpha(\mu_1) = \beta. \)
An ideal test is one for which the Operating characteristic function satisfies the relation

\[ \alpha(\mu_0) = 1 \text{ when } H_0 \text{ is true} \]
\[ \alpha(\mu_1) = 0 \text{ when } H_1 \text{ is true.} \]

However such ideal tests seldom exist.

6.3 Randomized Tests

Definition:

A randomized test assigns to any point \( x \), the probability \( T(x) \) of rejecting \( H_0 \) when \( x \) has been observed or \( T(x) \) is the conditional probability of rejecting \( H_0 \) provided \( x \) has been observed. The function \( T(x) \) is called 'Test function' or 'critical function'.

Thus \( \mathbb{E}_{H_0}[T(x)] = \alpha \)

and

\( \mathbb{E}_{H_1}[1-T(x)] = \beta. \)

"A test \( T \) is said to be the best of size \( \alpha \) for testing \( H_0 \) against \( H_1 \), if \( \mathbb{E}_{H_0}[T(x)] = \alpha \) and for every other test \( T' \), the relations

(i) \( \mathbb{E}_{H_0}[T'(x)] \leq \alpha \)

and

(ii) \( \mathbb{E}_{H_1}[1-T(x)] \leq \mathbb{E}_{H_1}[1-T'(x)] \)

are satisfied."
The method of finding the best test of a simple hypothesis against a simple alternative is given by Neyman and Pearson (1933).

6.4 The Neyman-Pearson Fundamental Lemma

Suppose that $\mu = \{\mu_0, \mu_1\}$ and that the distribution of X have densities $f(x / \mu_0) = f_0(x)$ and $f(x / \mu_1) = f_1(x)$.

(i) Any test $T(x)$ of the form

$$T(x) = \begin{cases} 
1 & \text{if } f_1(x) > K f_0(x) \\
\gamma(x) & \text{if } f_1(x) = K f_0(x) \\
0 & \text{if } f_1(x) < K f_0(x)
\end{cases} \quad (6.4.1)$$

for some $K > 0$ and $0 \leq \gamma(x) \leq 1$, is the best of its size for testing $H_0: \mu = \mu_0$ against $H_1: \mu = \mu_1$.

Corresponding to $K = \infty$, the test

$$T(x) = \begin{cases} 
1 & \text{if } f_0(x) = 0 \\
0 & \text{if } f_0(x) > 0
\end{cases} \quad (6.4.2)$$

is the best of size zero for testing $H_0$ against $H_1$.

(ii) Existence: For every $\alpha$, $0 < \alpha \leq 1$, there exists a test as shown above with $\gamma(x) = \gamma$, a constant, for which $E_{H_0}[T(x)] = \alpha$.

(iii) Uniqueness: If $T^{1}$ is the best test of size $\alpha$ for testing $H_0$ against $H_1$, then it has the form (6.4.1) or (6.4.2), except perhaps for a set of $x$ with probability zero under $H_0$ and under $H_1$.

6.5 Generalized Neyman - Pearson Lemma

Let $f_0(x), f_1(x), \ldots, f_n(x)$ be integrable functions on the real line $R$. Let $T$ be any integrable function of the form
\[ T_0(x) = \begin{cases} \gamma(x) & \text{if } f_0(x) = \sum_{j=1}^{n} k_j f_j(x) \\ 0 & \text{if } f_0(x) < \sum_{j=1}^{n} k_j f_j(x) \end{cases} \quad \ldots \quad (6.5.1) \]

where \( 0 \leq \gamma(x) \leq 1 \), then \( T_0 \) maximizes the integral

\[
\int T(x) f_0(x) \, dx \quad \text{out of all function } T, \quad 0 \leq T \leq 1.
\]

such that

\[
\int T(x) f_j(x) \, dx = \int T_0(x) f_j(x) \, dx \quad \ldots \quad (6.5.2)
\]

for \( j = 1, 2, \ldots, n \).

If \( K \geq 0 \) for \( j = 1, 2, \ldots, n \), then \( T_0 \) maximizes

\[
\int T(x) f_0(x) \, dx \quad \text{out of all functions } T \text{ such that}
\]

\[
\int T(x) f_j(x) \, dx \leq \int T_0(x) f_j(x) \, dx \quad \text{for } j = 1, 2, \ldots, n.
\]

Since the likelihood functions of various distributions from location-scale families such as Normal distribution and Logistic distribution due to Harter and Moore (1967), Gupta (1952), David (1957) Tiku (1968), Double exponential distribution due to Govindarajulu (1966), Rectangular distribution due to Sarhan and Greenberg (1959) etc. are available in literature, an attempt is made in this chapter to extend classical test procedures on the basis of metrically censored samples. In section 6.6, the trimmed sample space, size and power of the test based on trimmed sample space and the classical Neyman-Pearson test procedure based on trimmed sample space are
explained. The size and power of the test in cases where \( r_1 = 0 = r_2 , r_1 \neq 0 \neq r_2 , r_1 = 0 , r_2 \neq 0 , r_1 \neq 0 , r_2 = 0 \) are derived. The shape of power curves in the above situations are mentioned and loss in efficiency due to censoring is intuitively explained by drawing a vertical line on x - axis.

A test procedure which is closely related to likelihood ratios, called Likelihood Ratio Test, is explained in the section 6.8. The shape of the distribution of likelihood ratio function \( \lambda(x) \), under Null hypothesis is mentioned in the Fig no.1. In case sup \( \{ L(x , \theta) ; \theta \in \omega_T \} \) is very small when compared to sup \( \{ L(x , \theta) ; \theta \in \omega'_T \} \) some modifications are carried out.

Section 6.9 deals with the approximate distribution of Likelihood Ratio test statistic when censored samples are from location-scale families; other than Normal.

### 6.6 Trimmed Sample Space

**Definition:**

A sample space generated by trimmed sample observations \( (x_{r+1}, \ldots, x_{n-r}) \) is called a trimmed sample space and is denoted by \( \Omega_T \).

Let \( \omega_T \) be a subset of \( \Omega_T \). If \( \mu_0 \) is the true value of \( \mu \) of the population, then form a statistical hypothesis (H\(_0\)) such that

\[
\mu_0 \in \omega_T \quad \text{against the alternative hypothesis} \quad \mu_0 \in \omega'_T
\]

where \( \omega'_T = \Omega_T - \omega_T \).

It is customary to say that \( H_0 \) is true if \( \mu_0 \in \omega_T \) and \( H_0 \) is false if \( \mu_0 \in \omega'_T \).

The region \( \omega_T \) is called a critical region and the region \( \omega'_T \) is called acceptance Region. Define \( \alpha \) and \( \beta \) as

\[
P \{ x \in \omega_T \mid H_0 \} = \alpha
\]
and
\[ P \{ x \in \omega_T \mid H_1 \} = \beta \]

### 6.7 Hypothesis Testing over Trimmed Sample Space

Let \( x_{(r_1 + 1)}, x_{(r_2 + 2)}, \ldots, x_{(n-r_2)} \) be \( r_1 - r_2 \) metrically censored ordered sample of size \( n - r_1 - r_2 \) from location-scale families with mean as location parameter.

The trimmed mean of above sample is defined as
\[
\hat{\mu}_t = \frac{\sum_{i=r_1 + 1}^{n-r_2} x(i)}{n-r_1 - r_2} 
\]  

(6.7.1)

Let \( \mu_0 \) and \( \mu_1 \) (\( \mu_0 < \mu_1 \)) be two admissible values of the unknown parameter \( \mu \). To test the hypothesis

\[ H_0 : \mu = \mu_0 \] against \( H_1 : \mu = \mu_1 \), consider the ratio of likelihood functions

\[ L(X; \mu, \sigma \mid H_0) \text{ and } L(X; \mu, \sigma \mid H_1) \] where \( L(X; \mu, \sigma^2 \mid H_0) \)

\[ = L \left( x_{(r_1 + 1)}, \ldots, x_{(n-r_2)} ; \mu_0 ; \sigma^2 \right) \]

\[
= \frac{n!}{r_1! r_2!} \left[ \Phi_{H_0} \left( \frac{x_{(r_1 + 1)} - \mu}{\sigma} \right) \right]^{r_1} \left[ 1 - \Phi_{H_0} \left( \frac{x_{(n-r_2)} - \mu}{\sigma} \right) \right]^{r_2} 
\]

and

\[ L(X; \mu, \sigma^2 \mid H_1) = L(x_{(r_1 + 1)}, \ldots, x_{(n-r_2)} ; \mu_1 ; \sigma^2) \]
where $\phi(x)$ is the density function and $\Phi(x)$ is the distribution function of the characteristic $X$ respectively.

By Neyman-Pearson lemma, choose $\omega_t$ as long as the inequality
\[
\frac{L(X; \mu, \sigma | H_1)}{L(X; \mu, \sigma | H_0)} \geq c \text{ is satisfied} \quad \ldots \quad (6.7.4)
\]

If the sample observations are from normal populations then from (6.7.1) to (6.7.4), the inequality
\[
e^{n-r_1-r_2} \left( \mu_1 \left( \frac{\mu_1 - \mu_0}{\sigma^2} - \frac{\mu_1^2 - \mu_0^2}{2\sigma^2} \right) \right) \geq C \left[ \frac{\Phi \left( \frac{x_{(r_1+1)} - \mu_0}{\sigma} \right)}{\Phi \left( \frac{x_{(r_1+1)} - \mu_1}{\sigma} \right)} \right]^{r_1} \quad \ldots \quad (6.7.5)
\]

Set $\Phi \left( \frac{x_{(r_1+1)} - \mu_1}{\sigma} \right) = \Phi (\xi_0)$, $\Phi \left( \frac{x_{(r_1+1)} - \mu_1}{\sigma} \right) = \Phi (\xi_1)$

and
\[
\Phi \left( \frac{x_{(n-r_2)} - \mu_0}{\sigma} \right) = \Phi (\eta_0), \quad \Phi \left( \frac{x_{(n-r_2)} - \mu_1}{\sigma} \right) = \Phi (\eta_1)
\]
Expanding \( \Phi(\xi_0), \Phi(\eta_0) \) by Taylor's series

\[
\Phi(\xi_0) = \Phi(\xi_1) + \frac{(\xi_0 - \xi_1)}{1!} \Phi'(\xi_1) \bigg|_{\xi_1 = \xi_0} + \frac{(\xi_0 - \xi_1)^2}{2!} \Phi''(\xi_1) \bigg|_{\xi_1 = \xi_0} + R_1(\xi_1) \quad \ldots \ldots (6.7.6)
\]

and

\[
\Phi(\eta_0) = \Phi(\eta_1) + \frac{(\eta_0 - \eta_1)}{1!} \Phi'(\eta_1) \bigg|_{\eta_1 = \eta_0} + \frac{(\eta_0 - \eta_1)^2}{2!} \Phi''(\eta_1) \bigg|_{\eta_1 = \eta_0} + R_2(\eta_1) \quad \ldots \ldots (6.7.7)
\]

where the remainders \( R_1(\xi_1) \) and \( R_2(\eta_1) \) tend to zero as \( n \to \infty \). Using this in (6.7.5) and taking logarithms, the inequality (6.7.5) takes the form

\[
\hat{\mu}_t \geq \frac{\sigma^2 \log c}{n' (\mu_1 - \mu_0)} + \frac{r_1}{n'} \frac{\phi(\xi_1)}{\Phi(\xi_1)} - \frac{r_2}{n'} \frac{\phi(\eta_1)}{1 - \Phi(\eta_1)} + \frac{\mu_0 + \mu_1}{2}, \quad \ldots (6.7.8)
\]

where \( n' = n - r_1 - r_2 \).

Hence choose \( \omega_T \) for testing the hypothesis

\( H_0 : \mu = \mu_0 \) against \( H_1 : \mu = \mu_1 \) as long as the inequality (6.7.5) is satisfied.

**Note(i):** when \( r_1 = r_2 = 0 \) then \( n' = n \) and \( \hat{\mu}_t = \hat{\mu} \)

The inequality (6.7.6) agrees with

\[
\hat{\mu}_t = \hat{\mu} \geq \frac{\sigma^2 \log c}{n (\mu_1 - \mu_0)} + \frac{\mu_0 + \mu_1}{2} = A_1 \quad \ldots \ldots (6.7.9)
\]

and
The power of the test is

\[
\lim_{n \to \infty} P \left[ \hat{\mu} \geq A_1 \mid H_0 \right] = \frac{\sqrt{n}}{\sigma \sqrt{2\pi}} \int_{A_1}^{\infty} e^{- \frac{n}{2\sigma^2} (\hat{\mu} - \mu_0)^2} \, d\hat{\mu} = 1 - \alpha \quad \ldots \quad (6.7.10)
\]

The value of \( A_1 \) can be determined from the equation (6.7.10) or from (6.7.11). For instance, if \( \alpha = 0.05 \), \( n = 30 \), \( \mu_0 = 0 \) and \( \mu_1 = 1 \) then \( A_1 \approx 0.3 \) one can verify that by Neyman-Pearson’s lemma, the most powerful test for testing \( H_0 : \mu = \mu_0 \) against \( H_1 : \mu = \mu_1 \) is the same for all values of \( \mu \) greater than \( \mu_1 \). This test consists of those points of \( x \) for which the inequality (6.7.9) is satisfied. If the admissible alternative hypothesis are \( H_1 : \mu = \mu_1 \) where \( \mu_1 > \mu_0 \) then the critical region can be determined from (6.7.10) and (6.7.11).

The following figure indicates the shape of power function of the test with 25 sample observations from Normal population. Tests are \( H_0 : \mu = 0; \mu_1 = 1 \) and \( \sigma = 5 \).

Fig. 6.7.1. Shape of power curve when sampled from Normal populations
Since \( \frac{\phi (\xi_i)}{\Phi (\xi_i)} \) and \( \frac{\phi (\eta_i)}{1 - \Phi (\eta_i)} \) are ratios of ordinates and probability integrals of Normal distribution, Tiku (1967) suggestions can be brought to solve (6.7.8).

Write \( g(x) = \frac{\phi (\xi)}{\Phi (\xi)} \), then \( g(\xi) \approx a + b \xi \).

It can be verified empirically that the points

\( g(\xi) \) over the interval \( l_1 \leq \xi \leq l_2 \) of finite length lie very close to the line \( g(\xi) = a + b \xi \).

where \( b = \frac{g(l_2) - g(l_1)}{l_2 - l_1} \)

and \( a = g(l_1) - l_1 b \).

Using this in (6.7.8), we have the inequality

\[
\hat{\mu}_t \geq \frac{\sigma^2 \log c}{n} \left( \mu_1 - \mu_0 \right) + \frac{\mu_0 + \mu_1}{2} + p \left[ a_1 + b_1 \xi_i \right] - q \left[ a_2 + b_2 \eta_i \right] \tag{6.7.12}
\]

where \( p = \frac{r_1}{n} \); \( q = \frac{r_2}{n} \)

\[
b_2 = \frac{g'(h_2) - g'(h_1)}{h_2 - h_1} \tag{6.7.13}
\]

\[
b_1 = \frac{g(k_2) - g(k_1)}{k_2 - k_1} \tag{6.7.14}
\]
\[ a_2 = g'(h_1) - h_1 b_2' \]  \hspace{1cm} (6.7.15)

\[ a_1 = g(k_1) - k_1 b_1 \]  \hspace{1cm} (6.7.16)

and \( g(\xi) \) refers to \( \frac{\phi(k)}{\Phi(\xi)} \) and \( g'(\alpha) \) refers to \( \frac{\phi(\eta)}{1 - \Phi(\eta)} \). The intervals \((h_1, h_2)\) and \((k_1, k_2)\) are wide enough to cover \( \frac{x_{(1)} - \mu_1}{\sigma} \) and \( \frac{x_{(r_1 + 1)} - \mu_1}{\sigma} \).

The values of \((a_2, b_2)\) are calculated by Tiku (1967) for various values of \(n, p, \) and \( q\). For \(n=10, p=0, q=0.3\), the values of \((a_2, b_2)\) are \((0.7862, 0.7402)\).

When \(q=0\) is in case of right censoring, the inequality (6.7.13) gives,

\[ \hat{\mu}_t \geq \frac{\sigma^2 \log c}{n^1} \frac{\mu_0 + \mu_1}{2} + p [a_1 + b_1 \xi_1] = A_2 \]  \hspace{1cm} (6.7.17)

Since \( \hat{\mu}_t \) is asymptotically normal

\[ \lim_{n \to \infty} p[\hat{\mu}_t \geq A_2 | H_0] = \frac{\sqrt{n}}{\sigma p \sqrt{2\pi}} \int_{A_2}^\infty e^{-\frac{n}{2 \sigma^2} (\hat{\mu}_t - \mu_0)^2} d\hat{\mu}_t = \alpha^* \]  \hspace{1cm} (6.7.18)

and power of the test is

\[ \lim_{n \to \infty} p[\hat{\mu}_t \geq A_2 | H_1] = 1 - \beta^* \]  \hspace{1cm} (6.7.19)

When \( p=0 \) is in case of left censoring, the inequality (6.7.12) gives

\[ \hat{\mu}_t \geq \frac{\sigma^2 \log c}{n^1} \frac{\mu_0 + \mu_1}{2} - q [a_2 + b_2 \eta_1] = A_3 \]  \hspace{1cm} (6.7.20)

In such case, the size of the test is
\[
\lim_{n' \to \infty} p \left[ \hat{\mu}_{i} \geq \Lambda_{3} \mid \mu_{i} \right] = \alpha^{*} \text{ and power of the test is}
\]
\[
\lim_{n' \to \infty} p \left[ \hat{\mu}_{i} \geq \Lambda_{3} \mid \mu_{i} \right] = 1 - \beta^{*} \quad \ldots \ldots \quad (6.7.22)
\]

**Fig. 6.72** Shape of power curve when \( p = 0 \)  

**Fig. 6.73** Shape of power curve when \( q = 0 \)

From the above figures any vertical line on x-axis indicate the loss in power of the test due to censoring.

### 6.8 Likelihood Ratio (LR) Tests and their Applications

A practical procedure for testing simple and composite statistical hypothesis is described. Since this procedure is closely related to likelihood ratios, it is called LR test and is introduced by Neyman and Pearson (1937). It has good general properties, as among them: if a sufficient statistic for a parameter exists, the LR produces a test based on it, and if a uniformly most powerful test exists, the LR often leads to it.

Suppose we have \( r_{1} \leq r_{2} \) metrically censored samples \( (x_{(r_{1}+1)}, \ldots, x_{(n-r_{2})}) \) from location-scale families with the location scale parameter \( \theta = (\mu, \sigma) \);
and we wish to test the hypothesis $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$. The test statistic to consider in the Likelihood Ratio is given by

$$\lambda(x) = \frac{\sup (L(x; \theta) ; \theta \in \Theta)}{\sup (L(x; \theta) ; \theta \in \Theta_1)} . . . \quad (6.8.1)$$

where $L(x, \theta)$ denotes the LR function of ordered observations $(x_{(1)}^t, \ldots, x_{(n-r_0)})$. $(6.8.1)$ can be written as

$$\lambda(x) = \frac{\sup (L(x, \theta) ; \theta \in \Omega_T)}{\sup (L(x, \theta) ; \theta \in \Omega')}. \quad . . . \quad (6.8.2)$$

where $\Omega_T$ denote the critical region of size $\alpha$ in the trimmed sample space $\Omega_T$.

Clearly $\lambda(x)$ cannot be less than zero nor greater than one. If for a particular observed value of $\theta$, the value of $\lambda(x)$ is nearly one, i.e, if $\sup (L(x, \theta) ; \theta \in \Omega_T)$ is nearly equal to $\sup (L(x, \theta) ; \theta \in \Omega')$, the hypothesis $H_0$ warrants support for the maximum probability density associated with the set $\theta$ in the region $\Omega_T$ further it cannot be much increased by shifting from them to other values of $\theta$ in $\Omega'$. If $\lambda(x)$ is near zero, the contrary is true; the null hypothesis $H_0$ doesn't warrant support. Thus $\lambda(x)$ is a random variable. The distribution of $\lambda(x)$, for $H_0$ true, is illustrated in the following figure.

![Fig: 6.8.1 Distribution of $\lambda(x)$ under $H_0$.](image-url)
Using LR test, the probability \( \alpha \) of an error of the first kind is evidently
\[
\int_0^{\lambda_a} f(L, \theta \in \omega_T) \, dL \quad \ldots \ldots \quad (6.8.3)
\]
while the power of the LR test is
\[
\int_0^{\lambda_a} f(L, \theta \in \omega'_{T'}) \, dL , \quad \ldots \ldots \quad (6.8.4)
\]
which is evidently a function of the possible values of \( \theta \). For a simple \( H_0' \): \( \theta = \theta_0' \), when it is true

The quantity \(-2 \log \lambda\) follows \( \chi^2 \) distribution with 1 degree of freedom. In particular cases, the LR tests have weak optimality properties.

Suppose that \( \sup \{ L(x, \theta) ; \theta \in \omega_T \} \) is very small when compared to \( \sup \{ L(x, \theta) ; \theta \in \omega'_{T'} \} \), in such case, the modified test statistic is defined as
\[
\lambda^*(x) = \frac{\sup \{ L(x, \theta) ; \theta \in \omega_T \}}{\sup \{ L(x, \theta) ; \theta \in \omega_{T'} \}} \quad \ldots \ldots \quad (6.8.5)
\]
where \( \lambda^*(x) = \max \{ \lambda(x) / H_1 \} \).

To test the hypothesis in the above situations, find a function \( h \) which is strictly increasing on the range of \( \lambda^* \) such that \( h(\lambda^*(x)) \) has a simple form and specify the size \( \alpha \) of the LR test through the test statistic \( \lambda^* \), using the relation
\[
\int_0^{\lambda_a^*} f(L, \theta \in \omega_{T'}) \, dL = \alpha \quad \ldots \ldots \quad (6.8.6)
\]
The power of the LR test is
\[
\int_0^{\lambda_a^*} f(L, \theta \in \omega_{T'}) \, dL \quad \ldots \ldots \quad (6.8.7)
\]
Note (i) If \{ L(x, \theta) ; \theta \in \Omega_T \} is a family of density functions and 
\lambda^*(x) is a Maximum Likelihood Ratio and the function \( h(\lambda^*(x)) \) of \( \lambda^*(x) \) is 
continuous in \( t \) for each \( \theta \), then 

a) there exists a uniformly most accurate \( \gamma \) and the lower confidence 
limit \( \hat{Q}(\lambda^*) \) for all \( \gamma ; 0 < \gamma < 1 \).

b) If the root \( \hat{\theta}_r(\lambda^*) \) of the equation \( h(\lambda^*, \theta) = \gamma \) belongs to \( \theta \), then

\[ \hat{Q}(\lambda^*) = \hat{\theta}_\gamma(\lambda^*) \]

Note (ii) If \( x \) is a discrete random variable, we cannot apply the LR 
criterion to find confidence intervals since \( \lambda^*(x) \) is a step function. In order 
to overcome this difficulty, we generally consider randomized test functions 
which have the same power efficiency when compared to any other test 
function.

6.9 Approximations to the Distributions of Test Statistics when 
samples are from Location-Scale families other than Normal.

Suppose we want to test the hypothesis

\[ H_0 : \mu = \mu_0 \text{ against } H_1 : \mu = \mu_1 \]

on the basis of \( r_1 - r_2 \) metrically censored ordered samples \( x_{(r_1+1)} \ldots \)
\( x_{(n-r_2)} \) from the population with \( \mu \) as its location parameter, then by 
Neyman-Pearson lemma, we generally consider the ratio

\[ \frac{L(x, \mu_1)}{L(x, \mu_0)} \] \quad \quad \quad \quad \quad \quad (6.9.1)

If the ratio \( \frac{L(x, \mu_1)}{L(x, \mu_0)} \geq c \), a constant, we generally reject the Null 
hypothesis.
To find the approximate distribution of the ratio (6.9.1) when Null hypothesis is true, write

\[
L(x ; \mu_0, \mu_1) = \prod_{i=r_1+1}^{n-r_2} \frac{f(x_i, \mu_1)}{f(x_i, \mu_0)} \tag{6.9.2}
\]

\[
\log L(x, \mu_0, \mu_1) = \log \left\{ \prod_{i=r_1+1}^{n-r_2} \frac{f(x_i, \mu_1)}{f(x_i, \mu_0)} \right\} = \sum_{i=r_1+1}^{n-r_2} L_1(x_i) \tag{6.9.3}
\]

Define Kullback-Leibler information number as

\[
I(\mu, \eta) = E_H \log \frac{f(x, \mu)}{f(x, \eta)} = \int_{-\infty}^{\infty} \log \left( \frac{f(x, \mu)}{f(x, \eta)} \right) f(x, \mu) \, dx \tag{6.9.4}
\]

if the family is of continuous type.

\[
= \sum_x \log \frac{p(x, \mu)}{p(x, \eta)} p(x, \mu) \tag{6.9.5}
\]

if the family is of discrete type.

It can be shown that Kullback-Leibler information number is always defined, though possibly infinite. More over \( I(\mu, \eta) \geq 0 \).

In cases where \( f_\mu = f_\eta \) (or) \( P_\mu = P_\eta \) then

\[
I(\mu, \eta) = 0
\]

Further \( E_{H_1} [L_1(x)] = I(\mu_1, \mu_0) \) \tag{6.9.6}
and \[ \mathbb{E}_{H_0}[L_1(x)] = -\mathbb{E}_{H_0}[-L_1(x)] \]

\[ = -I(\mu_0, \mu_1) \quad \ldots \ldots \quad (6.9.7) \]

Suppose if the variance of \( L_1(x) \) is finite and positive for all \( \mu \), and if \( \sigma_0^2 \) denote variance of \( L_1(x) \) under \( H_0 \), then

\[ P_{H_0}[\log L(x; \mu_0, \mu_1) \geq -n I(\mu_0, \mu_1) + \sigma_0 \sqrt{n} z] \]

\[ = P_{H_0}\left\{ \frac{1}{\sigma_0 \sqrt{n}} \sum_j [L_1(x_j) - \mathbb{E}_{H_0}[L_1(x_j)]] \geq Z \right\} \]

\[ \to 1 - \Phi(z) \text{ as } n \to \infty \quad \ldots \ldots \quad (6.9.8) \]

where \( \Phi(z) = \frac{1}{\sqrt{2\pi}} \int e^{-t^2/2} \, dt \).

Hence, if we use the test statistic \( L \), we can approximate the critical value of Most Powerful test of size \( \alpha \) by

\[ \exp \left\{ -n I(\mu_0, \mu_1) + \sigma_0 \sqrt{n} z (1-\alpha) \right\} \text{ iff } I(\mu_0, \mu_1) \neq \infty \quad \ldots \ldots \quad (6.9.9) \]