Appendix
ON QUADRATIC DIOPHANTINE EQUATION WITH FOUR VARIABLES

M.A.Gopalan, Manju Somanath* and N.Vanitha*
Department of Mathematics, National College, Trichy.

ABSTRACT
We obtain two parametric solutions of quadratic Diophantine Equation with four
variables of the form
\[ x^2 + xy + y^2 = A^2 + AB + B^2 \]
a few interesting relations among the solutions are
given. We also deduce integral solutions of some special quartic Diophantine
equations with four variables.

Key words : Multivariate quadratic equation, Integral solutions.

INTRODUCTION
At the 35th annual Australian Mathematics Society Conference the Diophantine Equation
\[ x^2 + xy + y^2 = A^2 + AB + B^2 \]  
was discussed informally as an equation which probably is so difficult to study that it may require
new Mathematics to prove that it has no nontrivial solutions or even only a finite number of non
trivial solutions.

Sloss [1] in his research report "ON A DIOPHANTINE EQUATION WITH FOUR
VARIABLES" has shown that the Diophantine Equation (1) does not have solutions in some
special cases. Any attempt of this problem will result in the development of many interesting new
fields of Mathematics.

In this paper we obtain two parametric solutions of quadratic Diophantine equation with
four variables of the form (1). A few interesting relations among the solutions are given. We also
deduce integral solutions of some special quartic Diophantine Equations with four variables.

METHOD OF ANALYSIS
Consider the quadratic Diophantine Equation with four variables
\[ x^2 + xy + y^2 = A^2 + AB + B^2 \]

Putting
\[ x = u + v, y = u - v, A = r + u, B = r - u \]  
in (1), it reduces to
\[ v^2 = 3u^2 - 2u^2 \]  
which is satisfied by
\[ r = p^2 + 3q^2 - 2pq \]
\[ u = 3q^2 - p^2 \]
\[ v = p^2 + 3q^2 - 6pq \]  
where \( p \neq q \).

Thus in view of (2), solution of (1) is
\[ x = 6q(p-q) \quad A = 2q(3q-p) \]
\[ y = 2p(3q-p) \quad B = 2p(p-q) \]

*Department of Mathematics, Cauvery College for Women, Trichy, India
<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>x</th>
<th>y</th>
<th>A</th>
<th>B</th>
<th>DIOPHANTINE EQUATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>2r^2-3s^2</td>
<td>2r^2</td>
<td>6rs</td>
<td>16r^4-12r^2s^2-9s^2</td>
<td>16r^4+12r^2s^2</td>
<td>18rs-12r^2s^2</td>
<td>x^2+x^2y^2=A^2+AB+B^2</td>
</tr>
<tr>
<td>2r^2-3s^2</td>
<td>-3s^2</td>
<td>6rs</td>
<td>36s^4-16r^2</td>
<td>36s^4+12r^2s^2</td>
<td>8r^4-12r^2s^2</td>
<td>x^2+x^2y^2=A^2+AB+B^2</td>
</tr>
<tr>
<td>18r^2</td>
<td>6r^2+3s^2</td>
<td>36s^5+216r^3s^2+288r^4</td>
<td>18rs</td>
<td>108r^3s^2+54s^3</td>
<td>432r^4108r^3s^2</td>
<td>x^2+x^2y^2=A^2+AB+B^2</td>
</tr>
<tr>
<td>9s^4</td>
<td>6r^2+3s^2</td>
<td>216r^4-108r^2s+108s^4</td>
<td>18rs</td>
<td>216r^4+108r^2s^2</td>
<td>108s^4-108r^2s^2</td>
<td>x^2+x^2y^2=A^2+AB+B^2</td>
</tr>
<tr>
<td>6r^2+9s^4</td>
<td>-3s^2</td>
<td>108r^5-108s^4</td>
<td>108r^5s^2-72s^4</td>
<td>6rs</td>
<td>54r^2-180r^3s^2+108s^4</td>
<td>x^2+x^2y^2=A^2+AB+B^2</td>
</tr>
<tr>
<td>6r^2+9s^4</td>
<td>2r^2</td>
<td>108r^5s^2-48r^4</td>
<td>108r^5s^2-162s^4</td>
<td>6rs</td>
<td>48r^2-108r^3s^2+162s^4</td>
<td>x^2+x^2y^2=A^2+AB+B^2</td>
</tr>
<tr>
<td>2r^2</td>
<td>2r^2-3s^2</td>
<td>6s^5-12r^4s^2-16r^4-12r^2s^2-9s^2</td>
<td>6s^5-20r^3s^2+6s^3</td>
<td>2rs</td>
<td>x^2+x^2y^2=A^2+AB+B^2+2</td>
<td></td>
</tr>
<tr>
<td>-s^4</td>
<td>2r^2-3s^2</td>
<td>24r^3s^2-12s^4</td>
<td>4s^5-12r^3s^2</td>
<td>24r^3-20r^3s^2+4s^2</td>
<td>2rs</td>
<td>x^2+x^2y^2=A^2+AB+B^2+2</td>
</tr>
<tr>
<td>p</td>
<td>3p</td>
<td>6p</td>
<td>4p</td>
<td>48p^2</td>
<td>-4p^2</td>
<td>x^2+x^2y^2=A^2+AB+B^2</td>
</tr>
</tbody>
</table>
On quadratic diophantine equation with four variables

Relations among the solutions
1. $pA = qy$
2. $px + 3Bq = 0$
3. $xy + 3AB = 0$
4. $x + y = A-B$
5. $p^n q^n (x^n ± y^n) = (-3Bq^3)^n ± (p^3A)^n, n = 1, 2, 3, ....$

DEDUCTIONS

From the solutions (3), choosing the values of $p$ and $q$ in such a way that one may obtain solutions of a quartic equation. In what follows, solutions of certain quartic equations with four variables are presented.

REMARK

However, apart from (4), we have three more solution patterns for (1) which are given by
1. $x = 6pq, y = 2q(p+q), A = 2(q^2 - 2p^2 - pq), B = 2p(2p-q)$
2. $x = 5p^2 + 2q^2, y = -p^2 - 6q^2, A = 6p^2 - q^2, B = 2p^2 + 5q^2$
3. $x = 4p^2, y = -2p^2 - 2q^2, A = 4p^2, B = 2p^2 + 2q^2$

REFERENCE

1. Sloss, B G 1997 On a Diophantine equation with four variables, Research report no 5, Department of Mathematics, Advanced College, Royal Melbourne Institute of Technology, Australia, April 1997

391
A method of generating infinitely many solutions of the Diophantine equation \(A^2 + B^2 + C^2 - D^2 = P^3 + Q^3\) is presented and the solutions have a two parametric representation.

**KEY WORDS:** Diophantine equation, equal sums and unlike powers.

**INTRODUCTION:** Diophantine equations of the form \(\sum_{i=1}^{m} a_i^k = \sum_{j=1}^{n} b_j^k\)

have been studied by numerous mathematicians for many years and by various methods. For an elaborate survey of equal sums of like powers one may refer [1, 2, 3, 4]. It seems that much work has not been done related to equal sums of unlike powers. It is therefore, to this end, we in this paper, propose to analyse the Diophantine equation \(A^2 + B^2 + C^2 - D^2 = P^3 + Q^3\) for its nontrivial integral solutions. Interesting relations among the solutions are obtained. Further, integral solutions of equations of the forms \(A^2 + B^2 + C^2 - D^2 = P^3 - Q^3\), \(A^2 - B^2 + C^2 - D^2 = P^3 - Q^3\) and \(A^2 - B^2 - C^2 + D^2 = P^3 - Q^3\) have also been presented.

**METHOD OF ANALYSIS:** Consider the identity

\[(x^2 + y^2)(x + y) = x^3 + y^3 + xy(x + y) \quad \ldots (1)\]

We write

\[x + y = l^2\]
\[x - y = m^2\] \(\ldots (2)\)

so that

\[x = (l^2 + m^2)/2\]
\[y = (l^2 - m^2)/2\]
\[xy = (l^4 - m^4)/4 \quad \ldots (3)\]

The substitution of (2) and (3) in (1) leads to

\[A^2 + B^2 + C^2 - D^2 = P^3 + Q^3 \quad \ldots (4)\]

where

\[A = l(l^2 + m^2)/2\]
\[B = l(l^2 + m^2)/2\]
\[C = lm^2/2\]
\[D = l^3/2\]
\[P = (l^2 + m^2)/2\]
\[Q = (l^2 - m^2)/2 \quad \ldots (5)\]

in which \(l\) and \(m\) are distinct non zero integers.

Since our interest centres on finding non trivial integral solutions we note that the above values are integers only when the values of \(l\) and \(m\) are even. Therefore taking \(l = 2L, m = 2M\) the integral solutions of (4) are given by

\[A = 4L(L^2 + M^2)\]
\[B = 4L(L^2 - M^2)\]
\[ C = 4LM^2 \]
\[ D = 4L^3 \]
\[ P = 2 \left( L^2 + M^2 \right) \]
\[ Q = 2 \left( L^2 - M^2 \right) \]
\[ L \neq M \]

where

In what follows we present various interesting relations among the solutions of (4)

1. When \( M = 1 \), \( P + Q + C \) divides \( B \)
2. \( P + Q \) divides \( D \)
3. When \( M \) divides \( L \), \( D/C \) is a perfect square
4. \( A/P = B/Q \)
5. \( A = C + D \)
6. \( B + C = D \)
7. \( A = B + 2C \)
8. \( (A + B) = 2D \)
9. \( AB = D^2 - C^2 \)
10. \( A^2 = B^2 + (8L^2M)^2 \)
   \[ P^2 = Q^2 + (4LM)^2 \]
   which leads to the fact that the triples \((Q, 4LM, P)\) and \((B, 8L^2M, A)\) form Pythagorean triangles.

11. \( P^3 + Q^3 = AB + 4C^2 \)
12. \( (P^3 - Q^3) D = 3C^2D + C^3 \)

**DEDUCTIONS**:

1. Interchanging \( L \) and \( M \) in (6), the solutions of \( A^2 + B^2 + C^2 - D^2 = P^3 - Q^3 \) are obtained
2. Interchanging \( L \) and \( M \) in \( C \) and \( D \) of (6), we obtain the solutions of the equations
   \[ A^2 + C^2 - B^2 - D^2 = P^3 - Q^3 \]
3. Interchanging \( L \) and \( M \) in \( A \) and \( B \) of (6), we obtain the solutions of the equation
   \[ A^2 - B^2 - C^2 + D^2 = P^3 - Q^3 \]

**REFERENCES**:

2. Jean-Joel Delorme, On the Diophantine equation \( x_1^6 + x_2^6 + x_3^6 = x_4^6 + y_2^6 + y_3^6 \), *Math Comp.*, Vol 59, 703-715 (1992)
Integral Solutions of $ax^2 + by^2 = w^2 - z^2$

Gopalan M.A.

Department of Mathematics, National College, Trichy, India
E-mail: gopalanma@yahoo.com

Manju Somanath & Vanitha N.

Department of Mathematics, Cauvery College for Women, Trichy, India
E-mail: manjuajil@yahoo.com, vanitha.1978@yahoo.co.in

Abstract

Two different patterns of integral solutions for the equation $ax^2 + by^2 = w^2 - z^2$

are presented. A few relations among the solutions are also given.

AMS Mathematics Subject Classification:

Keywords: Quadratic Diophantine equation with four unknowns, integral solutions.

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, quadratic Diophantine equations have been an interest to Mathematicians since antiquity [1, 2]. In [3], Quadratic Diophantine equations with four unknowns are obtained.

In this communication, we consider the quadratic Diophantine equations with four unknowns represented by

$$ax^2 + by^2 = w^2 - z^2$$

and two different patterns of integral solutions are obtained. Also, a few relations among the solutions are given.
2. **Method of Analysis**

The equation to be solved is

\[ ax^2 + by^2 = u^2 - z^2 \]  \hspace{1cm} (1)

**Pattern I**

Substituting the linear transformations

\[ x = X + k, \; y = Y + k, \; z = k - a, \; w = k - b. \; k \neq 0 \]

in (1), we have,

\[ X^2 = b - a - k^2 \]  \hspace{1cm} (2)

Choose \( a, b \) such that

\[ b - a = (n^2 + 1)k^2 \]  \hspace{1cm} (3)

Therefore the solutions of (1) are

\[
\begin{array}{cccc}
  x & y & z & w \\
  k(n+1) & k(n-1) & k^2n - a & k^2n - b \\
  k(1-n) & -k(n+1) & -k^2n - a & -k^2n - b \\
\end{array}
\]

**Observations**

A few interesting properties observed among the above solutions are given below

1. Each of the expressions \( x^2 - y^2, \; \frac{k(x+y)}{w+b}, \; \text{and} \; \frac{k(x+y)}{z+a} \) are congruent to zero under mod 2

2. Each of the expressions \( \frac{(x+y)^2}{n(z+a)} \; \text{and} \; \frac{(x+y)^2}{n(w+b)} \) are congruent to zero under modulo 4

3. Each of the expressions \( \frac{x+y}{x-y}, \frac{4(z+a)}{(x-y)^2}, \; \text{and} \; \frac{4(w+b)}{(x+y)^2} \) represents an integer.

4. \( n^2(z+a) - (w+b) = nxy \)

5. \( n^2(w+b) - (z+a) = nxy \)

**Pattern II**

Let

\[ x = 2rs, \; z = ar^2 - s^2 \]  \hspace{1cm} (4)
Integral Solutions of $ax^2 + by^2 = w^2 - z^2$

where $r$ and $s$ are non zero integers. Using (4) in (1), we have

$$w^2 = by^2 + (ar^2 + s^2)^2$$  \hspace{2cm} (5)

If $b$ is positive and square free then the equation (5) is in the form of the general Pellian equation. Employing the standard procedure the sequence of values of $y$ and $w$ satisfying the equation (5) are found to be

$$w_n = (1/2)(ar^2 + s^2) \{(w_0 + \sqrt{by_0})^{n+1} + (w_0 - \sqrt{by_0})^{n+1}\}$$
$$y_n = (1/2\sqrt{b})(ar^2 + s^2) \{(w_0 + \sqrt{by_0})^{n+1} - (w_0 - \sqrt{by_0})^{n+1}\}$$  \hspace{2cm} (6)

where $n = 0, 1, 2, \ldots$ and $(w_0, y_0)$ is the least positive integer solution of the Pell’s equation

$$w^2 = by^2 + 1$$

Thus equations (4) and (6) represent another pattern of solutions of equation (1)

The values of $y$ and $w$ satisfy the recurrence relations

$$y_{n+2} - 2w_0y_{n+1} + y_n = 0$$
$$w_{n+2} - 2w_0w_{n+1} + w_n = 0$$

A few numerical examples are represented below

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>6</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>36</td>
<td>51</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1224</td>
<td>1731</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>7134</td>
<td>10089</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>9</td>
<td>18</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>36</td>
<td>63</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>135</td>
<td>234</td>
<td>7</td>
</tr>
</tbody>
</table>

Note that when $b$ is a perfect square, the equation (5) is similar to the pythagorean equation for which the solutions may be obtained for suitable values of $r$ and $s$

When $a = 3, b = 4, r = 2$ and $s = 1$, equation (5) becomes

$$w^2 = (2y)^2 + 13^2$$

which is satisfied by

$$y = 42 \quad \text{and} \quad w = 85$$  \hspace{2cm} (7)

Also from (4),

$$x = 4 \quad \text{and} \quad z = 11$$  \hspace{2cm} (8)
References


ON THE BIVARIATE CUBIC EQUATION \((x + y)^3 = xy\)

M.A. GOPALAN, MANJU SOMANATH AND N.VANITHA*

RECEIVED : 8th February, 2005

Non-zero integral solutions to \((x + y)^3 = xy\) are obtained. Solutions to various patterns of the above equation are deduced.

INTRODUCTION: Cubic Diophantine equations with two unknowns have engaged the attention of Mathematicians since antiquity as can be seen from [1, 2] and they are rich in variety. In this communication, we consider the bivariate cubic equation represented by \((x + y)^3 = xy\) and its solutions are represented through Triangular numbers. The solutions of other patterns of similar cubic equations are deduced.

METHOD OF ANALYSIS

Consider the equation

\[(x + y)^3 = xy\]  \(\ldots (1)\)

The substitution of the linear transformation

\[x = u + v \text{ and } y = u - v\]  \(\ldots (2)\)

reduces (1) to

\[v^2 = n^2(1 - 8n)\]  \(\ldots (3)\)

The term \(1 - 8n\) is a square when \(1 - 8n\) is a square when

\[u = - T_n\]

where \(T_n\) is the \(n\)th triangular number given by \(T_n = \frac{n(n+1)}{2}, n = 1, 2, 3, \ldots\)

Therefore (3) gives

\[v = \pm (2n + 1). T_n\]  \(\ldots (4)\)

Thus the general solution of (1) is

\[x_n = \pm 2n. T_n\]

\[y_n = \mp 2(n + 1). T_n\]  \(\ldots (6)\)

From the solutions (6), the following relations are observed:

1. \(n(x_n + y_n) = -x_n\)
2. \(n(x_n - y_n) = (2n + 1)x_n\)
3. \(x_n/y_n = -n/n + 1\)
4. \(x_n + y_n = (2n + 1) + 2(n + 1)^2\)
GENERATING OF SOLUTIONS: Knowing any two solutions \((x_n, y_n)\) and \((x_{n+1}, y_{n+1})\), one can generate a third solution given by
\[
x_{n+2} = \pm x_n \cdot y_{n+1}, \quad y_{n+2} = \pm x_{n+1} \cdot y_n
\]
By repeating the above process, an infinite number of solutions can be obtained.

CONCLUSION: Proceeding in the similar way, the general solution of \((x - y)^3 = xy\) can also be calculated.

REFERENCES:
PARAMETRIC INTEGRAL SOLUTIONS OF THE CUBIC DIOPHANTINE EQUATION $x^2 + y^2 = (x - my)^3$

M.A. GOPALAN

Department of Mathematics, National College, Trichy, India

MANJU SOMANATH

and

N. VANITHA

Department of Mathematics, Cauvery College for Women, Trichy, India

(Received on 20th May 2005)

Parametric integral solutions for the cubic Diophantine equation $x^2 + y^2 = (x - my)^3$, where $m$ is any integer, in general form is obtained. The solutions of some similar equations are deduced.

1. INTRODUCTION

Diophantine problems have variety of range and richness. Especially, the determination of integral solutions for cubic, homogeneous or non-homogeneous Diophantine equations has been an interest to Mathematicians since antiquity as can be seen from [1, 2, 3]. In particular, in reference [2] two double parameter solutions for the equation $x^2 + y^2 = z^3$ are given. In this communication an attempt has been made to obtain a general form of integral solutions for the cubic Diophantine equation $x^2 + y^2 = (x - my)^3$, where $m$ is any positive integer. A few numerical examples are also presented.

2. METHOD OF ANALYSIS

Consider the equation

\[ x^2 + y^2 = (x - my)^3 \]  

KEY WORDS: Cubic Diophantine equation, Integral Solutions
The substitution

\[ x - my = h \]

in (1) leads to

\[(m^2 + 1)y^2 + 2myh + h^2(1-h) = 0\]

This is a quadratic in \(y\) and is satisfied by

\[ y = \frac{h}{m^2 + 1} \left[ -m \pm \left\{ (m^2 + 1)h - 1 \right\}^{1/2} \right] \tag{2} \]

Since we are interested in finding integral solutions, in (2) we choose \(h\) in such a way that \(\{(m^2 + 1)h - 1\}\) is positive square integer.

The following table represents the solutions of (1) for some particular values of \(m\) along with their corresponding \(h\) values.

<table>
<thead>
<tr>
<th>(m)</th>
<th>(h)</th>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2\alpha^2 - 2\alpha + 1)</td>
<td>(\alpha(2\alpha^2 - 2\alpha + 1))</td>
<td>((\alpha - 1)(2\alpha^2 - 2\alpha + 1))</td>
</tr>
<tr>
<td>2</td>
<td>(5\alpha^2 - 6\alpha + 2) (5\alpha^2 - 4\alpha + 1)</td>
<td>((2\alpha - 1)(5\alpha^2 - 6\alpha + 2)) ((1 - 2\alpha)(5\alpha^2 + 4\alpha + 1))</td>
<td>((\alpha - 1)(5\alpha^2 - 6\alpha + 2)) (-\alpha(5\alpha^2 - 4\alpha + 1))</td>
</tr>
<tr>
<td>3</td>
<td>(10\alpha^2 - 14\alpha + 5) (10\alpha^2 - 6\alpha + 1)</td>
<td>((3\alpha - 2)(10\alpha^2 - 14\alpha + 5)) ((1 - 3\alpha)(10\alpha^2 - 6\alpha + 1))</td>
<td>((\alpha - 1)(10\alpha^2 - 14\alpha + 5)) (-\alpha(10\alpha^2 - 6\alpha + 1))</td>
</tr>
<tr>
<td>4</td>
<td>(17\alpha^2 - 26\alpha + 10) (17\alpha^2 - 8\alpha + 1)</td>
<td>((4\alpha - 3)(17\alpha^2 - 26\alpha + 10)) ((1 - 4\alpha)(17\alpha^2 - 8\alpha + 1))</td>
<td>((\alpha - 1)(17\alpha^2 - 26\alpha + 10)) (-\alpha(17\alpha^2 - 8\alpha + 1))</td>
</tr>
<tr>
<td>5</td>
<td>(26\alpha^2 - 42\alpha + 17) (26\alpha^2 - 10\alpha + 1)</td>
<td>((5\alpha - 4)(26\alpha^2 - 42\alpha + 17)) ((1 - 5\alpha)(26\alpha^2 - 10\alpha + 1))</td>
<td>((\alpha - 1)(26\alpha^2 - 42\alpha + 17)) (-\alpha(26\alpha^2 - 10\alpha + 1))</td>
</tr>
</tbody>
</table>
From the above illustrations, the solutions of (1) in the general form are:

I. \[ h = (m^2+1)\alpha^2 - 2(m^2-m+1)\alpha + (m^2-2m+2) \]
\[ x = h(m\alpha-m+1) \]
\[ y = h(\alpha-1) \]

II. \[ h = (m^2+1)\alpha^2 \pm 2m\alpha + 1 \]
\[ x = h(1\pm m\alpha) \]
\[ y = \pm h\alpha \]

Some numerical examples are given below:

<table>
<thead>
<tr>
<th>m</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>68</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>130</td>
<td>26</td>
</tr>
<tr>
<td>5</td>
<td>222</td>
<td>37</td>
</tr>
<tr>
<td>6</td>
<td>350</td>
<td>50</td>
</tr>
<tr>
<td>7</td>
<td>520</td>
<td>65</td>
</tr>
<tr>
<td>8</td>
<td>738</td>
<td>82</td>
</tr>
<tr>
<td>9</td>
<td>1010</td>
<td>101</td>
</tr>
<tr>
<td>10</td>
<td>1342</td>
<td>122</td>
</tr>
</tbody>
</table>

From the above table it seems that when \( m \) is odd the value of \( x \) is even and \( y \) is odd whereas when \( m \) is even both \( x \) and \( y \) are even. When \( m \) is even the product \( xy \) is written as the difference of two squares. When \( m \) is odd the product \( xy \) is written as the sum of two squares.
3. DEDUCTIONS

1) Changing \( x \) by \(-x\) and \( y \) by \(-y\), we get the solutions of the equation\[x^2 + y^2 = (x-my)^3\]

2) Changing \( y \) by \(-y\), we get the solutions of the equation\[x^2 + y^2 = (-x +my)^3\]

REFERENCES


ON A QUADRATIC DIOPHANTINE EQUATION WITH FOUR VARIABLES

M.A. GOPALAN

Department of Mathematics, National College, Trichy, T.N., India
(E-Mail: gopalanma@yahoo.com)

MANJU SOMANATH*
(E-Mail: manjuajil@yahoo.com)

and

N. VANITHA*
(E-Mail: vanitha_1978@yahoo.co.in)

Department of Mathematics, Cauvery College for Women, Trichy, T.N., India

(Received on 31st October 2005)

We obtain two parametric solutions of quadratic Diophantine equation with four variables of the form \( x^2 + axy + y^2 = u^2 + \beta uv + v^2 \), \( a \neq \beta \). A few interesting relations among the solutions are also given.

INTRODUCTION

At the 35th annual Australian Mathematics Society Conference, the Diophantine equation,

\[ x^2 + xy + y^2 = u^2 + \beta uv + v^2 \]  \( \tag{1} \)

was discussed informally as an equation which probably is so difficult to study that it may require new Mathematics to prove that it has no non-trivial solutions or even only a finite number of non-trivial solutions.

B.G. Sloss [1] in his research report “On a Diophantine equation with four variables” has shown that the Diophantine equation (1) does not have solutions in some special cases.

Any attempt at solution of this problem will resulted in the development of many interesting new fields of mathematics.

In this paper we obtain two parametric solutions of Quadratic Diophantine Equation with four variables of the form.
Method of analysis

Consider the quadratic Diophantine Equation with four variables.

\[ x^2 + axy + y^2 = u^2 + 

Putting \( x = p + q, y = p - q, u = p + s, v = p - s \) in (2), it reduces to

\[ (\beta - \alpha)p^2 = (\beta - 2)s^2 - (\alpha - 2)q^2 \]  

Set \( s = x + (\alpha - 2) \) and \( q = x + (\beta - 2) \) in (3), it reduces to

\[ x^2 - p^2 = (\alpha - 2)(\beta - 2) \]

Method 1.

Let \( x - p = \beta - 2 \) and \( x + p = \alpha - 2 \)

Then, \( p = (\alpha - \beta)/2 \) \( x = (\alpha + \beta - 4)/2 \)

For integral solutions, \( \alpha \) and \( \beta \) should be both odd or even.

Case 1.

Let \( \alpha \) and \( \beta \) be both odd.

Take \( \alpha = 2k + 1 \) and \( \beta = 2t + 1, \ k \neq t \)

Solutions of equation (2) are given by

\[ x = 2k + 2t - 2 \]
\[ y = 2 - 4t \]
\[ u = 4k - 2 \]
\[ v = 2 - 2k - 2t \]

Case 2.

Let \( \alpha \) and \( \beta \) be both even.

Take \( \alpha = 2k \) and \( \beta = 2t, \ k \neq t \)
The solutions of equation (2) are given by

\[
\begin{align*}
x &= 2k + 2t - 4 \\
y &= 4 - 4t \\
u &= 4k - 4 \\
v &= 4 - 2k - 2t
\end{align*}
\]

Conditions:
1. The solutions are always even.
2. Values of \(x\) and \(u\) in case(i) exceed that of \(x\) and \(u\) in case(ii) by +2 respectively.
3. Values of \(y\) and \(v\) in case(i) exceed that of \(y\) and \(v\) in case(ii) by -2 respectively.

Relations:

\[
\begin{align*}
(1) \quad x + v &= 0 \\
(2) \quad y &= u + 2v \\
(3) \quad y^3 - u^3 - 8v^3 &= 6yuv \\
(4) \quad x + y &= u + v
\end{align*}
\]

Another pattern of solutions of equation (2) is obtained by solving equation (4) in a different approach which we present below.

Given \((\alpha - 2)(\beta - 2)\) is odd i.e., \((\alpha - 2)(\beta - 2) = 2w + 1\), which implies

\[
w = \frac{1}{2} [\alpha(3 - 2\alpha - 2\beta + 3)]
\]

an integer when \(\alpha, \beta\) are odd.

\[
\begin{align*}
\alpha &= 2k - 1 \\
\beta &= 2t + 1
\end{align*}
\]

so that, \(w = 2kt - k - t\)

1. reduces to \(x^2 - p^2 = 2w + 1\), which gives \(x = w + 1\), \(p = w\)

Hence, solutions of equation (2) are

\[
\begin{align*}
x &= 4kt - 2k \\
y &= -2t \\
u &= 4kt - 2t \\
v &= -2k
\end{align*}
\]

Given \((\alpha - 2)(\beta - 2)\) is even i.e., \((\alpha - 2)(\beta - 2) = 4(w + 1)\), which implies
\[ w = \frac{1}{4} (\alpha \beta - 2\alpha - 2\beta) \]

Here \( w \) is an integer when \( \alpha, \beta \) even or \( \alpha \) is even and \( \beta \) is odd or \( \alpha \) is odd and \( \beta \) is even.

Equation (4) reduces to \( x^{2} + p^{2} = 4(w + 1) \), which gives
\[ x = w \cdot 2 \cdot \beta \quad w \]

**Subcase (i).**

Suppose that \( \alpha \) and \( \beta \) are even.

Take \( \alpha = 2k, \beta = 2t \), which implies \( w = kt - k - t \)

**Hence the solutions of equation (2) are**

\[
\begin{align*}
x &= 2kt - 2k \\
y &= -2t \\
u &= 2kt - 2k - 2t - 2 \\
v &= 2
\end{align*}
\]

**Subcase (ii).**

Suppose that \( \alpha \) is even and \( \beta \) is odd.

Take \( \alpha = 2k, \beta = 2t + 1 \), which implies \( w = \frac{1}{4}(4kt - 2k - 4t - 2) \)

**Hence the solutions of equation (2) are**

\[
\begin{align*}
x &= 2kt - k \\
y &= -2t - 1 \\
u &= 2kt + k - 2t - 1 \\
v &= 2k
\end{align*}
\]

**Subcase (iii).**

Suppose that \( \alpha \) is odd and \( \beta \) is even.

Take \( \alpha = 2k + 1, \beta = 2t \), which implies \( w = (1/4)(4kt - 2t - 4k - 2) \)

**Solutions of (2) are**

\[
\begin{align*}
x &= 2kt + t - 2k - 1 \\
y &= -2t \\
u &= 2kt - 4 \\
v &= -2k - 1
\end{align*}
\]
REFERENCES


On Ternary Cubic Diophantine Equation

\[ x^2 + y^2 = 2z^3 \]

Gopalan M.A.
Department of Mathematics, National College, Trichy, India
E-mail: gopalanma@yahoo.com

Manju Somanath and Vanitha N.
Department of Mathematics, Cauvery College for Women, Trichy, India
E-mail: manjuajil@yahoo.com, vanitha.1978@yahoo.co.in

Abstract
Different patterns of non-zero integral solutions of the ternary cubic diophantine equation \( x^2 + y^2 = 2z^3 \) are obtained. A few relations among the solutions are presented. Also, solutions of a few higher degree equations are exhibited.

AMS Mathematics Subject Classification:

Keywords:

1. Introduction

Diophantine equations are numerously rich because of its variety. There are fascinating problems on higher degree Diophantine equations. In particular, cubic Diophantine equations with three unknowns have attracted the attention of Mathematicians since antiquity as can be seen form [1-3].

In this communication, the ternary cubic Diophantine equation of the form

\[ x^2 + y^2 = 2z^3 \]

is considered. This equation has

\((0, 0, 0), (0, 2^{3n-1}, 2^{2n-1}), (2^{3n-1}, 0, 2^{2n-1}), (\alpha^3, \alpha^3, \alpha^2)\)

as trivial solutions. Therefore towards this end, we search for non-trivial integral solutions of the above equation and different patterns of two parametric integral solutions are obtained. A few relations among the solutions are given. Also, solutions of a few higher degree equations are exhibited.
2. Method of Analysis

Let \( x, y, z \) be three non zero distinct integers such that \( x^2, y^3, z^4 \) are in arithmetic progression, which is represented as,

\[
x^2 + y^2 = 2z^3
\]  

(1)

Taking

\[
x = u + v \quad \text{and} \quad y = u - v
\]

(2)

where \( u \) and \( v \) are non-zero distinct integers, equation (1) reduces to.

\[
u^2 + v^2 = z^3
\]

(3)

which is satisfied by

\[
u = m^3 + mn^2
\]

\[
v = m^2n + n^3
\]

\[
z = m^2 + n^2
\]

in which \( m, n \in \mathbb{Z}^+ - \{0\} \).

In view of (2) the solutions of equation (1) are given by

\[
\begin{align*}
x &= (m + n)(m^2 + n^2) \\
y &= (m - n)(m^2 + n^2) \\
z &= m^2 + n^2
\end{align*}
\]

(4)

These solutions satisfy the following relations.

1. \( x^2 = y^2 (\text{mod} z^2) \)
2. \( (m - n)x = (m + n)y \)
3. \( x^6 + x^4y^2 - x^2y^4 - y^6 = 4(x^3 - y^3)z^6 \)
4. \( n(x^3 + y^3) + m(x^3 - y^3) = 2(x^2 - y^2)z^2 \)
5. \( mz(x^3 + y^3) - m(z(x^3 - y^3)) = x^2y^2(x^2 - y^2)^2 \)

It is noted that equation (3) is also satisfied by

\[
u = m^3 - 3mn^2
\]

\[
v = 3m^2n - n^3
\]

\[
z = m^2 + n^2
\]
On Ternary Cubic Diophantine Equation \( x^2 + y^2 = 2z^2 \)

and hence the corresponding solutions of equation (1) are obtained as

\[
\begin{align*}
  x &= (m+n)(m^2 + 4mn + n^2) \\
  y &= (m+n)(m^2 - 4mn + n^2) \\
  z &= m^2 + n^2
\end{align*}
\]  (5)

It is worth to mention here that choosing the values of \( m \) and \( n \) suitably in the above solutions, one obtains integral solutions of higher degree equations.

A few illustrations are shown below:

**Illustration 1:**

Set \( m = 2pq, n = p^2 - q^2, (p \neq q) \) in (4) and (5). Therefore, integral solutions of the equation \( x^2 + y^2 = 2z^2 \) are respectively

\[
\begin{align*}
  x &= (p^2 + 2pq - q^2)(p^2 + q^2)^2 \\
  y &= (q^2 + 2pq - p^2)(p^2 + q^2)^2 \\
  z &= (p^2 + q^2)
\end{align*}
\]  (6)

and

\[
\begin{align*}
  x &= (q^2 + 2pq - p^2)((p^2 + q^2)^2 + 8pq(p^2 - q^2)) \\
  y &= (p^2 + 2pq - q^2)((p^2 + q^2)^2 - 8pq(p^2 - q^2)) \\
  z &= (p^2 + q^2)
\end{align*}
\]  (7)

**Illustration 2:**

Take \( m = p^3 + pq^2, n = p^2 q + q^3 \) in (4) and (5). For this choice, the integral solutions of the equation \( x^2 + y^2 = 2z^2 \) are respectively given by,

\[
\begin{align*}
  x &= (p + q)(p^2 + q^2)^4 \\
  y &= (p - q)(p^2 + q^2)^4 \\
  z &= (p^2 + q^2)
\end{align*}
\]  (8)

and

\[
\begin{align*}
  x &= (p + q)(p^2 + q^2)^3(p^2 + 4pq + q^2) \\
  y &= (p - q)(p^2 + q^2)^3(p^2 - 4pq + q^2) \\
  z &= (p^2 + q^2)
\end{align*}
\]  (9)

Observe that by substituting \( m = p^3 - 3pq^2, n = 3p^2 q - q^3 \) in (4) and (5), the integral solutions of the equation \( x^2 + y^2 = 2z^2 \) are respectively given by,

\[
\begin{align*}
  x &= (p - q)(p^2 + q^2)(p^2 + 4pq + q^2) \\
  y &= (p + q)(p^2 + q^2)(p^2 - 4pq + q^2) \\
  z &= (p^2 + q^2)
\end{align*}
\]  (10)
and
\[
\begin{align*}
x &= (p + q)(p' + q')^5 + 4pq(3p' - 10p^2q^2 + 3q')^5 + 4pq(3p'^3 - 10p^2q'^2 + 3q'^3) \\
y &= (p - q)(p^2 + 4pq + q^2)^5 - 4pq(3p'^3 - 10p^2q'^2 + 3q'^3) \\
z &= (p^2 + q^2)
\end{align*}
\]

It is interesting to note here that, the solutions (9) and (10) are the same. In addition to the above two sets of solutions, we also have other patterns of solutions of (1) which are presented below:

**Pattern I:**

The substitution \( x = pz \) and \( y = qz \) in (1) leads to
\[
z = \frac{(p^2 + q^2)}{2}
\]
which is an integer when both \( p \) and \( q \) are even or odd.

**Case I:**

Let
\[
p = 2\alpha \quad q = 2\beta
\]
From (12) and (13), we get \( z = 2(\alpha^2 + \beta^2) \) and hence the values of \( x \) and \( y \) are
\[
\begin{align*}
x &= 4\alpha(\alpha^2 + \beta^2) \\
y &= 4\beta(\alpha^2 + \beta^2)
\end{align*}
\]
As an immediate consequence, we observe that
\[
\alpha x = \beta y (mod\ 4)
\]

**Case II:** Let
\[
p = 2\alpha - 1 \quad q = 2\beta - 1
\]
Using (6) in (4), we get \( z = 2(\alpha^2 + \beta^2 - (\alpha + \beta) + 1) \) and therefore
\[
\begin{align*}
x &= 2(2\alpha - 1)(\alpha^2 + \beta^2 - (\alpha + \beta) + 1) \\
y &= 2(2\beta - 1)(\alpha^2 + \beta^2 - (\alpha + \beta) + 1)
\end{align*}
\]
More generally, we assume
\[
p = m^\alpha - n^\beta, q = m^\alpha + n^\beta
\]
The solution of (1) is
\[
\begin{align*}
x &= (m^\alpha - n^\beta)(m^{2\alpha} + n^{2\beta}) \\
y &= (m^\alpha + n^\beta)(m^{2\alpha} + n^{2\beta}) \\
z &= m^{2\alpha} + n^{2\beta}
\end{align*}
\]
A few relations among the above solutions are given below

1. \( x^1 + y^1 \equiv 0 \pmod{2} \)
2. \( x y \) is written as the difference of two squares
3. \( x^3 + y^3 \equiv 0 \pmod{2} \)
4. \( x^3 + y^3 \equiv 0 \pmod{z^3} \)
5. \( x + y + z^2 \equiv 0 \pmod{2} \)
6. \( x + y + z^2 \equiv 0 \pmod{z} \)
7. \( x = y \pmod{2} \)
8. \( x = y \pmod{z} \)

**Pattern II:**
The substitution \( x = py \) and \( z = qy \) in (1) leads to
\[
y = \frac{q^4 + 1}{2p^3}
\]

The value of \( y \) is an integral only when \( p = 1 \) and \( q \) is odd.

Writing, \( q = 2\alpha - 1 \) the solutions of (1) are obtained as
\[
\begin{align*}
x & = (2\alpha^2 - 2\alpha + 1) \\
y & = (2\alpha^2 - 2\alpha + 1) \\
z & = (2\alpha - 1)(2\alpha^2 - 2\alpha + 1)
\end{align*}
\]

**References**


Abstract

Three parametric solutions of the quadratic Diophantine equation
\[ \sum_{i=1}^{4} x_i^2 = \sum_{i=1}^{4} y_i^2 \]
are presented. Also we deduce the solutions of
\[ \sum_{i=1}^{3} x_i^2 = \sum_{i=1}^{3} y_i^2 \]

Key words

Multivariable quadratic equation, Integer parametric solutions

Introduction

The numbers that can be expressed as the sum of squares of two numbers in two ways are called Ramanujan numbers and the determination of nonzero integer quadruples \(a^2 + b^2 = c^2 + d^2\) is well known.\(^1\)

In this communication we search for two \((x_1, x_2, x_3, x_4)\) and \((y_1, y_2, y_3, y_4)\) of integer quadruples such that
\[ \sum_{i=1}^{4} x_i^2 = \sum_{i=1}^{4} y_i^2 \]

Also we deduce the solutions of
\[ \sum_{i=1}^{3} x_i^2 = \sum_{i=1}^{3} y_i^2 \]
Method of analysis

Consider the identity

\[(a \cdot ab) + (c + bc) - (c + bc) = 4b(a^2 - c) \quad \text{------------------(1)}\]

where \(a, b, c\) are nonzero integers.

Setting

\[a = B - A, \quad b = A - C, \quad c = 1, \quad \lambda = \lambda^1\]

we have

\[4\lambda(a^2 - c) = 4B(A^2 - C)\]

Thus employing equation (1), we obtain

\[\lambda = \lambda, \quad \lambda + \lambda + \lambda^1 \quad \text{------------------(2)}\]

where

\[\lambda = (B - A + C)(A + C - 1), \quad \lambda = 2(A \cdot AB), \quad \lambda = 2(AB - A), \quad \lambda = 2(A \cdot BC)\]

Also observed the following relations are noticed

1) \[\sum_i \lambda_i = \sum_j \lambda_j \quad 1 \leq i, j \leq 4\]

2) \[\sum_i \lambda_i - \sum_j \lambda_j = \sum_{ij} \lambda_{ij} \quad 1 \leq i, j \leq 4\]

In the following table we present a few examples

| \(|55^2 + 81^2 + 48 + 96 = 64^2 + 72^2 + 45^2 + 99^2 \)
| \(|55^2 + 120^2 + 60^2 + 126 = 86^2 + 96^2 + 48^2 + 140^2 \)
| \(|90^2 + 176^2 + 98^2 + 180 = 126 + 140^2 + 80^2 + 198^2 \)

Deductions

After performing some algebra the equation (2) can be written as

\[(B + A - C)^2 - (A + C)^2 \cdot [(B - A + C)(A + C - 1)]^2 + 4[C + BC]^2 = [(B - A + C)(1 + A + C)]^2 + 16A \cdot B \cdot C \cdot 1 \lambda^1\]

Choosing

\[A = C = \alpha, \quad B = -\beta^2\]
we obtain the solutions of
\[ x_t = 2at(\beta^2 + A - C) \quad : \quad y_t = 2t(\beta^2 - C - C) \]

where
\[ x_t = \frac{\beta^2 - \gamma + C}{(\gamma - 1)} \quad \gamma = \frac{1 - A - t}{1 - a^2} \]

Some examples are given below.
1. \( 92^2 + 81^2 + 156^2 = 144^2 + 135^2 + 20^2 \)
2. \( 312^2 + 368^2 + 300^2 = 228^2 + 460^2 + 168^2 \)
3. \( 410^2 + 744^2 + 740^2 = 700^2 + 806^2 + 360^2 \)

**Conclusion**

By regrouping the terms of equation (2) other patterns of solutions of equation (3) may be obtained.

**References**