Exponentially localized solutions of Mel’nikov equation

C. Senthil Kumar a, R. Radha b, M. Lakshmanan a,∗

a Centre for Nonlinear Dynamics, Department of Physics, Bharathidasan University, Tiruchirapalli 620 024, India
b Department of Physics, Government College for Women, Kumbakonam 612 001, India

Accepted 2 March 2004

Abstract

The Mel’nikov equation is a (2+1) dimensional nonlinear evolution equation admitting boomeron type solutions. In this paper, after showing that it satisfies the Painlevé property, we obtain exponentially localized dromion type solutions from the bilinearized version which have not been reported so far. We also obtain more general dromion type solutions with spatially varying amplitude as well as induced multi-dromion solutions.

© 2004 Elsevier Ltd. All rights reserved.

1. Introduction

The identification of dromions which are exponentially localized solutions in (2+1) dimensional soliton equations [1–6] has been one of the most interesting developments in soliton theory in recent times, which has given a fillip to the understanding of integrable systems in (2+1) dimensions. Essentially, these localized solutions arise due to the presence of some additional nonlocal terms or effective local fields associated with “boundaries”. Further, the advent of “explode decay dromions” which are again exponentially localized solutions with time varying amplitudes [7] and induced dromions [6,8] using arbitrary functions of space and time variables have set in motion the process of unearthing more and more novel localized entities in (2+1) dimensional nonlinear systems.

1.1. Mel’nikov equation

An interesting evolution equation in (2+1) dimensions which we consider here is the one proposed by Mel’nikov [9,10] that describes (under certain conditions) the interaction of two waves on the x-axis. This equation is of the form

\[ 3u_t - \left[ u_x + \left( 3u^2 + u_{xx} + 8\kappa |\chi|^2 \right)_x \right]_x = 0, \]

\[ i\chi_t = u_x + \chi_{xx}, \]

where \( u \) is the long wave amplitude (real), \( \chi \) is the complex short wave envelope, and the parameter \( \kappa \) satisfies the condition \( \kappa^2 = 1 \). Eq. (1) may be considered either as a generalization of the K–P equation with the addition of a complex scalar field or as a generalization of the NLS equation with a real scalar field (after suitable interchange of coordinates \( y \) and \( t \)). Mel’nikov [10] has pointed out that Eq. (1) admits boomeron type solutions, which can be realized from an asymptotic analysis of the two soliton solution.

It is expected that the investigation of this equation may have wider ramifications in plasma physics, nonlinear optics and hydrodynamics. It is this diverse presence of this equation which prompts one to make a detailed investigation of

∗Corresponding author. Fax: +91-431-2407093/660245.
E-mail address: lakshman@cnld.bdu.ac.in (M. Lakshmanan).
their dynamics, particularly to identify whether localized solutions exist in this system. For this purpose, we first carry out a Painlevé singularity structure analysis and confirm that Eq. (1) does indeed satisfy the Painlevé property. Then bilinearizing the evolution equation and making use of certain arbitrary functions present in the solution, we obtain a large class of exponentially localized dromion solutions.

2. Singularity structure analysis of Mel’nikov equation

We explore the singularity structure of Eq. (1), by rewriting $x = a$ and $x^r = b$ as

$$3u_x - u_{xy} - 6u_x^2 - 6u_{xx} - u_{xxx} - 8\kappa(a_x b + 2a_x b_x + ab_2) = 0,$$  \hspace{1cm} (2a)

$$ia_t = u_a + a_x,$$  \hspace{1cm} (2b)

$$-ib_t = ub + b_x.$$  \hspace{1cm} (2c)

We now effect a local Laurent expansion in the neighbourhood of a noncharacteristic singular manifold $\{x, y, t\} = 0, \psi \neq 0, \phi \neq 0$. Assuming the leading orders of the solutions of Eq. (2) to have the form

$$u = u_0 \phi^a, \quad a = a_0 \phi^b, \quad b = b_0 \phi^c,$$  \hspace{1cm} (3)

where $u_0, a_0$ and $b_0$ are analytic functions of $\{x, y, t\}$ and $\alpha, \beta, \gamma$ are integers to be determined, we now substitute (3) into (2) and balance the most dominant terms to get

$$a = \beta = \gamma = -2,$$  \hspace{1cm} (4)

with the condition

$$\kappa a_0 b_0 = -9 \phi_x^2, \quad u_0 = -6 \phi_x^2.$$  \hspace{1cm} (5)

Now, considering the generalized Laurent expansion of the solutions in the neighbourhood of the singular manifold

$$u = u_0 \phi^a + \cdots + u_r \phi^{ar} + \cdots,$$  \hspace{1cm} (6a)

$$a = a_0 \phi^b + \cdots + a_r \phi^{br} + \cdots,$$  \hspace{1cm} (6b)

$$b = b_0 \phi^c + \cdots + b_r \phi^{cr} + \cdots,$$  \hspace{1cm} (6c)

the resonances (powers) at which arbitrary functions enter into (6) can be determined by substituting (6) into Eq. (2) and comparing the coefficients of $\phi^{ar}, \phi^{br}, \phi^{cr}$ to give

$$\left( \begin{array}{ccc} \phi_x^2 (r^2 - 5r - 30) \hat{r} & 8ab_0 \hat{r} & 8\kappa a_0 \hat{r} \\ a_0 & \phi_x^2 (r^2 - 5r) & 0 \\ b_0 & 0 & \phi_x^2 (r^2 - 5r) \end{array} \right) \begin{pmatrix} u_r \\ a_r \\ b_r \end{pmatrix} = 0,$$  \hspace{1cm} (7)

where $\hat{r} = (r - 4)(r - 5)$. Solving Eq. (7), one gets the resonance values as

$$r = -3, -1, 0, 4, 5, 6, 8.$$  \hspace{1cm} (8)

The resonance at $r = -1$ naturally represents the arbitrariness of the manifold $\phi(x, y, t) = 0$. In order to prove the existence of arbitrary functions at the other resonance values, we now substitute the full Laurent series

$$u = u_0 \phi^a + \sum_r u_r \phi^{ar},$$  \hspace{1cm} (9a)

$$a = a_0 \phi^b + \sum_r a_r \phi^{br},$$  \hspace{1cm} (9b)

$$b = b_0 \phi^c + \sum_r b_r \phi^{cr},$$  \hspace{1cm} (9c)

into Eq. (2). Now collecting the coefficients of $\phi^{-6}, \phi^{-4}, \phi^{-2}$ and solving them, we obtain the relations (5), implying a resonance at $r = 0$.

Similarly collecting the following coefficients, we obtain the necessary information about the positive resonances:
(i) coefficients of \( (\phi^{-5}, \phi^{-3}, \phi^{-1}) \):
\[
u_1 = 0, \quad a_1 = \frac{i\alpha \phi_1}{2} - \alpha_0, \quad b_1 = -\frac{ib_0 \phi_1}{2} - b_0.
\]
(10)

(ii) coefficients of \( (\phi^{-4}, \phi^{-6}, \phi^{-2}) \): \( a_2, a_3, \) and \( b_2 \) can be uniquely determined,

(iii) coefficients of \( (\phi^{-3}, \phi^{-1}, \phi^{-1}) \): \( a_3, a_5, \) and \( b_3 \) can be uniquely determined,

(iv) coefficients of \( (\phi^{-2}, \phi^0, \phi^0) \): only two equations result for three unknowns \( u_4, a_4, b_4 \) and so one of them is arbitrary, corresponding to a resonance at \( r = 4 \),

(v) coefficients of \( (\phi^{-1}, \phi^1, \phi^1) \): only \( u_5 \) is determined, while \( a_5 \) and \( b_5 \) are arbitrary corresponding to double resonance at \( r = (5,5) \),

(vi) coefficients of \( (\phi^0, \phi^2, \phi^2) \): only two equations result for three unknowns \( u_6, a_6, b_6 \) and so one of them is arbitrary, corresponding to \( r = 6 \),

(vii) coefficients of \( (\phi^1, \phi^3, \phi^3) \): \( u_7, a_7, \) and \( b_7 \) can be determined in terms of earlier coefficients,

(viii) coefficients of \( (\phi^2, \phi^4, \phi^4) \): only two equations result for three unknowns \( u_8, a_8, b_8 \) and so one of them is arbitrary, corresponding to \( r = 8 \).

For the negative resonance \( r = -3 \), following the approach of Conte et al. [11], we demand that both the solution of Mel'nikov Eq. (2) and the solution close to it represented by a perturbation series in a small parameter \( \epsilon \) are free from movable critical manifolds. We identify that the first order perturbed series does admit an arbitrary function corresponding to a movable pole at the resonance \( r = -3 \). Consequently, for each of the eight resonances given by Eq. (8), one can associate an arbitrary function in the solution (and close to it) without introducing movable critical manifolds.

It must be mentioned that the above system (2) admits another leading order behaviour with \( a = -2, \beta = y = -1, \alpha_0 = -2i\alpha_0, \) and \( b_0 \) are arbitrary. The Laurent series with the above leading order leads to resonances 0, 0, -1, 3, 3, 4, 5, 6, corresponding to a normal branch and the existence of sufficient number of arbitrary functions can be established at these resonance values. This can also be verified from the corresponding bilinear form as studied by Grammaticos et al. [12]. Consequently one can be assured that the Mel'nikov equations (1) or (2) indeed satisfies the Painlevé property.

3. Bilinearization of Mel'nikov equation and localized solutions

We next Hitora bilinearize the Mel'nikov Eq. (1) to bring out the existence of exponentially localized solutions of Mel'nikov equation. Making the transformation
\[
u = 2 \frac{\delta^2}{\delta x^2} \ln G, \quad \chi = \frac{G}{F},
\]
identifiable from the Painlevé analysis, Eq. (1) gets converted into the following Hirota bilinear form,
\[
(3D_x^2 - D_x D_y - D_y^2)F \cdot F = 8\kappa |G|^2,
\]
\[
iD_x G \cdot F = D_y^2 G \cdot F,
\]
where the \( D \)'s are the usual bilinear operators. However, this bilinearization have been already done by Hase et al. [13] where they have given soliton solutions whereas we have brought out localized solutions here. For a completion, we proceed as follows. Introducing the series expansion,
\[
G = \epsilon^0 + \epsilon^1 G^{(3)} + \cdots, \quad F = 1 + \epsilon^1 f^{(2)} + \epsilon^2 f^{(4)} + \cdots
\]
into the above bilinear form and gathering terms with various powers of the small parameter \( \epsilon \), we obtain the following set of equations,
\[
O(\epsilon) : iG^{(1)} = G^{(1)}_{xx},
\]
\[
O(\epsilon^2) : 3f_x^{(2)} - f_y^{(2)} - f_{xx}^{(2)} = 4\kappa g^{(1)} g^{(1)*}.
\]
et al. / Chaos, Solitons and Fractals 22 (2004) 705-712

eqn. (14a), we can immediately write down the following solution,
\[
g^{(1)} = \sum_{j=1}^{N} \exp(\psi_j), \quad \psi_j = l_j x + m_j y + \omega_j t + \psi_j^{(0)}, \quad \text{io} \omega_j = l_j^2,
\]  
where the spectral parameters \( l_j, m_j, \omega_j \) and \( \psi_j^{(0)} \) are all complex. Confining to \( N = 1 \) in Eq. (15) and substituting (15) into (14b), we obtain
\[
f^{(2)} = \exp(\psi_1 + \psi_1^* + 2\Delta), \quad \exp(2\Delta) = \frac{\kappa}{\left(3\omega_1^2 - l_1^2 m_1 - 4l_1^2\right)}.
\]
Here \( l_1 = l_{1R} + il_{1I}, m_1 = m_{1R} + im_{1I} \) and \( \omega_1 = \omega_{1R} + i\omega_{1I} \) and also \( \text{io}_1 = l_1^2 \). Choosing \( g^{(i+1)} = 0, f^{(i)} = 0, \text{for } j > 1, \) in Eq. (13) and using Eqs. (15) and (16) along with the transformation (11), the physical field \( u \) and the potential \( \chi \) can be easily seen to be driven by the envelope soliton and pulse soliton respectively as
\[
\chi = \left(\frac{\sqrt{l_{1R}}(12l_{1R}^2 - m_{1R} - 4l_{1I}^2)}{2\sqrt{\kappa}}\right) \text{sech}(\psi_{1R} + A) e^{i\phi_u},
\]
\[
u = 2l_{1R}^2 \text{sech}^2(\psi_{1R} + c),
\]
where \( \psi_{1R} = l_{1R} x + m_{1R} y + 2l_{1R}l_{1I} t + \psi_{1I}^{(0)} \) and \( \psi_{1I} = l_{1I} x + m_{1I} y - (l_{1R}^2 - l_{1I}^2) t + \psi_{1I}^{(0)} \). One can proceed further in the standard way to obtain higher order soliton solutions also.

### 3.1. Dromions

Looking at the above solutions (17), we realize the fact that as the parameter \( l_{1R} \to 0 \), both \( u \) and \( \chi \) vanish. But, when \( m_{1R} \to 12l_{1R}^2 - 4l_{1I}^2 \), the potential \( \chi \) vanishes, whereas the physical field \( u \) survives and is driven by a ghost soliton of the form,
\[
u = 2l_{1R}^2 \text{sech}^2(\psi_{1R} + c),
\]
where \( c \) is a new constant. This predicts the existence of exponentially localized solution for the complex field variable \( \chi \) in Eq. (1).

#### 3.1.1. \((1,1)\) Dromion

To generate a \((1,1)\) dromion, we now make the ansatz
\[
P = 1 + e^{\psi_1} + e^{\psi_2} + e^{\psi_1^*} + e^{\psi_2^*} + K e^{\psi_1 + \psi_2 + \psi_1^* + \psi_2^*},
\]
where
\[
\tilde{\psi}_1 = px + wt,
\]
\[
\tilde{\psi}_2 = qy.
\]
Here \( K \) is a real constant and \( p, \omega \) and \( q \) are complex constants. Substituting (19) in (12a), we obtain
\[
G = \rho e^{\tilde{\psi}_1 + \tilde{\psi}_2},
\]
\[
4\kappa|\rho|^2 = (p + p^*)(q + q^*)(1 - K),
\]
for the parametric choice \( 3\omega_{1R}^2 = 4p_{1R}^2 \). Substituting (19) and (21) in (12b), we find that \( \omega = -ip^2 \), where \( p_{1R} = \pm \sqrt{3}p_{1I}, p_{1R} \)

and \( p_{1I} \) are the real and imaginary parts, respectively of \( p \). Hence, the exponentially localized solution with one bound state for the potential field \( \chi \) for the above choice of \( \omega \) and \( p \) takes the form
\[
\chi = \frac{\rho e^{\tilde{\psi}_1 + \tilde{\psi}_2}}{1 + e^{\psi_1} + e^{\psi_2} + e^{\psi_1^*} + e^{\psi_2^*} + K e^{\psi_1 + \psi_2 + \psi_1^* + \psi_2^*}},
\]
while the scalar field has the form
which is always bounded, but localized everywhere except in the neighbourhood of the line $\tilde{\psi}_1 + \tilde{\psi}_1 = 0$ in the $(x,y)$ plane. A snapshot of the $(1,1)$ dromion solution for the magnitude of the potential field $\chi$ is shown in Fig. 1. One can proceed to find multi-dromion solutions also, generalizing the above $(1,1)$ dromions.

3.1.2. Dromions with spatially varying amplitude

It can be seen from Eq. (14a) that the differential equation for $g$ involves only the variables $t$ and $x$. Hence, an arbitrary function of $y$ can also enter into its solution so that the most general form of it can be given as

$$g(x,t) = \exp(\psi), \quad \psi = \lambda x + \beta t + \gamma(y),$$

(24)

where $f_0(y)$s are arbitrary functions of $y$. This fact can be harnessed in a suitable way to construct a more general class of localized solutions. Following the above procedure to derive the $(1,1)$ dromion solution (22), one can easily obtain the generalized dromion solution involving arbitrary function of $y$ in the same form as Eq. (22) except that $\tilde{\psi}_2$ is now given by

$$\tilde{\psi}_2 = f(y),$$

(25)

instead of (20b), where the arbitrary function $f(y)$ is in general complex. The amplitude of the localized solution with arbitrary function of $y$ is now defined by the equation

$$4\kappa|\rho|^2 = 2(\rho + \rho^*)(1 - K)f'_0(y),$$

(26)

where $f'_0(y)$ is the derivative of the real part of $f$ with respect to $y$. Thus, the amplitude of the above localized solution varies with the spatial coordinate $y$ by virtue of Eq. (26). This situation is reminiscent of the explode-decay dromion of the variable coefficient DSJ equation [7] where the amplitude varies with time. It should be mentioned that, to our knowledge this is the first time the amplitude of a localized solution of a $(2+1)$ dimensional nonlinear partial differential equation has been found to vary as a function of the spatial coordinate $y$. This can be easily generalized to multidromions with spatially varying amplitude.

3.1.3. Induced dromions

We also wish to point out that the existence of an arbitrary function in the solution of $g^{(1)}$ in Eq. (14a) can be further utilized to obtain new induced dromion solutions [6] for $\chi$. For example, Eqs. (14a) and (14b) can also be solved in terms of arbitrary functions as

$$g^{(1)} = a(y)e^{\tilde{\psi}_1}, \quad \tilde{\psi}_1 = lx + \omega_1 t,$$

(27)

where $a(y)$ is an arbitrary complex function of $y$ and $l_1$ and $\omega_1$ are complex constants constrained by the condition $i\omega_1 = l_1$. Substituting the form (27) into (14b) and solving the resultant equation, we obtain

$$f^{(1)} = b(y)e^{\tilde{\psi}_1},$$

(28)

Fig. 1. Snapshot of the $(1,1)$ dromion solution of the Mel'nikov equation (see Eq. (22)).
where \( b(y) \) is a real function of \( y \) and is given by the condition
\[
6 \omega_{1R}^2 b(y) - l_{1R} b_y - 8 l_{1R}^4 b(y) = 2 \kappa |a(y)|^2.
\]

The above solutions can then be used to generate curved line soliton for the field variable and the potential as
\[
\chi = \frac{a(y)}{\sqrt{b(y)}} e^{\psi_{1R}} \frac{e^{\psi_{1I}}}{\text{sech} \left[ \frac{1}{2} \log b(y) \right]},
\]

\[
u = 2 l_{1R}^2 \frac{e^{\psi_{1R}}}{\text{sech} \left[ \frac{1}{2} \log b(y) \right]},
\]

where \( \psi_{1R} \) and \( \psi_{1I} \) are the real and imaginary parts, respectively of \( \psi_{1} \). Thus, by choosing the arbitrary functions \( a(y) \) and \( b(y) \) suitably which are constrained by the Eq. (29), one can induce localized solutions for the field variable \( \chi \) as in the case of Zakharov–Strachan equation [6]. Even though there exists two functions \( a(y) \) and \( b(y) \), only one of them is found to be arbitrary, which is evident from the Eq. (29) above. For example, by choosing
\[
\frac{a(y)}{\sqrt{b(y)}} = 2 \text{sech}(m_1 y),
\]

where \( m_1 \) is a real constant, we can find from Eq. (29) that
\[
b(y) = \exp \left( -\frac{1}{l_{1R}} (8 \kappa \tanh m_1 y + (8 l_{1R}^4 - 6 \omega_{1R}^2) y) \right).
\]

Then we obtain the induced localized solution
\[
\chi = e^{i \psi_{1R}} \text{sech}(m_1 y) \text{sech} \left[ l_{1R} (x + 2 l_{1R}) - \frac{1}{2 l_{1R}} (8 \kappa \tanh m_1 y + (8 l_{1R}^4 - 6 \omega_{1R}^2) y) \right].
\]

Thus, by choosing \( a(y) \) and \( b(y) \) suitably, one can induce a wide class of localized solutions for the Mel'nikov Eq. (1). For example, choosing an algebraic form
\[
\frac{a(y)}{\sqrt{b(y)}} = \frac{2}{(y + y_0)^2 + 1},
\]

we obtain an algebraically decaying localized solution
\[
\chi = \frac{1}{(y + y_0)^2 + 1} e^{i \psi_{1R}} \text{sech} \left[ l_{1R} (x + 2 l_{1R}) - \frac{1}{2 l_{1R}} \left( (8 l_{1R}^4 - 6 \omega_{1R}^2) y + 8 \kappa \int \frac{dy}{(y + y_0)^2 + 1} \right) \right].
\]

One can as well generalize this procedure to construct even wider class of localized solutions. In fact, multi-induced dromions take the simple form
\[
\chi_N = \frac{\sum_{j=1}^{N} a_j(y)}{\sqrt{b(y)}} e^{\psi_{1R}} \text{sech} \left[ \psi_{1R} + \frac{1}{2} \log b(y) \right],
\]

where \( a_j, j = 1, 2, \ldots, N \) are arbitrary functions of \( y \) and they are related to \( b(y) \) by the relation
\[
6 \omega_{1R}^2 b(y) - l_{1R} b_y - 8 l_{1R}^4 b(y) = 2 \kappa \sum_{j=1}^{N} \sum_{q=1}^{N} a_q(y) a_{q^*}^*(y).
\]

Note that the sum of the arbitrary functions of \( y \) on the right hand side of Eq. (36) becomes possible due to the structure of Eq. (1). In practice one can choose \( \sum_{j=1}^{N} a_j(y) \) conveniently, for example, as a combination of algebraic and hyperbolic functions. With \( N = 2 \), choosing the functions as
\[
\frac{a_1(y)}{\sqrt{b(y)}} = 2 \text{sech}(m_1 y + \delta_1), \quad \frac{a_2(y)}{\sqrt{b(y)}} = 2 \text{sech}(m_2 y + \delta_2),
\]

where \( m_1, m_2, \delta_1 \) and \( \delta_2 \) are parameters, solving Eq. (29), one can obtain
Fig. 2. Snapshot of the induced two dromion solution of the Mel'nikov equation (see Eq. (40)).

\[ b(y) = \exp \left( -\frac{1}{L_{1y}} \left( 8 f_{1y}^2 - 6 \omega_{1y}^2 \right) y + 8 \kappa \left( \tanh(m_1y + \delta_1) + \tanh(m_2y + \delta_2) \right) \right) \]
\[ + 2 \int \frac{\sech(m_1y + \delta_1) \sech(m_2y + \delta_2) dy}{\text{sech}^2(y)} \]  

Then, the induced two dromion solution is given by

\[ X_2 = e^{b(y)} \left( \sech(m_1y + \delta_1) + \sech(m_2y + \delta_2) \right) \sech \left[ \frac{1}{2} \left( 8 f_{1y}^2 - 6 \omega_{1y}^2 \right) y \right] \]
\[ + 8 \kappa \left( \tanh(m_1y + \delta_1) + \tanh(m_2y + \delta_2) + 2 \int \frac{\sech(m_1y + \delta_1) \sech(m_2y + \delta_2) dy}{\text{sech}^2(y)} \right) \]  

and is shown in Fig. 2. One can identify the mutual influence of one dromion over the other from the additional terms occurring in the square bracket of Eq. (40).

If one chooses both the functions \( a_j, j = 1, 2 \), algebraically,

\[ \frac{a_1(y)}{\sqrt{b(y)}} = \frac{2}{(m_1y + \gamma_{10})^2 + 1}, \quad \frac{a_2(y)}{\sqrt{b(y)}} = \frac{2}{(m_2y + \gamma_{20})^2 + 1} \]  

or one of the functions to be algebraic and the other one to be hyperbolic

\[ \frac{a_1(y)}{\sqrt{b(y)}} = 2 \sech(m_1y + \delta_1), \quad \frac{a_2(y)}{\sqrt{b(y)}} = \frac{2}{(m_2y + \gamma_{20})^2 + 1} \]  

one can generate different kinds of induced lump-lump or dromion-lump solutions, respectively.

We have also tried to generate more general two soliton solutions by choosing

\[ g^{(1)} = a_1(y)e^{b_1} + a_2(y)e^{b_2}, \quad \hat{\gamma}_1 = l_1x + \omega_1t, \quad \hat{\gamma}_2 = l_2x + \omega_2t, \]  

which lead to a condition \( l_1 = l_2 \) thereby reducing to our original form (36) for \( N = 2 \). This is also true for \( N > 2 \). Thus we believe that the solution (36) constitutes the most general localized solution we could construct for the Mel'nikov equation through our procedure.

4. Conclusion

In this paper, we have pointed out the interesting fact that the (2+1) dimensional Mel'nikov equation admits exponentially localized solutions of different classes. We have also checked its integrability through Painleve analysis. We have in particular constructed localized dromion solutions and obtained new classes of localized solutions such as dromions with spatially varying amplitude and induced dromions.

Acknowledgements

The work of C.S. and M.L. form part of a Department of Science and Technology, Govt. of India sponsored research project.
References

Periodic and localized solutions of the long wave–short wave resonance interaction equation

R Radha, C Senthil Kumar, M Lakshmanan, X Y Tang and S Y Lou

1 Department of Physics, Government College for Women, Kumbakonam 612 001, India
2 Centre for Nonlinear Dynamics, Department of Physics, Bharathidasan University, Truchirapalli 620 024, India
3 Department of Physics, Shangai Jiao Tong University, Shanghai 200030, People’s Republic of China
4 Center of Nonlinear Science, Ningbo University, Ningbo 315211, People’s Republic of China

Abstract

In this paper, we investigate the (2+1)-dimensional long wave–short wave resonance interaction (LSRI) equation and show that it possess the Painlevé property. We then solve the LSRI equation using Painlevé truncation approach through which we are able to construct solution in terms of three arbitrary functions. Utilizing the arbitrary functions present in the solution, we have generated a wide class of elliptic function periodic wave solutions and exponentially localized solutions, such as dromions, multidromions, instantons, multi-instantons and bounded solitary wave solutions.

PACS numbers: 02.30.Jr, 02.30.Ik, 05.45.Yv

1. Introduction

The identification of dromions which are exponentially decaying solutions in the Davey–Stewartson I and other equations [1–6] has triggered a renewed interest in the study of integrable models in (2+1) dimensions. Dromions arise essentially by virtue of coupling the field variable to a mean field/potential, thereby preventing wave collapse in (2+1) dimensions and they can, in general, undergo inelastic collision unlike one-dimensional solitons. The identification of a large number of arbitrary functions in the solutions of (2+1)-dimensional integrable models has only added to the richness in the structure of them and hence construction of localized excitations in (2+1) dimensions continues to be a challenging and rewarding contemporary problem.

In this paper, we consider the existence of localized structures in the long wave–short wave resonance interaction equation of the form,
where the fields $S$ and $L$ denote short surface wave packets and long interfacial waves respectively, while $*$ stands for the complex conjugation. The above equation has recently been studied [7, 8] and its posion and one dromion solutions have been generated through Hirota method. However, no further general solutions could be constructed through this procedure for equations (1). In this contribution, we develop a very simple and straightforward procedure to generate a rather extended class of generic solutions of physical interest. For this purpose, first we carry out the singularity structure analysis to the LSRI equation and confirm its Painlevé nature. We utilize the local Laurent expansion of the general solution and truncate it at the constant level term (Painlevé truncation approach) and obtain solutions in terms of arbitrary functions. Through this procedure, we generate various periodic and exponentially localized solutions to equation (1). The novelty here is that the solution is generated through a very simple procedure but the solution obtained is rich in structure because of the arbitrary functions [9–14] present in the solution.

The plan of the paper is as follows. In section 2, we present the singularity structure analysis of the LSRI equation. Using these results, in section 3, we have shown the construction of solutions for the LSRI equation through the Painlevé truncation approach. Section 4 contains a wide class of localized solutions of the LSRI equation, both periodic and exponentially localized ones, through a judicial choice of the arbitrary functions. In section 5, we summarize our results. The appendix contains the one dromion solution of LSRI equation obtained through Hirota bilinearization approach for comparison.

2. Singularity structure analysis

To explore the singularity structure of equation (1), we rewrite $S = q$ and $S^* = r$ to obtain the following set of coupled equations:

\[
\begin{align*}
\text{i}(q_t + q_y) - q_{xx} + Lq &= 0, \\
-i(r_t + r_y) - r_{xx} + Lr &= 0, \\
L_t &= (2qr)_x.
\end{align*}
\]

(2a) (2b) (2c)

We now effect a local Laurent expansion in the neighbourhood of a noncharacteristic singular manifold $\phi(x, y, t) = 0$, $\phi_x \neq 0$, $\phi_y \neq 0$. Assuming the leading orders of the solutions of equation (2) to have the form

\[
q = q_0\phi^\alpha, \quad r = r_0\phi^\beta, \quad L = L_0\phi^\gamma,
\]

(3)

where $q_0$, $r_0$ and $L_0$ are analytic functions of $(x, y, t)$ and $\alpha$, $\beta$, $\gamma$ are integers to be determined, we substitute (3) into (2) and balance the most dominant terms to obtain

\[
\alpha = \beta = -1, \quad \gamma = -2,
\]

(4)

with the condition

\[
q_0r_0 = \phi_x\phi_t, \quad L_0 = 2\phi_x^2.
\]

(5)

Now, considering the generalized Laurent expansion of the solutions in the neighbourhood of the singular manifold,

\[
q = q_0\phi^\alpha + \cdots + q_1\phi^{r+\alpha} + \cdots
\]

(6a)
Periodic and localized solutions of the long wave–short wave resonance interaction equation

\[ r = r_0 \phi^\beta + \cdots + r_j \phi^{i+j} \beta + \cdots, \quad (6b) \]
\[ L = L_0 \phi^\gamma + \cdots + L_j \phi^{i+j} \gamma + \cdots, \quad (6c) \]

the resonances which are the powers at which arbitrary functions enter into (6) can be determined by substituting (6) into (2). Vanishing of the coefficients of \( (\phi^{i-3}, \phi^{i-3}, \phi^{i-3}) \) leads to the condition

\[
\begin{pmatrix}
- j(j - 3) \phi_x^2 & 0 & q_0 \\
0 & - j(j - 3) \phi_x^2 & r_0 \\
2(j - 2) r_0 \phi_x & 2(j - 2) q_0 \phi_x & -(j - 2) \phi_x \\
\end{pmatrix}
\begin{pmatrix}
q_j \\
r_j \\
L_j \\
\end{pmatrix}
= 0. \quad (7)
\]

From equation (7), one gets the resonance values as

\[ j = -1, 0, 2, 3, 4. \quad (8) \]

The resonance at \( j = -1 \) naturally represents the arbitrariness of the manifold \( \phi(x, y, t) = 0 \).

In order to prove the existence of arbitrary functions at the other resonance values, we now substitute the full Laurent series,

\[ q = q_0 \phi^\alpha + \sum_j q_j \phi^{i+\alpha}, \quad (9a) \]
\[ b = r_0 \phi^\beta + \sum_j r_j \phi^{i+\beta}, \quad (9b) \]
\[ L = L_0 \phi^\gamma + \sum_j L_j \phi^{i+\gamma}, \quad (9c) \]

into equation (2). Now collecting the coefficients of \( (\phi^{-3}, \phi^{-3}, \phi^{-3}) \) and solving the resultant equation, we obtain equation (5), implying the existence of a resonance at \( j = 0 \).

Similarly, collecting the coefficients of \( (\phi^{-2}, \phi^{-2}, \phi^{-2}) \) and solving the resultant equations by using the Kruskal ansatz, \( \phi(x, y, t) = x + \psi(y, t) \), we get

\[ q_1 = \frac{1}{2} [i q_0 (\psi_x + \psi_y) - 2 q_{0x}], \quad (10a) \]
\[ r_1 = \frac{1}{2} [-i r_0 (\psi_x + \psi_y) - 2 r_{0x}], \quad (10b) \]
\[ L_1 = 0. \quad (10c) \]

Collecting the coefficients of \( (\phi^{-1}, \phi^{-1}, \phi^{-1}) \), we have

\[ i(q_{0x} + q_{0y}) - q_{0xx} + L_0 q_2 + L_1 q_1 + L_2 q_0 = 0, \quad (11a) \]
\[ -i(r_{0x} + r_{0y}) - r_{0xx} + L_0 r_2 + L_1 r_1 + L_2 r_0 = 0, \quad (11b) \]
\[ L_{1r} = 2 [q_{0x} r_1 + r_{0x} q_1 + q_{1x} r_0 + q_1 r_{0x}] = 0. \quad (11c) \]

From (11a) and (11b), we can eliminate \( L_2 \) to obtain a single equation for the two unknowns \( q_2 \) and \( r_2 \).

\[ L_0 (r_0 q_2 - q_0 r_2) - (r_0 q_{0xx} - q_0 r_{0xx}) + i(r_0 (q_{0x} + q_{0y}) + q_0 (r_{0x} + r_{0y})) = 0. \quad (11d) \]

which ensures that either \( q_2 \) or \( r_2 \) is arbitrary. Obviously, \( L_2 \) itself can be obtained either from (11a) or (11b). Similarly, collecting the coefficients of \( (\phi^0, \phi^0, \phi^0) \), we have

\[ i(q_{1x} + q_{2x} \psi_x) + i(q_{1y} + q_{2y} \psi_y) - (q_{1xx} + 2 q_{2x}) + L_2 q_1 + L_3 q_0 = 0, \quad (12a) \]
\[ -i(r_{1x} + r_{2x} \psi_x) - i(r_{1y} + r_{2y} \psi_y) - (r_{1xx} + 2 r_{2x}) + L_2 r_1 + L_3 r_0 = 0, \quad (12b) \]
\[ L_{2t} + L_3 \psi_t = 2[q_{0x}r_2 + (q_{1x} + q_2)r_1 + (q_{2x} + q_3)r_0 + r_0q_2 + (r_{1x} + r_2)q_1 + (r_{2x} + 2r_3)q_0]. \]  
\hspace{1em} (12c)

We rewrite equations (12a) and (12b) as

\[ L_3 = \frac{1}{q_0}(-i(q_{1t} + q_2q_1) - i(q_{1y} + q_2q_3) + (q_{1xx} + 2q_{2x}) - L_{2q_1}); \]  
\hspace{1em} (12d)

\[ L_3 = \frac{1}{r_0}(+i(r_{1t} + r_2q_1) + i(r_{1y} + r_2q_3) + (r_{1xx} + 2r_{2x}) - L_{2r_1}); \]  
\hspace{1em} (12e)

Making use of the earlier relations (5), (10b) and (11b), we find that the right-hand sides of equations (12d) and (12e) are equal. Then, we are left with two equations for three unknowns. So, one of the three coefficients \( q_3, r_3 \) or \( L_3 \) is arbitrary. Collecting now the coefficients of \( (0, 0, 0) \), we have

\[ i(q_{2t} + 2q_3q_1) + i(q_{2y} + 2q_3q_3) - (q_{2xx} + 4q_{3x} + 6q_4) + L_{0q_4} + L_{2q_2} + L_{3q_1} + L_{4q_0} = 0, \]  
\hspace{1em} (13a)

\[ -i(r_{2t} + 2r_3q_1) - i(r_{2y} + 2r_3q_3) - (r_{2xx} + 4r_{3x} + 6r_4) + L_{0r_4} + L_{2r_2} + L_{3r_1} + L_{4r_0} = 0, \]  
\hspace{1em} (13b)

\[ L_{3t} + 2L_4 \psi_t = 2[q_{0x}r_3 - q_0r_4 + (q_{1x} + q_2)r_2 + (q_{2x} + 2q_3)r_1 + (q_{3x} + 3q_4)r_0 + r_0q_3 - r_0q_4 + (r_{1x} + r_2)q_2 + (r_{2x} + 2r_3)q_1 + (r_{3x} + 3r_4)q_0]. \]  
\hspace{1em} (13c)

Here also, the above set of three equations reduces to two equations. So, one of the three functions \( q_4, r_4 \) or \( L_4 \) is arbitrary. One can proceed further to determine all other coefficients of the Laurent expansions (9) without the introduction of any movable critical singular manifold. Thus, the LSRI equation indeed satisfies the Painlevé property.

3. Painlevé truncation approach

To generate the solutions of LSRI equation, we suitably harness the results of the Painlevé analysis. Truncating the Laurent series of the solutions of the LSRI equation at the constant level term, we have the Bäcklund transformation

\[ q = \frac{q_0}{\phi} + q_1, \]  
\hspace{1em} (14a)

\[ r = \frac{r_0}{\phi} + r_1, \]  
\hspace{1em} (14b)

\[ L = \frac{L_0}{\phi^2} + \frac{L_1}{\phi} + L_2. \]  
\hspace{1em} (14c)

Assuming a seed solution given by

\[ q_1 = r_1 = 0, \quad L_2 = L_2(x, y), \]  
\hspace{1em} (15)

we now substitute (14) with the above seed solution (15) into equations (2) to obtain the following system of equations by equating the coefficients of \( (\phi^{-3}, \phi^{-3}, \phi^{-3}) \):

\[ -2q_0\phi_2^2 + L_0q_0 = 0, \]  
\hspace{1em} (16a)

\[ -2r_0\phi_2^2 + L_0r_0 = 0, \]  
\hspace{1em} (16b)
Solving the above system of equations, we obtain the leading order coefficients already given by equation (5), namely $q_0 r_0 = \phi_t \phi_t$ and $L_0 = 2\phi_x^2$. Now collecting the coefficients ($\phi^{-2}$, $\phi^{-2}$, $\phi^{-2}$) we have the following system of equations:

\begin{align}
-2q_0 \phi_t - iq_0 \phi_y + 2q_0 \phi_x + q_0 \phi_{xx} + L_1 q_0 = 0, \\
ir_0 \phi_t - ir_0 \phi_y + 2r_0 \phi_x + r_0 \phi_{xx} + L_1 r_0 = 0, \\
L_0 - L_1 \phi_t = 2(q_0 r_0)_x.
\end{align}

From equation (17c), we have

\begin{equation}
L_1 = -2 \left[ \frac{\phi_{xx} + \phi_x \phi_{tx}}{\phi_t} \right].
\end{equation}

Using (18) in equation (17a) or (17b), one can easily obtain the relation

\begin{equation}
q_0 x = \frac{1}{2} \left[ i(\phi_t + \phi_y) + \phi_{xx} - \frac{2\phi_x \phi_{tx}}{\phi_t} \right].
\end{equation}

On integration, we obtain

\begin{equation}
q_0 = F(y,t) \exp \left[ \frac{1}{2} \int \frac{i(\phi_t + \phi_y) + \phi_{xx} - \frac{2\phi_x \phi_{tx}}{\phi_t}}{\phi_x} \, dx \right],
\end{equation}

where $F(y,t)$ is an arbitrary function of $y$ and $t$. Obviously the above solution is consistent with (17).

Again collecting the coefficients of ($\phi^{-1}$, $\phi^{-1}$, $\phi^{-1}$), we have the following set of equations:

\begin{align}
-2q_0 \phi_t - iq_0 \phi_y + 2q_0 \phi_x + q_0 \phi_{xx} + L_2 q_0 = 0, \\
-ir_0 \phi_t - ir_0 \phi_y + 2r_0 \phi_x + r_0 \phi_{xx} + L_2 r_0 = 0, \\
L_1 t = 0.
\end{align}

Using (18), we rewrite equation (21c) to obtain the trilinear form

\begin{equation}
\phi_t^2 \phi_{xx} - \phi_x \phi_{tx} \phi_t + \frac{\phi_x^2 \phi_t}{\phi_{tx}} + \frac{\phi_x \phi_{tx} \phi_t}{\phi_t} = 0.
\end{equation}

The structure of the trilinear equation (22) suggests that a specific solution can be given in the form

\begin{equation}
\phi = \phi_1(x,y) + \phi_2(y,t),
\end{equation}

where $\phi_1(x,y)$ and $\phi_2(y,t)$ are arbitrary functions in the indicated variables. Using (23) in equations (18) and (20), one can obtain the functions $q_0$ and $L_1$ as

\begin{align}
q_0 &= F(y,t) \exp \left[ \frac{1}{2} \int \frac{i(\phi_2 t + \phi_1 y + \phi_2 y) + \phi_{1xx}}{\phi_{1x}} \, dx \right], \\
L_1 &= -2\phi_{1xx}.
\end{align}

From (24b), we find that equation (21c) is an identity. Using (24a), equations (21a) and (21b) can be reduced to the form

\begin{equation}
\phi_{2tt} + \phi_{2ty} = 0.
\end{equation}
Equation (25) can be solved readily to express the submanifold $\phi_2 (y, t)$ in the form
\[ \phi_2 = F_2(y) + F_3(t - y), \]  
where $F_2(y)$ and $F_3(t - y)$ are arbitrary functions in $y$ and $(t - y)$, respectively.

Finally, collecting the coefficients of $(\phi^0, \phi^0, \phi^0)$, we have only one equation
\[ L_{2t} = 0. \]  
Using (21a) for $L_2$, (27) reduces to the form
\[ (F_{1t} + F_{2y})F + (F_t + F_y)F_t = 0. \]  
Equation (28) can be solved to obtain the form for $F(y, t)$ as
\[ F(y, t) = F_1(t - y). \]  
Thus the LSRI equation (1) has been solved by the truncated Painlevé approach and the fields $q$ and $r$ can be given in terms of the arbitrary functions as
\[ q = \frac{q_0}{\phi_1(x, y) + F_2(y) + F_3(t - y)}, \]  
\[ r = \frac{\phi_{1x} \phi_{2t}}{q_0 (\phi_1(x, y) + F_2(y) + F_3(t - y))} \]  
and
\[ L = \frac{2 \phi_{1x}^2}{(\phi_1(x, y) + F_2(y) + F_3(t - y))^2} - \frac{2 \phi_{1xx}}{(\phi_1(x, y) + F_2(y) + F_3(t - y))} + L_2, \]  
where
\[ L_2 = \int \frac{1}{2} \left( \frac{i(\phi_{1yy} + F_{2yy}) + \phi_{1xx}}{\phi_{1x}} - \frac{i(\phi_{1y} + F_{2y}) + \phi_{1xx}}{\phi_{1x}^2} - \frac{1}{4} \left( \phi_{1y} + F_{2y} \right)^2 + \phi_{1xx}^2 \right) dx \]  
Here the function $q_2(y, t)$ is given by equation (26) and $q_0$ by (24a), while the functions $\phi_1(x, y), F_2(y), F_3(t - y)$ are themselves arbitrary in the indicated variables.

4. Novel exact solutions of LSRI equation

Now we make use of the above-truncated Laurent expansion solution to obtain exact solutions of the LSRI equation (1) for the variables $S(x, y, t)$ and $L(x, y, t)$.

Taking into account our notation in equation (2), that is $q = S(x, y, t)$ and $r = S^*(x, y, t)$, we have $q = r^*$ as far as equation (1) is concerned. Using this condition in equations (30a) and (30b), we obtain the condition
\[ |F_1(t - y)|^2 = F_{3t}. \]  
Thus, from the results of the previous section, we find that the solution of the original variable $S(x, y, t)$ takes the form
\[ S(x, y, t) = \frac{\sqrt{F_{3t} \phi_{1x}}}{} \exp \left( \int \frac{1}{2} \frac{i(\phi_{1y} + F_{2y})}{\phi_{1x}} dx \right) \left( \phi_1(x, y) + F_2(y) + F_3(t - y) \right), \]  
while its squared magnitude takes the form
\[ |S|^2 = \frac{\phi_{1x} F_{3t}}{(\phi_1(x, y) + F_2(y) + F_3(t - y))^2}. \]
The form of $L(x, y, t)$ remains the same as given in equation (31). With the above general form of the solutions, we now identify interesting classes of exact solutions to equation (1), including periodic and localized solutions by giving specific forms for the three arbitrary functions $\phi_1(x, y)$, $F_2(y)$ and $F_3(t - y)$.

4.1. Periodic solutions and localized dromion solutions

Let us now choose the arbitrary functions $\phi_1$ and $F_3$ to be Jacobian elliptic functions, namely $sn$ or $cn$ functions. The motivation behind this choice of arbitrary function stems from the fact that the limiting forms of these functions happen to be localized functions. Hence, a choice of $cn$ and $sn$ functions can yield periodic solutions which are more general than exponentially localized solutions (dromions). We choose, for example,

$$\phi_1 = \text{sn}(ax + by + c_1; m_1), \quad F_2 = 4, \quad F_3 = \text{sn}(t - y + c_2; m_2),$$

so that

$$S(x, y, t) = \sqrt{a \text{cn}(u_1; m_1) \text{dn}(u_1; m_1) \text{cn}(u_2; m_2) \text{dn}(u_2; m_2)} e^{\frac{iy}{b}x},$$

where $u_1 = ax + by + c_1$ and $u_2 = t - y + c_2$. In equations (36) and (37), the quantities $m_1$ and $m_2$ are the modulus parameters of the respective Jacobian elliptic functions while $a$, $b$, $c_1$ and $c_2$ are arbitrary constants. The corresponding expression for $|S(x, y, t)|^2$ takes the form

$$|S|^2 = \frac{|a \text{cn}(u_1; m_1) \text{dn}(u_1; m_1) \text{cn}(u_2; m_2) \text{dn}(u_2; m_2)|}{(4 + \text{sn}(u_2; m_2) + \text{sn}(u_1; m_1))^2}.$$  

The profile of the above solution for the parametric choice $a = b = 1, c_1 = c_2 = 0, m_1 = 0.2, m_2 = 0.3, t = 0$ is shown in figure 1(a). Note that the periodic wave moves with unit phase velocity.

4.1.1. $(1, 1)$ dromion solution. As a limiting case of the periodic solution given by equation (38), when $m_1, m_2 \rightarrow 1$, the above solution degenerates into an exponentially localized solution (dromion). Noting that $\text{cn}(u; 1) = \text{dn}(u; 1) = \text{sech} u$ and $\text{sn}(u; 1) = \text{tanh} u$, the limiting forms corresponding to $(1, 1)$ dromion take the expressions

$$S = \frac{\sqrt{a} \text{sech}(t - y + c_2) \text{sech}(ax + by + c_1)}{4 + \text{tanh}(ax + by + c_1) + \text{tanh}(t - y + c_2)} e^{\frac{iy}{b}x}$$

and

$$|S|^2 = \frac{a \text{sech}^2(t - y + c_2) \text{sech}^2(ax + by + c_1)}{(4 + \text{tanh}(ax + by + c_1) + \text{tanh}(t - y + c_2))^2}.$$  

The variable $L$ then takes the form (using expression (31))

$$L = \frac{2a^2 \text{sech}^4(ax + by + c_1)}{(4 + \text{tanh}(ax + by + c_1) + \text{tanh}(t - y + c_2))^2} - \frac{2 \text{sech}^2(t - y + c_2)}{4 + \text{tanh}(ax + by + c_1) + \text{tanh}(t - y + c_2)}$$

$$- \text{tanh}(ax + by + c_1) - i \text{tanh}(ax + by + c_1)$$

$$+ \text{sech}^2(ax + by + c_1) - \text{sech}^2(ax + by + c_1) - \frac{1}{4}.$$  

A schematic form of the $(1, 1)$ dromion for the parametric choice $a = b = 1, c_1 = c_2 = 0$ is shown in figure 1(b). Again note that the dromion travels with a unit velocity in a diagonal
direction in the $x$-$y$ plane. One can check that the $(1,1)$ dromion obtained by Lai and Chow in [8] using the Hirota bilinear method is a special case of the above solution (40) by fixing the parameters $a$, $b$, $c_1$ and $c_2$ suitably. However, the later method is unable to give more general solutions (see also the appendix).

4.2. More general periodic solutions and higher order dromion solutions

4.2.1. Periodic solution and (2, 1) dromion. Next we obtain a more general periodic solution by choosing further general forms for the arbitrary functions. As an example, we choose

$$
\phi_1 = d_1 \text{sn}(c_1 + a_1 x + b_1 y; m_1) + d_2 \text{sn}(c_2 + a_2 x + b_2 y; m_2),
$$

$$
F_2 = 4, \quad F_3 = d_3 \text{sn}(c_3 + t - y; m_3),
$$

where $a_i$, $b_i$, $c_i$ and $d_i$ are arbitrary constants and $m_i$'s are modulus parameters ($i = 1, 2, 3$). Then

$$
|S|^2 = \frac{q_1}{q_2},
$$

where $q_1 = |(d_1 a_1 \text{cn}(u_1; m_1) \text{dn}(u_1; m_1) + d_2 a_2 \text{cn}(u_2; m_2) \text{dn}(u_2; m_2))d_3 \text{cn}(u_3; m_3) \text{dn}(u_3; m_3)|$, $q_2 = (4 + d_1 \text{sn}(u_1; m_1) + d_2 \text{sn}(u_2; m_2) + d_3 \text{sn}(u_3; m_3))^2$, $u_1 = c_1 + a_1 x + b_1 y$, $u_2 = c_2 + a_2 x + b_2 y$ and $u_3 = c_3 + t - y$ with corresponding expressions for $S(x, y, t)$. The profile of the above solution for the parametric choice $a_1 = 1, b_1 = 1, a_2 = 1, b_2 = -1, d_1 = 5, d_2 = 4, d_3 = 0.5, c_1 = 0, c_2 = c_3 = 5, m_1 = 0.2, m_2 = 0.3, m_3 = 0.4, t = 0$ is shown in

Figure 1. (a) Elliptic function solution (38), (b) localized dromion solution (40) for the variable $|S(x, y, t)|^2$ and (c) the corresponding magnitude of the variable $L(x, y, t)$ given by (41).
Periodic and localized solutions of the long wave–short wave resonance interaction equation

Figure 2. (a) Elliptic function solution (43), (b)–(d) (2, 1) dromion solution (44) and its interaction at time intervals (b) \( t = -10 \), (c) \( t = 0 \) and (d) \( t = 10 \).

Figure 2(a). As \( m_1, m_2, m_3 \to 1 \), the above solution, namely equation (43), degenerates into a (2, 1) dromion solution given by

\[
|S|^2 = \frac{(d_1 a_1 \text{sech}^2 u_1 + d_2 a_2 \text{sech}^2 u_2) d_3 \text{sech}^2 u_3}{(4 + d_1 \tanh u_1 + d_2 \tanh u_2 + d_3 \tanh u_3)^2}
\]

where \( u_1 = c_1 + a_1 x + b_1 y \), \( u_2 = c_2 + a_2 x + b_2 y \) and \( u_3 = c_3 + t - y \). The dromion interaction for the parametric choice \( a_1 = b_1 = a_2 = 1, b_2 = -1, d_1 = 0.5, d_2 = d_3 = 1, c_1 = c_2 = c_3 = 0 \) is shown in figures 2(b)–(d) for different time intervals. Here both the dromions travel with equal velocity but along opposite diagonals in the \( x-y \) plane. The interaction is elastic for this choice. The variable \( L \) can be evaluated again using equation (31), which we desist from presenting here.

4.2.2. Periodic solution and (2, 2) dromion. Another example for more general periodic solution is given by choosing

\[
\phi_1 = d_1 \text{sn}(c_1 + a_1 x + b_1 y; m_1) + d_2 \text{sn}(c_2 + a_2 x + b_2 y; m_2),
\]

\[
F_2 = 4, \quad F_3 = d_3 \text{sn}(c_3 + t - y; m_3) + d_4 \text{sn}(c_4 + t - y; m_4).
\]

In equation (45), we choose \( m_1, m_2, m_3 \to 1 \), to obtain (2, 2) dromion solution given by

\[
|S|^2 = \frac{(d_1 a_1 \text{sech}^2 u_1 + d_2 a_2 \text{sech}^2 u_2)(d_3 \text{sech}^2 u_3 + d_4 \text{sech}^2 u_4)}{(4 + d_1 \tanh u_1 + d_2 \tanh u_2 + d_3 \tanh u_3 + d_4 \tanh u_4)^2},
\]

where \( u_1 = c_1 + a_1 x + b_1 y \), \( u_2 = c_2 + a_2 x + b_2 y \), \( u_3 = c_3 + t - y \) and \( u_4 = c_4 + t - y \). The solution of (2, 2) dromion for the parametric choice \( a_1 = b_1 = a_2 = b_2 = 1, d_1 = 0.5, d_2 = d_3 = d_4 = 1, c_1 = c_2 = c_3 = c_4 = 0 \) is plotted in figure 3 for various time intervals. We find that there are two sets of dromions, each set containing two dromions one followed by the other. The two sets of dromions are travelling with the same velocity in opposite diagonals of the \( x-y \) plane. The dromions interact and move forward as time progresses.
Figure 3. (a)-(e) (2, 2) dromion solution (46) interaction at time intervals (a) \( t = -8 \), (b) \( t = -4 \), (c) \( t = 0 \), (d) \( t = 4 \) and (e) \( t = 8 \).

4.2.3. \((M, N)\) dromion. To generalize the above solutions, we choose

\[
\phi_1 = \sum_{j=1}^{M} d_j \text{sn}(c_j + a_j x + b_j y; m_j),
\]

\[
F_2 = 4, \quad F_3 = \sum_{k=1}^{N} d_k \text{sn}(c_k + t - y; m_k),
\]

where \(a_j, b_j, c_j, d_j, c_k, d_k\) are arbitrary constants and all \(m_j\)'s and \(m_k\)'s take values between 0 and 1 for periodic solutions and equal to 1 for dromion solutions. We proceed as above to construct periodic and dromion solutions, respectively, as

\[
|S|^2 = \frac{1}{4 + \sum_{j=1}^{M} d_j \text{sn}(u_1; m_j) + \sum_{k=1}^{N} d_k \text{sn}(u_2; m_k)}
\]

and

\[
|S|^2 = \frac{\sum_{j=1}^{M} d_j a_j \text{sech}^2(c_j + a_j x + b_j y) \sum_{k=1}^{N} d_k \text{sech}^2(c_k + t - y)}{4 + \sum_{j=1}^{M} d_j \text{tanh}(c_j + a_j x + b_j y) + \sum_{k=1}^{N} d_k \text{tanh}(c_k + t - y)},
\]

where \(u_1 = c_j + a_j x + b_j y\) and \(u_2 = c_k + t - y\).
Periodic and localized solutions of the long wave–short wave resonance interaction equation

4.3. Instanton-type solutions

Another type of elliptic function solution can be chosen as

\[
\phi_1 = \text{sn}(ax + c_1; m_1) \text{cn}(by + c_2; m_2) , \quad F_2 = 4 , \quad F_3 = \text{sn}(t - y + c_3; m_3).
\]  

(50)

Then

\[
|S|^2 = \frac{|a \text{cn}(u_1; m_1) \text{dn}(u_1; m_1) \text{cn}(u_2; m_2) \text{cn}(u_3; m_3) \text{dn}(u_3; m_3)|}{(4 + \text{sn}(u_1; m_1) \text{cn}(u_2; m_2) + \text{sn}(u_3; m_3))^2}.
\]

(51)

where \(u_1 = ax + c_1, u_2 = by + c_2\) and \(u_3 = t - y + c_3\). The profile of the above periodic solution for the parametric choices \(a = 1, b = -1, c_1 = c_2 = c_3 = 0, m_1 = 0.2, m_2 = 0.3, m_3 = 0.4\) is shown in figure 4(a).

As \(m_1, m_2, m_3 \to 1\), equation (51) degenerates into an instanton-type solution,

\[
|S|^2 = \frac{a \text{sech}^2(t - y + c_3) \text{sech}^2(ax + c_1) \text{sech}(by + c_2)}{(4 + \tanh(ax + c_1) \text{sech}(by + c_2) + \tanh(t - y + c_3))^2}.
\]

(52)

A schematic diagram of the instanton solution for the parametric choice \(a = 1, b = -1, c_1 = c_2 = 0, c_3 = 0.5\) is shown in figures 4(b)–(f) for various time intervals. We can see
that the instanton expressed by (52) has a maximum amplitude at \( t = 0 \) while the amplitude decays exponentially as time \( |t| \to \infty \).

### 4.4. Two-instanton solution

A more general form of (51) is given by

\[
\phi_1 = d_1 \text{sn}(c_1 + a_1 x; m_1) \text{cn}(b_1 y; m_2) + d_2 \text{sn}(c_2 + a_2 x; m_3) \text{cn}(b_2 y; m_4),
\]

\[
F_2 = 4, \quad F_3 = d_3 \text{sn}(c_3 + t - y; m_5).
\]

Then

\[
|S|^2 = \frac{f_1}{f_2}.
\]

Here \( f_1 = \left| (d_1 a_1 \text{cn}(u_1; m_1) \text{dn}(u_1; m_1) \text{cn}(b_1 y; m_2) + d_1 a_2 \text{cn}(u_2; m_3) \text{dn}(u_2; m_3) \text{cn}(b_2 y; m_4)) d_3 \text{cn}(u_3; m_5) \text{dn}(u_3; m_5) \right|, f_2 = (4 + d_1 \text{sn}(u_1; m_1) \text{cn}(b_1 y; m_2) + d_2 \text{sn}(u_2; m_3) \text{cn}(b_2 y; m_4) + d_3 \text{sn}(u_3; m_5))^2, u_1 = c_1 + a_1 x, u_2 = c_2 + a_2 x \text{ and } u_3 = c_3 + t - y. \) The above periodic solution for the parametric choices \( a_1 = b_1 = a_2 = 1, b_2 = -1, d_1 = d_2 = d_3 = 1, c_1 = c_3 = -5, c_2 = 0, m_1 = 0.2, m_2 = 0.3, m_3 = 0.4, m_4 = 0.2, m_5 = 0.3 \) is shown in figure 5(d).

As \( m_1, m_2, m_3, m_4, m_5 \to 1 \), equation (54) degenerates into two-instanton solution given by

\[
|S|^2 = \frac{(\text{sech}^2(a_1 x) \text{sech}(b_1 y) + \text{sech}^2(a_2 x) \text{sech}(b_2 y)) \text{sech}^2(t - y)}{(4 + \tanh(a_1 x) \text{sech}(b_1 y) + \tanh(a_2 x) \text{sech}(b_2 y) + \tanh(t - y))^2}.
\]

The time evolution of the solution (55) is shown in figures 5(b)–(f). Choosing the arbitrary constants appropriately, we have one of the instantons having a maximum amplitude at \( t = -2 \) while the other at \( t = 2 \) and decay exponentially as time \( |t| \to \infty \).

To generalize the above solutions, we choose

\[
\phi_1 = \sum_{j=1}^{M} d_j \text{sn}(c_j + a_j x; m_j) \text{cn}(f_j + b_j y; n_j),
\]

\[
F_2 = 4, \quad F_3 = \sum_{k=1}^{N} d_k \text{sn}(c_k + t - y; m_k)
\]

where \( a_j, b_j, c_j, d_j, f_j, c_k, d_k \) are arbitrary constants, \( m_j, n_j \) and \( m_k \) take values between 0 and 1. One can construct multi-instanton solution by choosing all the values of \( m_j, n_j \) and \( m_k \) to be equal to 1.

### 4.5. Bounded multiple solitary waves

In expression (35), one can also easily identify bounded multiple solitary waves all moving with the same velocity. For instance, using the Jacobian elliptic function form (45) with \( d_2 = 0 \) in the limit \( m_1, m_3, m_4 \to 1 \), one can obtain multiple solitary waves which are bounded. Figure 6 displays the structure of a two-soliton solution expressed by

\[
|S|^2 = \frac{1}{3} \frac{\text{sech}^2(t - y + 5) + 2 \text{sech}^2(t - y - 5)}{\left[ \tanh \frac{x+y}{3} + 8 + \tanh(t - y + 5) + 2 \tanh(t - y - 5) \right]^2},
\]
Periodic and localized solutions of the long wave–short wave resonance interaction equation

\[ \phi_1 = \frac{1}{2} \tanh \frac{x + y}{3}, \quad F_2 = 4, \quad F_3 = \frac{1}{2} \tanh(t - y + 5) + \tanh(t - y - 5). \]

The figure shows that one of the solitary waves follows the other one but both are travelling with equal velocity. Hence, there will not be any interaction between them.
Finally, one can obtain other interesting classes of solutions for different choices of the arbitrary functions in equations (34), (35) and (31).

5. Conclusion

In summary, we have investigated the singularity structure of the (2+1)-dimensional LSRI equation and confirmed that it satisfies the Painlevé property. The Painlevé truncation approach has been used to construct successfully a very wide class of solutions of the (2+1)-dimensional LSRI equation. The rich solution structure of the LSRI equation is revealed because of the entrance of three arbitrary functions in (34) and (31). Especially, Jacobian elliptic function periodic wave solutions and three special localized structures, namely dromion, dromion-type instanton and bounded dromion solutions are given explicitly. However, more general multiple non-bounded dromion solutions whose phase velocities differ from each other have not yet been obtained from the present approach. It appears that one has to solve equation (22) for more general solutions than the form (23) presented in this paper in order to deduce more general solution. This is an open problem at present.

Acknowledgments

The works of CS and ML form a part of a Department of Science and Technology, Government of India sponsored research project. The work of SYL was supported by the National Natural Science Foundations of China (nos. 90203001 and 10475077). RR wishes to thank Department of Science and Technology (DST), Government of India for sponsoring a major research project.

Appendix A. One dromion solution through Hirota bilinearization

Here we briefly point out how the (1,1) dromion solution can be obtained through the Hirota bilinearization method [8]. To bilinearize equation (1), we make the transformation

\[ S = \frac{g}{f}, \quad L = 2(\log f)_{xx}, \]  

(A.1)

which can be identified from the Painlevé analysis in section 2. The resultant bilinear form is given by

\[ (i(D_x + D_y) + D_x^2)g \cdot f = 0, \]  

(A.2a)

\[ D_x D_t f \cdot f = 2gg^*, \]  

(A.2b)

where \( D \)'s are the usual Hirota operators. To generate a (1,1) dromion, one considers the ansatz

\[ f = 1 + \exp(\psi_1 + \psi_1^*) + \exp(\psi_2 + \psi_2^*) + M \exp(\psi_1 + \psi_1^* + \psi_2 + \psi_2^*), \]  

(A.3)

where

\[ \psi_1 = px + qy, \]  

(A.4a)

\[ \psi_2 = \lambda y - \Omega t. \]  

(A.4b)

Here \( M \) is a real constant and \( p, \Omega, \lambda \) and \( q \) are complex constants. Substituting (A.3) into
and also the conditions $M < 1, q = ip^2$ and $\Omega = \lambda$. This is a special case of the dromion we have obtained in (40) with the constants $c_1 = c_2 = \frac{1}{2} \log \frac{3}{2}$ and by choosing $\psi_{1R} = -(ax + by + c_1)$ and $\psi_{2R} = -(t - y + c_2)$ and $M = \frac{1}{3}$, where $\psi_{1R}, \psi_{2R}$ are the real parts of $\psi_1, \psi_2$, respectively. Thus, equation (40) contains the solution of Lai and Chow [8] as a special case. No higher order solution has been constructed by this method.

References

Nonintegrability of (2 + 1)-dimensional continuum isotropic Heisenberg spin system: Painlevé analysis

C. Senthil Kumar, M. Lakshmanan, B. Grammaticos, A. Ramani

Centre for Nonlinear Dynamics, Department of Physics, Bharathidasan University, Tiruchirapalli 620 024, India
GMPIB, Université Paris VII, Tour 24-14, 75251 Paris, France
CPT, Ecole Polytechnique, CNRS, UMR 7644, 91128 Palaiseau, France

Received 13 March 2006; accepted 28 March 2006
Available online 18 April 2006
Communicated by A.R. Bishop

Abstract
While many integrable spin systems are known to exist in 1 + 1 and 2 + 1 dimensions, the integrability property of the physically important (2 + 1)-dimensional isotropic Heisenberg ferromagnetic spin system in the continuum limit has not been investigated in the literature. In this Letter, we show through a careful singularity structure analysis of the underlying nonlinear evolution equation that the system admits logarithmic type singular manifolds and so is of non-Painlevé type and is expected to be nonintegrable.

© 2006 Elsevier B.V. All rights reserved.
PACS: 02.30.Jr; 02.30.Ik; 75.10.Pq
Keywords: Painlevé property; Integrability

1. Introduction
The nonlinear dynamics underlying magnetic spin systems is a fascinating topic of study and it is of considerable interest especially from the points of view of soliton theory and condensed matter physics. The underlying evolution equations are highly nonlinear and they give rise to many integrable cases both in 1 + 1 and 2 + 1 dimensions.

The standing example of an integrable spin system in 1 + 1 dimensions is the isotropic Heisenberg ferromagnetic spin (IHFS) chain [1–3] in its continuum limit. The underlying spin evolution equation is

\[ S_t = S \wedge S_{xx} \]

where \( S = (S_1, S_2, S_3) \), \( S^2 = 1 \). It is equivalent geometrically [2] and through gauge transformation [3] to the ubiquitous soliton possessing nonlinear Schrödinger equation [2]. Also the corresponding spin evolution equation itself is associated with a Lax pair and the inverse scattering transform analysis can be carried out for the system directly [4].

Besides the isotropic spin system, there exists a number of other spin systems in 1 + 1 dimensions which possess Lax pairs, gauge-equivalent counterparts and complete integrability property. These include the addition of anisotropy and magnetic field to the isotropic case leading to the spin evolution equation [5],

\[ S_t = S \wedge S_{xx} - 2A(S \cdot n)n + \mu B \]

where \( n = (0, 0, 1), B = (0, 0, B) \), \( A \) is the strength of anisotropy and \( B \) is the strength of the magnetic field along the \( z \)-direction.

* Corresponding author.
E-mail address: lakshman@cnld.bdu.ac.in (M. Lakshmanan).

0375-9601/$ – see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.physleta.2006.03.074
One more interesting integrable spin evolution equation is the bianisotropic equation studied by Sklyanin [6],

\[ S_t = S \wedge S_{xx} + S \wedge JS, \]

where \( J = \text{diag}(J_1, J_2, J_3) \) is the anisotropic matrix. The above type of spin equations are also special cases of the Landau–Lifshitz (L–L) equation deduced from phenomenological arguments [7]. Besides the aforementioned systems, various higher order and inhomogeneous integrable extensions also exist. For example, the spin evolution equation

\[ S_t = (\nu + \mu x)S \wedge S_{xx} + \mu S \wedge S_x - (\nu + \mu x)S_x - \gamma \left[ S_{xx} + \frac{3}{2} (S_x)^2 \cdot S \right], \]

is integrable [8]. Also an SO(3)-invariant deformed Heisenberg spin system has been shown to be integrable [9]

\[ S_t = S \wedge S_{xx} + \alpha S_x (S_x)^2, \]

and it is equivalent to the integrable derivative NLS equation [10]

\[ iq_t + q_{xx} + 2|q|^2 q - 2\alpha (|q|^2 q)_x = 0. \]

Many other integrable generalizations have also been obtained by Myrzakulov and coworkers [11–14]. All the above equations admit Lax pairs and satisfy Painlevé property. Naturally, the question arises as to what is the situation in 2 + 1 dimensions. The well-known integrable generalization of Eq. (1) in 2 + 1 dimensions are the Ishimori equation [15],

\[ S_t = S \wedge (S_{xx} + \sigma^2 S_{yy}) + \phi_x S_x + \phi_y S_y, \]

\[ \phi_{xx} - \sigma^2 \phi_{yy} = -2\alpha^2 S \cdot S_x \wedge S_y, \]

where \( S = (S_1, S_2, S_3), \ S^2 = 1, \) and \( \phi(x, y, t) \) is a scalar field and \( \sigma^2 = \pm 1, \) and the Myrzakulov M–I equation [13]

\[ S_t = \{S \wedge (S + uS)\}_x, \]

\[ u_x = -S \cdot S_x \wedge S_y, \]

where \( u(x, y, t) \) is a scalar field.

Again these equations possess Lax pairs and admit the Painlevé property. However, till today the integrability nature of the physically interesting (2 + 1)-dimensional direct generalization of (1), namely,

\[ S_t = S \wedge (S_{xx} + S_{yy}), \]

where \( S = (S_1, S_2, S_3), \ S^2 = 1, \) has not been studied, though the special case of Eq. (9) with circular symmetry

\[ S_r = S \wedge \left( S_{rr} + \frac{1}{r} S_r \right), \]

where \( r = \sqrt{x^2 + y^2}, \) is known to be integrable [16].

In this Letter, we wish to investigate the singularity structure property of the isotropic Heisenberg spin equation (9) in 2 + 1 dimensions and prove that it is of non-Painlevé type and so is expected to be nonintegrable, even though the special cases (1) and (10) are of Painlevé type and so integrable. The Painlevé analysis of the Heisenberg spin type equations is rather tricky as was shown for the case of (1 + 1)-dimensional system with anisotropy and transverse magnetic field [17], where the “Taylor” type expansion can lead to logarithmic singular manifolds leading to nonintegrability.

In order to investigate the Painlevé singularity structure underlying Eq. (9), we first rewrite it in terms of the complex stereographic field variable \( \omega(x, y, t) \) through the transformation

\[ S^+ = S_1 + iS_2 = \frac{2\omega}{1 + |\omega|^2}, \quad S_3 = \frac{1 - |\omega|^2}{1 + |\omega|^2}. \]

In terms of this variable, the equation of motion for the (2 + 1)-dimensional Heisenberg spin system can be written as

\[ (1 + \omega \omega^*)[i\omega_t + \omega_{xx} + \omega_{yy}] - 2\omega^* (\omega_x^2 + \omega_y^2) = 0, \]

and its complex conjugate. Representing \( \omega \rightarrow F \) and \( \omega^* \rightarrow G, \) Eq. (12) and its complex conjugate equation can be written as

\[ (1 + FG)(iF_t + F_{xx} + F_{yy}) - 2G(F_x^2 + F_y^2) = 0, \]

\[ (1 + FG)(-iG_t + G_{xx} + G_{yy}) - 2F(G_x^2 + G_y^2) = 0. \]
We carry out a Painlevé analysis of Eqs. (13) by seeking a generalized Laurent expansion for each dependent variable in the form

\[ F = F_0 \phi^p + \sum_j F_j \phi^{p+j}, \quad F_0 \neq 0, \quad (14a) \]

\[ G = G_0 \phi^q + \sum_j G_j \phi^{q+j}, \quad G_0 \neq 0, \quad (14b) \]

in the neighbourhood of the noncharacteristic singular manifold \( \phi(x, y, t) = 0, \phi_x, \phi_y \neq 0 \). The results are as follows.

2. Leading order behaviour

Looking at the dominant terms, we distinguish the following possibilities corresponding to (i) \( p \leq 0, q \leq 0 \), (ii) \( p \leq 0, q \geq 0 \), (iii) \( p \geq 0, q \leq 0 \).

2.1. Case (i): \( p \leq 0, q \leq 0 \)

Upon using the leading order solution \( F = F_0 \phi^p, G = G_0 \phi^q \), substituting it in Eqs. (13), and balancing the most dominant terms, we obtain

\[ F_0^2 G_0^4 [p(p - 1) - 2p^2 + 2p^2 + 2p^2 - 2q^2] \phi^{2p + q - 2} = 0, \quad (15a) \]

\[ G_0^2 [q(q - 1) - 2q^2 + 2q^2 + 2q^2 + 2q^2 - 2p^2] \phi^{2q + p - 2} = 0. \quad (15b) \]

From the above, we have the following three possibilities of leading order behaviour:

Branch (i): \( p = -1, q = -1 \), \( F_0, G_0 \) arbitrary.

Branch (ii): \( p = -1, q = 0 \), \( F_0, G_0 \) arbitrary.

Branch (iii): \( p = 0, q = -1 \), \( F_0, G_0 \) arbitrary.

In addition, there is a possibility that \( p = 0, q = 0 \), which requires a more detailed analysis, see below.

2.2. Case (ii): \( p \geq 0, q \geq 0 \)

\[ (F_0(p - 1)\phi^p - F_0^2 G_0(p + 1)\phi^{2p+q-2}) (\phi_x^2 + \phi_y^2) = 0, \quad (16a) \]

\[ (G_0(q - 1)\phi^q - F_0 G_0^2 (q + 1)\phi^{2q+p-2}) (\phi_x^2 + \phi_y^2) = 0. \quad (16b) \]

From Eqs. (16a), (16b) we obtain \( p + q = 0 \) and \( F_0 G_0 = \frac{-1}{p+1} \) from Eq. (16a), and \( F_0 G_0 = \frac{-1}{q+1} \) from Eq. (16b), respectively. We also obtain the same result for the case \( p \geq 0, q \leq 0 \). This suggests that \( p = q = 0 \) is the only possibility here. Looking at this case more carefully, by using Eqs. (14) in (13), we obtain the following.

Branch (iv): \( p = 0, q = 0 \):

\[ (1 + F_0 G_0) [t(F_{0u} + F_{1u}\phi_x) + F_{0xx} + 2F_{1x}\phi_x + F_{1xx} + 2F_{2x}\phi_x^2 + F_{0yy} + 2F_{1y}\phi_y + F_{1yy} + 2F_{2y}\phi_y^2] \]

\[ - 2G_0 [F_{0u}^2 + F_{0y}^2 + F_1 (\phi_x^2 + \phi_y^2) + 2F_1 (F_{0u}\phi_x + F_{0y}\phi_y)] = 0, \quad (17a) \]

\[ (1 + F_0 G_0) [-i(G_{0u} + G_1\phi_x) + G_{0xx} + 2G_{1x}\phi_x + G_1\phi_{xx} + 2G_2\phi_x^2 + G_{0yy} + 2G_{1y}\phi_y + G_1\phi_{yy} + 2G_2\phi_y^2] \]

\[ - 2G_0 [G_{0u}^2 + G_{0y}^2 + G_1 (\phi_x^2 + \phi_y^2) + 2G_1 (G_{0u}\phi_x + G_{0y}\phi_y)] = 0. \quad (17b) \]

We consider two separate cases of the manifold: (i) \( F_0 G_0 \neq -1 \), (ii) \( F_0 G_0 = -1 \). In the former case, from Eqs. (17a) and (17b), the coefficient functions \( F_2 \) and \( G_2 \) can be expressed in terms of \( F_0, G_0, F_1 \) and \( G_1 \) leaving the later functions arbitrary. For the case \( (1 + F_0 G_0) = 0 \), we assume for simplicity the Kruskal's reduced manifold \( \phi(x, y, t) = x + \psi(y, t) = 0 \). Using this in (17), we find two sets of solutions.

2.2.1. Case 1

\[ F_1 = \frac{i F_{0y}}{(1 - i \psi_y)}, \quad (18a) \]

\[ G_1 = \frac{i G_{0y}}{(1 - i \psi_y)}. \quad (18b) \]
2.2.2. Case 2

\[ F_1 = \frac{i F_0 \gamma}{(1 - i \gamma)^4}, \quad (19a) \]
\[ G_1 = \frac{-i G_0 \gamma}{(1 + i \gamma)}, \quad (19b) \]

3. Resonances

To find the resonances, that is the powers of the Laurent series (14) at which arbitrary functions enter, for branches (i), (ii) and (iii) we expand

\[ F = F_0 \phi^p + \cdots + \alpha \phi^{p+r}, \quad (20a) \]
\[ G = G_0 \phi^q + \cdots + \beta \phi^{q+r} \]

\((\alpha, \beta \text{ not both zero})\) and substitute in Eqs. (13) containing the dominant terms alone to fix the values of \(r\). Detailed calculation leads to the following results:

Branch (i): \(p = -1, q = -1\): \(r = -1, 0, 0, 0, 1\),
Branch (ii): \(p = -1, q = 0\): \(r = -1, 0, 0, 1\),
Branch (iii): \(p = 0, q = -1\): \(r = -1, 0, 0, 1\).

For the case of branch (iv), \(p = 0, q = 0\), we proceed with the expansion

\[ F = F_0 + F_1 \phi + \cdots + F_r \phi^r, \quad (21a) \]
\[ G = G_0 + G_1 \phi + \cdots + G_r \phi^r \]

and substitute them into Eqs. (13) and collect the coefficients of \(\phi^{r-2}\) and \(\phi^{r-1}\) (after making use of Eqs. (17)).

3.1. Coefficients of \(\phi^{r-2}\)

When \((1 + F_0 G_0) \neq 0\), we have the condition

\[ (1 + F_0 G_0)(1 + \psi^2)(r - 1)F_r = 0, \quad (22a) \]
\[ (1 + F_0 G_0)(1 + \psi^2)(r - 1)G_r = 0. \quad (22b) \]

It follows that the resonance values are \(r = 0, 0, 1, 1\). For the case \((1 + F_0 G_0) = 0\), the conditions become identities.

3.2. Coefficients of \(\phi^{r-1}\)

When \((1 + F_0 G_0) \neq 0\), the resulting condition is in conformity with the resonance values \(r = 0, 0, 1, 1\) noted above. When \((1 + F_0 G_0) = 0\), we have

\[ r[(F_0 G_1 + F_1 G_0)(1 + \psi^2)(r - 1) - 4(G_0 F_1(1 + \psi^2) + G_0 F_0 \gamma \psi)]F_r = 0, \quad (23a) \]
\[ r[(F_0 G_1 + F_1 G_0)(1 + \psi^2)(r - 1) - 4(F_0 G_1(1 + \psi^2) + F_0 G_0 \gamma \psi)]G_r = 0. \quad (23b) \]

These equations reduce to the following forms for the cases 1 and 2, respectively.

3.2.1. Case 1

\[ 4ir G_0 F_0 \gamma = 0, \quad (24a) \]
\[ 4ir F_0 G_0 \gamma = 0. \quad (24b) \]

For this case, the resonance values are 0, 0.

3.2.2. Case 2

In this case, we have

\[ r[F_0 G_0 (r - 5) - F_0 G_0 \gamma (r - 1)] = 0, \quad (25a) \]
\[ r[G_0 F_0 (r - 5) - G_0 F_0 \gamma (r - 1)] = 0. \quad (25b) \]

Since \((1 + F_0 G_0) = 0\), \(F_0 G_0 + F_0 G_0 \gamma = 0\) and, consequently, from Eqs. (25), we find the resonance values to be \(r = 0, 0, 3, 3\).
4. Analysis of the Laurent expansion for arbitrary functions

In the case of the branches (i), (ii) and (iii) we have verified that the resonance conditions are indeed satisfied in the sense that apart from the arbitrariness of the singular manifold, required number of arbitrary functions occur at $r = 0$ and $r = 1$ in the Laurent series and also that no logarithmic singularity can occur in the leading order for the branch (i). We now carry out the calculations for the analysis of the Taylor like expansion corresponding to the branch (iv) (again in terms of the Kruskal’s reduced manifold $x + \psi (y, t) = 0$) by writing

$$F(x, y, t) = F_0(y, t) + F_1(y, t)\psi + F_2(y, t)\psi^2 + F_3(y, t)\psi^3 + \cdots,$$

$$G(x, y, t) = G_0(y, t) + G_1(y, t)\psi + G_2(y, t)\psi^2 + G_3(y, t)\psi^3 + \cdots.$$  

Substituting the above into Eq. (13), and collecting the coefficients of different powers of $\psi$ we obtain the following results.

4.1. Zeroth order in $\psi$

(a) For the manifold $F_0 G_0 \neq -1$, the Taylor-like series (26) can be easily shown not to admit any movable singular manifold, where four arbitrary functions can enter into the series (while the manifold $\phi$ can be absorbed into $F_1$ or $G_1$). This is in conformity with the resonance values $r = 0, 0, 1, 1$ pointed out after Eq. (22).

(b) For the manifold $F_0 G_0 = -1$, one can obtain two sets of the expression for $F_1$ and $G_1$ which are the same as cases 1 and 2 given by Eqs. (18) and (19), respectively. We will consider each of the cases separately.

4.2. Case 1—Eqs. (18)

4.2.1. First order in $\psi$

With $(1 + F_0 G_0) = 0$, we have

$$- 4F_0 \left[ F_0, F_1 \psi + 2F_1, F_2 \psi^2 + F_2 \psi F_2 \psi \right],$$

$$- 4G_0 \left[ F_0, F_1 \psi + 2F_1, F_2 \psi^2 + F_2 \psi F_2 \psi \right] - 2F_1 \left[ F_0^2 \psi + F_0 F_0 \psi \right] = 0.$$  

Using the results of the previous order for $F_1$ and $G_1$, we obtain

$$F_2 = \frac{-F_0, \psi - F_0 \psi, \psi}{2(1 - \psi, \psi)^2},$$

$$G_2 = \frac{-G_0, \psi - G_0 \psi, \psi}{2(1 - \psi, \psi)^2}.$$  

4.2.2. Second order in $\psi$

Here we obtain $F_3$ and $G_3$ as

$$F_3 = \frac{i}{12G_0 F_0} \left[ 2G_0 \left[ 4F_2^2 + (F_2, F_2 \psi^2 + 4F_1, F_2 \psi) + (F_0, F_0 \psi + F_1, F_2 \psi) \right] + 2G_1 \left[ 4F_1, F_2 + 2(F_0, F_1 \psi + F_1, F_2 \psi) \right] \right],$$

$$G_3 = \frac{i}{12G_0 F_0} \left[ 2G_0 \left[ 4G_2^2 + (G_2, F_2 \psi^2 + 4G_1, G_2 \psi) + (G_0, G_0 \psi + G_1, G_2 \psi) \right] + 2G_1 \left[ 4G_1, G_2 + 2(G_0, G_1 \psi + G_1, G_2 \psi) \right] \right].$$  

In a similar way, one can compute $(F_4, G_4), (F_5, G_5), \text{etc}$. No indeterminate coefficients appear in the series (at least up to the order deduced) and thus no possibility for singularity arises. We also note that either $F_0$ or $G_0$ and $\psi (y, t)$ are the only arbitrary functions in the Taylor like series (26) in conformity with the resonance values $r = 0, 0$. 

4.3. Case 2—Eqs. (19)

4.3.1. First order in $\phi$

From Eq. (13) we obtain

$$F_2 = \frac{i}{4F_0G_0} \left[ -i(F_0G_0 - F_0G_0) \left[ \left( F_0 + i\psi G_0 - F_0\psi G_0 \right) + F_{0yy} + \frac{2iF_{0yy}}{1 - i\psi G_0} \psi_{yy} - \frac{2F_{0yy}}{(1 - i\psi G_0)^2} \psi_{yy} + \frac{iF_{0yy}}{1 - i\psi G_0} \psi_{yy} \right] \right]$$

$$+ 4G_0 \left[ \frac{F_0}{1 - i\psi G_0} \left( iF_{0yy} - \frac{F_{0yy}}{1 - i\psi G_0} \right) \right]$$

(30a)

$$G_2 = \frac{-i}{4F_0G_0} \left[ -i(F_0G_0 - F_0G_0) \left[ -i(\psi_0 - iG_0) F_{0yy} + \frac{2iG_{0yy}}{(1 - i\psi G_0)^2} \psi_{yy} \right] - \frac{iG_{0yy}}{1 - i\psi G_0} \psi_{yy} \right]$$

$$+ 4F_0 \left[ \frac{G_{0yy}}{1 + i\psi G_0} - \frac{G_{0yy}}{(1 + i\psi G_0)^2} \psi_{yy} \right]$$

(30b)

4.3.2. Second order in $\phi$

$$2G_0 \left[ 4F_0^2 + \left( F_0^2 + 4F_1^2 \psi_0^2 + 4F_2^2 \psi_{0yy} \right) \right]$$

$$+ 2F_0 \left[ 4F_0F_2 + 2(F_0F_1 + 2F_0F_2 \psi G_0 + F_1F_2 \psi G_0 + 2F_2 \psi_{0yy} + F_1 \psi_{0yy} + 2F_2 \psi_{0yy} \right]$$

$$- (F_0G_1 + F_1G_0 + F_2G_0) \left[ \left( F_0G_0 + F_1 \psi G_0 + 2F_2 \psi_{0yy} + F_1 \psi_{0yy} + 2F_2 \psi_{0yy} \right) \right]$$

$$- (F_0G_1 + F_1G_0) \left[ \left( F_0G_0 + F_1 \psi G_0 + 2F_2 \psi_{0yy} + F_1 \psi_{0yy} + 2F_2 \psi_{0yy} \right) \right] = 0.$$  

(31a)

$$2F_0 \left[ 4G_0^2 + \left( G_0^2 + 4G_1^2 \psi_0^2 + 4G_2^2 \psi_{0yy} \right) \right]$$

$$+ 2F_0 \left[ 4G_0G_2 + 2(G_0G_1 + 2G_0G_2 \psi G_0 + G_1G_2 \psi_0 G_0 + 2G_2 \psi_{0yy} + G_1G_2 \psi_{0yy} + 2G_2 \psi_{0yy} \right]$$

$$- (F_0G_2 + G_1G_0) \left[ \left( -iG_0 + G_1 \psi G_0 + 2G_2 + (G_0G_2 + G_1G_2 \psi_0 G_0 + 2G_2 \psi_{0yy} + G_1G_2 \psi_{0yy} + 2G_2 \psi_{0yy} \right) \right]$$

$$- (F_0G_1 + G_1G_0) \left[ \left( -iG_0 + G_1 \psi G_0 + 2G_2 + (G_0G_2 + G_1G_2 \psi_0 G_0 + 2G_2 \psi_{0yy} + G_1G_2 \psi_{0yy} + 2G_2 \psi_{0yy} \right) \right] = 0.$$  

(31b)

It may be noted that in this order both $F_3$ and $G_3$ are absent indicating that they are arbitrary functions corresponding to the resonance values $r = 3, 3$. Note that from Eqs. (30) and the relation $(1 + F_0G_0) = 0$, two of the three functions $F_0$, $G_0$ and $\psi$ are arbitrary corresponding to the values $r = 0, 0$. However, simplifying the above set of equations (31) by using the expressions obtained for the coefficients $F_1$, $F_2$, $G_1$, $G_2$ in terms of $F_0$, $G_0$ and $\psi$ (see Eqs. (18), (19), (30)), we find that Eqs. (31) reduce to two nontrivial conditions which are incompatible, unless the $\psi$-dependence is dropped (corresponding (1 + 1)-dimensional system (1)) or one carries out the analysis with the radial variable $r = \sqrt{x^2 + y^2}$ (see Eq. (10)). As a consequence logarithmic singularity appears in the series expansion (26). Consequently, the (2 + 1)-dimensional continuum isotropic Heisenberg spin system (12) and so (9) does not satisfy the Painlevé property [18] and is expected to be nonintegrable.

One can also carry out the analysis with the general manifold $\phi(x, y, t)$ instead of the Kruskal’s reduced manifold and one can check that the same conclusion results in here also.

To conclude, in this Letter we have shown that the physically important (2 + 1)-dimensional isotropic Heisenberg continuum spin system (9) does not admit the Painleve property and so it belongs to the class of nonintegrable nonlinear evolution equations. It will be of considerable interest to investigate the underlying spatiotemporal structures of such a nonlinear evolution equation in detail.

Acknowledgements

The work of C.S. and M.L. forms part of a Department of Science and Technology, Government of India, sponsored research project.

References

Trilinearization and localized coherent structures and periodic solutions for the (2 + 1) dimensional K-dV and NNV equations

C. Senthil Kumar a, R. Radha b, M. Lakshmanan a,*

a Centre for Nonlinear Dynamics, School of Physics, Bharathidasan University, Tiruchirapalli 620 024, India
b Department of Physics, Government College for Women, Kumbakonam 612 001, India

Accepted 22 January 2007

Communicated by Prof. L. Marek-Crnjac

Abstract

In this paper, using a novel approach involving the truncated Laurent expansion in the Painlevé analysis of the (2 + 1) dimensional K-dV equation, we have trilinearized the evolution equation and obtained rather general classes of solutions in terms of arbitrary functions. The highlight of this method is that it allows us to construct generalized periodic structures corresponding to different manifolds in terms of Jacobian elliptic functions, and the exponentially decaying dromions turn out to be special cases of these solutions. We have also constructed multi-elliptic function solutions and multi-dromions and analysed their interactions. The analysis is also extended to the case of generalized Nizhnik–Novikov–Veselov (NNV) equation, which is also trilinearized and general class of solutions obtained.

© 2007 Elsevier Ltd. All rights reserved.

1. Introduction

The recent interest in the investigation of integrable models in (2 + 1) dimensions can be attributed to the identification of dromions in the Davey–Stewartson I equation [1,2]. These dromions which originate at the crosspoint of the intersection of two nonparallel ghost solitons decay exponentially in all directions and are driven by lower dimensional boundaries or velocity potentials (arbitrary functions) in the system. It must be mentioned that the presence of these so-called boundaries enriches the structure of (2 + 1) dimensional integrable models leading to the formation of localized solutions like dromions, lumps, breathers, positons, etc. [2,3]. Thus, a judicious harnessing of these lower dimensional arbitrary functions [4–9] of space and time may give rise to new spatio-temporal patterns in higher dimensions besides throwing more light on their integrability.

In this direction, we have recently identified a simple procedure to solve the (2 + 1) dimensional systems, namely the Painlevé truncation approach (PTA) [7]. This approach converts the given evolution equation into a multilinear (in

* Corresponding author. Tel./fax: +91 431 2407093.
E-mail address: lakshman@cnld.bdu.ac.in (M. Lakshmanan).

0960-0779/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.chaos.2007.01.066
general) equation in terms of the noncharacteristic manifold. This multilinear equation can be solved in terms of lower dimensional arbitrary functions of space and time. Using this approach, one can generate various generalized periodic and localized solutions. This was earlier demonstrated for the case of long wave–short wave resonance interaction equation in \((2 + 1)\) dimensions by converting it into a trilinear equation \([7]\). In this paper, we wish to show that through the PTA the \((2 + 1)\) dimensional K-dV equation, equation can be trilinearized. A four parameter special class of solutions involving three arbitrary functions \(g(y), h(y)\) and \(f(x, t)\) in the indicated variables naturally arises for this trilinear equation. This in turn leads to the universal form of solution obtained earlier by Tang et al. \([5]\) using a variable separation approach as a special case. However, there exists a possibility of obtaining more general solution to the trilinear equation, which remains to be explored. A large class of Jacobi elliptic function periodic solutions, multidromion and bound state solutions are also obtained. It is also shown that the procedure can be extended to the case of Nizhnik–Novikov–Veselov (NNV) equation as well.

The plan of the paper is as follows. In Section 2, we show how the \((2 + 1)\) dimensional K-dV equation can be trilinearized using the PTA and the solution obtained in terms of three arbitrary functions and four arbitrary parameters. Sections 3 and 4 contain two broad classes of solutions of the \((2 + 1)\) dimensional K-dV equation, both periodic and exponentially localized ones, through judicious choices of the arbitrary functions. In Section 5, we point out how the generalized NNV equation can be trilinearized through the PTA and how various periodic and localized solutions can be obtained. Finally in Section 6, we summarize our results. Appendix A contains the one dromion solution of the \((2 + 1)\) dimensional K-dV equation obtained through Hirota bilinearization approach for comparison.

2. The \((2 + 1)\) dimensional K-dV equation and construction of solutions

The \((2 + 1)\) dimensional K-dV equation introduced by Boiti et al. \([10]\) has the form
\[
q_x + q_{txt} = 3(q q_x q_x x - (1)
\]
This nonlocal equation (1) reduces to the K-dV equation
\[
t x = y.
\]
Introducing a potential function \(v(x, y, t)\) defined by
\[
\mathcal{L}, = q_x.
\]
Eq. (1) can be converted into a set of coupled equations of the form
\[
q_x + q_{txt} = 3(q v_x x)
\]
\[
v_x = q_x.
\]
The nature of (3b) permits the presence of an arbitrary function \(v'(x, t)\) as
\[
v(x, y, t) = \int_{-\infty}^{t} q_x dy' + v'(x, t),
\]
Expanding the physical field \(q\) and the potential \(v\) in the form of a Laurent series in the neighbourhood of the noncharacteristic manifold \(\phi(x, y, t) = 0, (\phi_x \neq 0, \phi_y \neq 0, \phi_z \neq 0)\) and utilising the results of the Painlevé analysis \([11]\), we obtain the following Bäcklund transformation by truncating the Laurent series at the constant level term
\[
q = q_0 \phi^{-2} + q_1 \phi^{-1} + q_2,
\]
\[
v = v_0 \phi^{-2} + v_1 \phi^{-1} + v_2,
\]
where \((q, v)\) and \((q_2, v_2)\) are different sets of solutions of the \((2 + 1)\) dimensional K-dV equation. In Ref. \([4]\) certain basic localized dromion solutions have been obtained using (3) and Hirota bilinearization. Here, we make use of the full power of Eq. (5) by treating it as a variable transformation, and obtain rather general classes of solutions. This new approach leads to a wide class of solutions not only for Eq. (1) but for other \((2 + 1)\) dimensional systems as well \([7]\).

Considering now a vacuum solution of the form
\[
q_2 = 0, \quad v_2 = v_2(x, t)
\]
and substituting (6) with (5) in Eq. (3), we obtain the following set of equations by equating the coefficients of \((\phi^{-2}, \phi^{-3})\) to zero,
\[
-24q_0 \phi_x^2 + 12q_0 v_0 \phi_x = 0, \quad (7a)
\]
\[
v_0 \phi_y + q_0 \phi_x = 0. \quad (7b)
\]
Solving the above system of equations, we obtain

\[ q_0 = 2\phi_x\phi_y, \quad v_0 = 2\phi_y^2. \]  

Again, collecting the coefficients of \((\phi^{-4}, \phi^{-2})\), we have

\[
18q_0\phi_x^2 + 18q_y\phi_x\phi_y - 6q_0\phi_x - 3(q_0v_0 - 2q_0v_1\phi_x - q_1\phi_yv_0) - 3(v_0q_0 - 2v_0q_1\phi_x - v_1\phi_yv_0) = 0, \\
v_0 - v_1\phi_y - (q_0 - q_1\phi_x) = 0.
\]

Solving the above system of equations, we obtain

\[ q_1 = -2\phi_{xy}, \quad v_1 = -2\phi_{yy}. \]

Next, collecting the coefficients of \((\phi^{-3}, \phi^{-1})\), we have

\[
-2q_0\phi_1 + (-6q_0\phi_x^2 - 6q_0\phi_x - 2q_0\phi_{xxx} + 6q_1\phi_x^2 + 6q_1\phi_y\phi_x) - 3(q_0v_1 - 2q_0\phi_xv_2 + q_1\phi_yv_0 - q_1v_1\phi_x) \\
-3(q_0v_1 + q_1v_0 - q_1v_1\phi_x) = 0, \\
v_0 - q_1 = 0.
\]

Here (11b) is an identity as may be verified using (10). Solving (11a) by using (8) and (10), we get the form of \(v_2\) as

\[ v_2 = \frac{\phi_1 + \phi_{xxx} - \phi_y\phi_{xy}}{3\phi_y} + \frac{\phi_1 + \phi_{xxx} - \phi_y\phi_{xy}}{\phi_y}. \]

Now, we collect the coefficients of next order \((\phi^{-2}, \phi^0)\) to obtain

\[
(q_0 - q_1\phi_x) + 3(q_0\phi_{xxx} - q_1\phi_y\phi_x - 3q_0\phi_x) - 3(q_0\phi_xv_2 + q_1v_1\phi_x) - 3(q_0\phi_{xx} + q_1v_1\phi_{xx}) = 0, \\
v_0 - q_1 = 0.
\]

On the other hand, substituting the expression for the quantities \(q_0, v_0, q_1, v_1\) and \(v_2\) in Eq. (13a), we obtain the trilinear equation

\[ \phi_x(\phi_y\phi_{xy} - \phi_{xx}\phi_y) + \phi_x(\phi_y\phi_{xy} - \phi_{xx}\phi_y) = 0. \]

Substituting (12) in (13b), we obtain

\[
(\phi_{xy} + \phi_{xxx})(\phi_x\phi_y^2 - (\phi_x + \phi_{xx})\phi_y \phi_y^2 + 3(\phi_x \phi_{xy} - \phi_{xx}\phi_y)\phi_y \phi_x \phi_y - 3(\phi_x \phi_{xy} - \phi_{xx}\phi_y)\phi_x \phi_y \\
-3(\phi_x \phi_{xy} - \phi_{xx}\phi_y)\phi_x \phi_y \phi_y = 0.
\]

Using Eq. (14) into (15), the later reduces to the form

\[ (\phi_{xy} + \phi_{xxx})(\phi_x \phi_y - (\phi_x + \phi_{xx})\phi_y \phi_y - 3(\phi_x \phi_{xy} - \phi_{xx}\phi_y)\phi_y \phi_y) = 0, \]

which is again in a trilinear form. Thus Eqs. (14) and (16) may be considered as the equivalent trilinear forms of Eqs. (3).

One can immediately observe that the set of trilinear equations (14) and (16) admit a set of two arbitrary functions and their products as solutions

\[ \phi = f(x, t), \quad \phi = g(y), \]

and

\[ \phi = f(x, t)g(y), \]

where \(f\) and \(g\) are arbitrary functions in the indicated variables. In fact a more general solution involving three arbitrary functions and four arbitrary constant parameters can be easily identified:

\[ \phi(x, y, t) = c_1 + c_2f(x, t) + c_3h(y) + c_4f(x, t)g(y), \]

where \(c_1, c_2, c_3, c_4\) are arbitrary constants.
where \( h(y) \) and \( g(y) \) are in general different arbitrary functions of \( y \). Here \( c_1, c_2, c_3, \) and \( c_4 \) are arbitrary parameters. The problem of identifying even more general solution to (14) and (16) remains to be investigated, while we will concentrate on the special solution (18) in this paper.

Collecting the coefficients \( \phi^{-1}, \phi \), we obtain

\[
q_t + q_{xxx} - 3q_t v_2 - 3q_v v_2 = 0.
\]

(19)

It can be checked that Eq. (1) is compatible with the earlier results.

Now substituting the above form (18) for the manifold \( \phi(x, y, t) \) into the truncated Painlevé series (5) for the functions \( q(x, y, t) \) and \( v(x, y, t) \), along with the expressions for the coefficient functions \( q_0, v_0, q_1, v_1 \) and \( v_2 \) given above, we finally obtain the solution to Eqs. (3) as

\[
q(x, y, t) = \frac{2f_s[(c_2 + c_4)h - (c_1 + c_3)g]}{[c_1 + c_2f(x, t) + c_3h(y) + c_4f(x, t)g(y)]^2},
\]

(20)

\[
v(x, y, t) = \frac{2(c_2 + c_4)h v_2}{[c_1 + c_2f(x, t) + c_3h(y) + c_4f(x, t)g(y)]^2} - \frac{2(c_2 + c_4)f_{xx}}{f_x} + \frac{f_t + f_{xxx}}{3f_x}.
\]

(21)

It may be noted that the expressions given in (20) and (21) coincide exactly with the forms obtained by Tang et al. [5] using the method of variable separation for the special case \( g(y) = h(y) \) in Eq. (18). Another special case \( c_1 = c_2 = 0, c_3 = c_4 = 1 \) was identified by Peng recently [9]. Also, in our case it is clear that finding any solution to (14) and (16) which is more general than (18) will lead to more general solution than (20) and (21). This problem is being investigated further. In the following, we will investigate the nature of solutions (20) and (21) corresponding to periodic and localized solutions. For this purpose, we shall consider the special cases (i) \( c_4 = 0 \) and (ii) \( c_2 = c_3 = 0 \) in (18) or (20) and (21), corresponding to the sum and product of arbitrary functions, respectively, and investigate the nature of periodic and localized solutions. It may be noted that there is no difficulty in proceeding with the general form (20) and (21) also except for the fact that the expression will be more lengthy which we desist from presenting here.

### 3. Periodic and localized solutions corresponding to sum of arbitrary functions

In this section, we will assume \( c_1 = c, c_2 = c_3 = 1, c_4 = 0 \) in (18) or (20) and (21). Consequently, we have

\[
q = \frac{2f_s h_x}{(f + h + c)^2},
\]

(22a)

\[
v = \frac{2f_s^2}{(f + h + c)^2} - \frac{2f_{xx}}{(f + h + c)^2} + \frac{f_t + f_{xxx}}{3f_x}.
\]

(22b)

#### 3.1. Harnessing of arbitrary functions and novel solutions of \((2 + 1)\) dimensional K-dV equation

Let us now choose the arbitrary functions \( f(x, t) \) and \( h(y) \) to be Jacobian elliptic functions, namely \( \text{sn} \) or \( \text{cn} \) functions. The motivation behind this choice of arbitrary function stems from the fact that the limiting forms of these functions happen to be localized functions. Hence, a choice of \( \text{cn} \) and \( \text{sn} \) functions can yield periodic solutions which are more generalized than exponentially localized solutions (dromions). We choose, for example,

\[
f = \alpha \text{sn}(ax + c_1 t + d_1; m_1), \quad h = \beta \text{sn}(by + d_2; m_2)
\]

(23)

so that

\[
q(x, y, t) = \frac{2\alpha \beta ab \text{cn}(u_1; m_1) \text{dn}(u_1; m_1) \text{cn}(u_2; m_2) \text{dn}(u_2; m_2)}{(c + \alpha \text{sn}(u_1; m_1) + \beta \text{sn}(u_2; m_2))^2},
\]

(24)

where \( u_1 = ax + c_1 t + d_1 \) and \( u_2 = by + d_2 \). In Eqs. (23) and (24), the quantities \( m_1 \) and \( m_2 \) are moduli of the respective Jacobian elliptic functions while \( \alpha, \beta, a, b, c, c_1, d_1 \) and \( d_2 \) are arbitrary constants. We choose the constant parameters \( c, \alpha \) and \( \beta \) such that \( |c| > |a + b| \) for nonsingular solution. The profile of the above solution for the parametric choice \( \alpha = \beta = 1, a = 0.5, b = 0.4, c = -4, c_1 = -1, d_1 = d_2 = 0, m_1 = 0.2, m_2 = 0.3 \) at time \( t = 0 \) is shown in Fig. 1a. Note that the periodic wave travels along the \( x \)-direction only.
3.1.1. (1,1) Dromion solution

As a limiting case of the periodic solution given by Eq. (24), when \(m_1, m_2 \to 1\), the above solution degenerates into an exponentially localized solution (dromion). Noting that \(cn(u; 1) = dn(u; 1) = \sech(u)\) and \(sn(u; 1) = \tanh(u)\), the limiting form corresponding to (1,1) dromion takes the expression

\[
g = \frac{2ab\beta \sech^2(ax + c_1t + d_1) \sech^2(by + d_2)}{(c + \alpha \tanh(ax + c_1t + d_1) + \beta \tanh(by + d_2))^2} (|c| > |a + \beta|) \tag{25}
\]

and the variable \(v\) then takes the form (using expression (22b))

\[
v = \frac{2a^2c^2 \sech^4(ax + c_1t + d_1)}{(c + \alpha \tanh(ax + c_1t + d_1) + \beta \tanh(by + d_2))^2} + \frac{4a^2c^2 \sech^2(ax + c_1t + d_1) \tanh(ax + c_1t + d_1)}{(c + \alpha \tanh(ax + c_1t + d_1) + \beta \tanh(by + d_2))^2} + \frac{1}{3a} \\
\times [c_1 + 4a^2 \tanh^2(ax + c_1t + d_1) - 2a^3 \sech^2(ax + c_1t + d_1)]. \tag{26}
\]

Schematic form of the (1,1) dromion for the parametric choice \(\alpha = \beta = 1, a = 0.5, b = 0.4, c = -4, c_1 = -1, d_1 = d_2 = 0\) at time \(t = 0\) is shown in Fig. 1b. Again note that the dromion travels along the x-direction. One can obtain the (1,1) dromion given by Radha and Lakshmanan in Ref. [4] by fixing the parameters \(\alpha, \beta, a, b, c_1, d_1, d_2\) of Eq. (25) suitably (see also Appendix A).

3.1.2. More general periodic solution and (2,1) dromion

Next we obtain a more general periodic solution by choosing further general forms for the arbitrary functions. As an example, we choose
\begin{align}
\frac{f}{h} &= a_1 \sin(a_1 x + c_1 t + d_1; m_1) + a_2 \sin(a_2 x + c_2 t + d_2; m_2), \\
\frac{f}{h} &= \beta \sin(by + d_3; m_3),
\end{align}

where \( b, \beta, a_1, a_2, c, \) and \( d_j \) are arbitrary constants \((i = 1, 2; j = 1, 2, 3)\) and \( m_j \)'s are modulus parameters. Then

\begin{align}
q &= \frac{q_1}{q_2},
\end{align}

where \( q_1 = 2[(a_1 a_2 \cos(u_1; m_1) + a_2 a_2 \cos(u_2; m_2)) \frac{\beta b \cos(u_3; m_3)}{\cos(u_3; m_3)}, q_2 = (c + c_1 \tan(t) + a_2 \tan(u_2; m_2) + \beta \tan(u_3; m_3)),
\]

\( u_1 = a_1 x + c_1 t + d_1, u_2 = a_2 x + c_2 t + d_2 \) and \( u_3 = by + d_3 \) with corresponding expressions for \( q(x, y, t) \). We choose the constant \( c \) such that \( |c| > |a_1 + a_2 + \beta| \) for nonsingular solutions. The profile of the above solution for the parametric choice \( a_1 = a_2 = \beta = 1, a_1 = 0.5, a_2 = 0.5, b = 0.4, c = -4, c_1 = -1, c_2 = -2, d_1 = d_2 = d_3 = 0, m_1 = 0.2, m_2 = 0.3, m_3 = 0.4 \) at time \( t = 0 \) is shown in Fig. 2a. As \( m_1, m_2, m_3 \rightarrow 1 \), the above solution given by Eq. (28) degenerates into a \((2,1)\) dromion solution given by

\begin{align}
q &= \frac{2[a_1 a_2 \sech^2(u_1 + a_2 a_2 \sech^2(u_2) \sech^2(u_3))]}{c \frac{(a_1 a_2 + a_2 a_2 \tanh(u_1 + a_2 a_2 \tanh(u_2) + \beta \tanh(u_3))}(c > |a_1 + a_2 + \beta|),
\end{align}

where \( u_1 = a_1 x + c_1 t + d_1, u_2 = a_2 x + c_2 t + d_2 \) and \( u_3 = by + d_3 \). The solution is plotted for the parametric choice \( a_1 = a_2 = \beta = 1, a_1 = 0.5, a_2 = 0.5, b = 0.8, c = -4, c_1 = -1, c_2 = -2.5, d_1 = d_2 = d_3 = 0, \) for various values of \( t \) in Figs. 2b–d in order to bring out the dromion interaction clearly.

3.1.3. Asymptotic analysis for \((2,1)\) dromion solution

Since both the dromions in Figs. 2b–d corresponding to the solution (29) are travelling along the \( x \)-direction, following one another, it is enough to do this analysis for \( y = 0 \) (and \( d_1 = 0 \) so that \( u_1 = 0 \)). This analysis holds good for any other value of \( y \) and \( d_1 \neq 0 \). This restriction corresponds to the cross section of dromions, which are essentially solitons. We analyse the limits \( t \rightarrow -\infty \) and \( t \rightarrow +\infty \) separately so as to understand the interaction of dromions centered around \( u_1 \approx 0 \) or \( u_2 \approx 0 \). Without loss of generality, let us assume \( c_1 > c_2 \) and \( a_1 < a_2 \). Then, we find in the limit \( t \rightarrow -\infty \), \( u_1 = a_1 x + c_1 t + d_1 \) and \( u_2 = a_2 x + c_2 t + d_2 \) take the following limiting values.

![Fig. 2. (a) Elliptic function solution (28) for \( q(x,y,t) \), (b)-(d) (2,1) dromion solution (29) for \( q(x,y,t) \) and its interaction at time units
(b) \( t = -4 \), (c) \( t = 0 \) and (d) \( t = 4 \).](https://example.com/fig2.png)
(1) As \( t \to -\infty \):
\[
\begin{align*}
&u_1 \approx 0, \quad u_2 \to +\infty, \\
&u_2 \approx 0, \quad u_1 \to -\infty.
\end{align*}
\]
(2) As \( t \to +\infty \):
\[
\begin{align*}
&u_1 \approx 0, \quad u_2 \to -\infty, \\
&u_2 \approx 0, \quad u_1 \to +\infty.
\end{align*}
\]
1. Before interaction (as \( t \to -\infty \)):
For \( a_1 \approx 0, a_2 \to +\infty \) and \( \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1 \), the (2,1) dromion solution (29) becomes (soliton solution corresponding to dromion 1)
\[
q = \frac{2a_1b}{c(c+2)} \text{sech}^2(u_1 + \delta_1), \quad \delta_1 = \sqrt{\frac{c+2}{c}}.
\]
For \( u_2 \approx 0, u_1 \to -\infty, \) (29) becomes (soliton solution corresponding to dromion 2)
\[
q = \frac{2a_2b}{c(c-2)} \text{sech}^2(u_2 + \delta_2), \quad \delta_2 = \sqrt{\frac{c}{c-2}}.
\]
2. After interaction (as \( t \to +\infty \)):
For \( a_1 \approx 0, a_2 \to +\infty, \) (29) becomes (soliton solution corresponding to dromion 1)
\[
q = \frac{2a_1b}{c(c-2)} \text{sech}^2(u_1 + \delta_2).
\]
For \( u_2 \approx 0, u_1 \to +\infty, \) (29) becomes (soliton solution corresponding to dromion 2)
\[
q = \frac{2a_2b}{c(c+2)} \text{sech}^2(u_2 + \delta_1).
\]

The above results can be interpreted in the following way. Before interaction, the dromion 1 has a larger amplitude, travelling slower and the dromion 2 has a shorter amplitude, travelling faster. During interaction (at time \( t = 0 \)), an exchange of energy between the dromions takes place. This results in the gain in amplitude of dromion 2 and fall in amplitude of dromion 1. But there is no change in velocity of the dromions and there is only a change in phase.

3.1.4. Bounded multiple solitary waves

One can also construct bounded two solitary waves by choosing
\[
\begin{align*}
f &= a \text{sn}(ax + c_1t + d_1; m_1), \\
h &= \beta_1 \text{sn}(b_1y + d_2; m_2) + \beta_2 \text{sn}(b_2y + d_3; m_3),
\end{align*}
\]
where \( a, a_1, \beta_1, b_1, b_2, b_3 \) and \( d_i \) (\( i = 1, 2; j = 1, 2, 3 \)) are arbitrary constants and \( m_j \)'s are modulus parameters. Then
\[
q = \frac{q_3}{q_4},
\]
where
\[
q_3 = 2aa \text{cn}(u_1; m_1) \text{dn}(u_1; m_1) [\beta_1 b_1 \text{cn}(u_2; m_2) \text{dn}(u_2; m_2) + \beta_2 b_2 \text{cn}(u_3; m_3) \text{dn}(u_3; m_3)],
\]
\[
q_4 = (c + \alpha \text{sn}(u_1; m_1) + \beta_1 \text{sn}(u_2; m_2) + \beta_2 \text{sn}(u_3; m_3))^2,
\]
\[
u_1 = a_1 x + c_1 t + d_1, \quad \nu_2 = b_1 y + d_2 \quad \text{and} \quad \nu_3 = b_2 y + d_3.
\]
The above solution for the parametric choice \( \alpha = 1, \beta = 1, a = 0.5, b_1 = 2.5, b_2 = 1.7, c = -4, c_1 = 1, c_2 = -2, d_1 = d_2 = d_3 = 0, m_1 = 0.2, m_2 = 0.3, m_3 = 0.4, t = 0 \) is shown in Fig. 3a. As \( m_1, m_2, m_3 \to 1 \), the above solution, Eq. (35), degenerates into a bounded two dromion solution given by
\[
q = \frac{2aa \text{sech}^2 u_1 (\beta_1 b_1 \text{sech}^2 u_2 + \beta_2 b_2 \text{sech}^2 u_3)}{(c + \alpha \tanh u_1 + \beta_1 \tanh u_2 + \beta_2 \tanh u_3)^2},
\]
where \( u_1 = ax + c_1 t + d_1, u_2 = b_1 y + d_2 \) and \( u_3 = b_2 y + d_3 \). The above solution for the parametric choice \( \alpha = 1, \beta = 1, a = 0.5, b_1 = 2.5, b_2 = 1.7, c = -4, c_1 = 1, c_2 = -2, d_1 = d_2 = d_3 = 0 \) is shown in Fig. 3b. Here both the dromions travel with equal velocity along the \( x \) direction. Since they move parallel to each other there is no interaction between them.

Please cite this article in press as Senthil Kumar C et al., Bilinearization and localized coherent structures and periodic waves, Chaos, Solitons & Fractals (2007), doi:10.1016/j.chaos.2007.01.058.
3.2. \((N,M)\) dromion solution

Proceeding in a similar way as above, one can generate \((N,M)\) dromion solution by choosing

\[
\begin{align}
    f &= \sum_{i=1}^{N} \alpha_i \text{sn}(a_i x + c_i t + d_i; m_i), \\
    h &= \sum_{j=1}^{M} \beta_j \text{sn}(b_j y + d_j; m_j),
\end{align}
\]

(37a)

(37b)

where \(\alpha_i, \beta_j, a_i, b_j, c_i, d_i, d_j, m_i, m_j\) are arbitrary constants and \(m_i, m_j\) are modulus parameters and substituting them in the expression for \(q\) and \(u\) in (22). As the analysis is similar to the above, we do not present further details here.

3.3. Singular solutions

While presenting the explicit dromion solutions above, we have always assumed the magnitude of the parameters \(c\) in the solution (22) to be greater than a constant value in order to have nonsingular solutions. For example, for the \((1,1)\) case.

\[
\begin{align}
    f &= \sum_{i=1}^{N} \alpha_i \text{sn}(a_i x + c_i t + d_i; m_i), \\
    h &= \sum_{j=1}^{M} \beta_j \text{sn}(b_j y + d_j; m_j),
\end{align}
\]

(37a)

(37b)

where \(\alpha_i, \beta_j, a_i, b_j, c_i, d_i, d_j, m_i, m_j\) are arbitrary constants and \(m_i, m_j\) are modulus parameters and substituting them in the expression for \(q\) and \(u\) in (22). As the analysis is similar to the above, we do not present further details here.
dromion solution (25), we have taken $|c| > |x + \beta|$. For $|c| \leq |x + \beta|$, the solution (25) becomes singular and it is illustrated in Figs. 4a-c for different instants of time. Here we have chosen $\alpha = \beta = 1$, $c = 1$, $a = 0.5$, $b = 0.4$, $c_1 = d_1 = d_2 = 0$, $\alpha_1 = \alpha_2 = 1$.

4. Solutions corresponding to product of arbitrary functions

In this section, we will assume $c_2 = c_3 = 0$ and $C_4 = 1$ in (18) or (20) and (21) and call $c_1 = c$. Then the solution can be written as

\begin{align}
q &= \frac{-2c_1 g_p}{(fg + c)^2}, \\
v &= \frac{2f_2 g^2}{(fg + c)^2} - \frac{2f_x g}{(fg + c)} + \frac{f_x + f_{xx}}{3f_x}.
\end{align}

Now we will look for periodic and localized solutions.

4.1. Harnessing of arbitrary functions and novel solutions of (2 + 1) dimensional K-dV equation

As before, we choose the arbitrary functions $f$ and $g$ to be Jacobian elliptic functions, namely sn or cn functions,

\begin{align}
f &= a\text{sn}(ax + c_1 t + d_1; m_1), \\
g &= \beta\text{sn}(by + d_2; m_2),
\end{align}

so that

\begin{align}
q(x,y,t) &= \frac{q_5}{q_6},
\end{align}

where $q_5 = -2c_1 ab \text{sn}(u_1; m_1) \text{cn}(u_1; m_1) \text{sn}(u_2; m_2) \text{cn}(u_2; m_2)$, $q_6 = (c + \alpha \beta) \text{sn}(u_1; m_1) \text{sn}(u_2; m_2))^2$, $u_1 = ax + c_1 t + d_1$ and $u_2 = by + d_2$. In Eqs. (39) and (40), the quantities $m_1$ and $m_2$ are the moduli of the respective Jacobian elliptic functions while $a$, $\beta$, $\alpha$, $a$, $b$, $c$, $c_1$, $d_1$ and $d_2$ are arbitrary constants with $|c| > |\alpha \beta|$. The profile of the above solution for the parametric choice $a = \beta = 1$, $a = 0.5$, $b = 0.4$, $c = -4$, $c_1 = -1$, $d_1 = d_2 = 0$, $m_1 = 0.2$, $m_2 = 0.3$ at time $t = 0$ is shown in Fig. 5a. It can be observed that here again the periodic wave moves along the $x$-direction only.

4.1.1. (1,1) Dromion solution

As a limiting case of the periodic solution given by Eq. (40) when $m_1, m_2 \rightarrow 1$, the above solution degenerates into an exponentially localized solution (dromion). The limiting form corresponding to the (1,1) dromion takes the expression

\begin{align}
q &= \frac{-2c_1 ab \text{sech}^2(ax + c_1 t + d_1) \text{sech}^2(by + d_2) (c + \alpha \beta \text{tanh}(ax + c_1 t + d_1) \text{tanh}(by + d_2))^2}{(c + \alpha \beta \text{tanh}(ax + c_1 t + d_1) \text{tanh}(by + d_2))^2},
\end{align}

with $|c| > |\alpha \beta|$ and the variable $v$ then takes the form (using expression (38b))

\begin{align}
v &= \frac{2a^2 \beta \text{sech}^4(ax + c_1 t + d_1) \text{tan}^2(by + d_2)}{(c + \alpha \beta \text{tanh}(ax + c_1 t + d_1) \text{tanh}(by + d_2))^2}
&\quad + \frac{4a^2 \beta \text{sech}^2(ax + c_1 t + d_1) \text{tan}^2(by + d_2)}{(c + \alpha \beta \text{tanh}(ax + c_1 t + d_1) \text{tanh}(by + d_2))} + \frac{1}{3a} [c_1 + 4a^2 \text{tan}^2(ax + c_1 t + d_1)] \nonumber
&\quad - \frac{2a^2 \beta \text{sech}^2(ax + c_1 t + d_1)].
\end{align}

Schematic form of the (1,1) dromion for the parametric choice $a = \beta = 1$, $a = 0.5$, $b = 0.4$, $c = -4$, $c_1 = -1$, $d_1 = d_2 = 0$ at time $t = 0$ is shown in Fig. 5b. Again note that the dromion travels along the $x$-direction only.

4.1.2. More general periodic solution and (2,1) dromion

Next, we obtain more general periodic solution by choosing further general forms for the arbitrary functions. As an example, we choose

\begin{align}
f &= a_1 \text{sn}(ax + c_1 t + d_1; m_1) + a_2 \text{sn}(ax + c_1 t + d_2; m_2), \\
g &= \beta \text{sn}(by + d_1; m_1),
\end{align}

where $b, \beta, a_1, a_2, c_1$ and $d_1$ are arbitrary constants ($i = 1, 2; j = 1, 2, 3$) and $m_i$'s are modulus parameters. Then
where \( g_1 = -2c^b(\alpha_1\alpha_1 \text{cn}(u_1; m_1) \text{dn}(u_1; m_1) + \alpha_2\alpha_2 \text{cn}(u_2; m_2) \text{dn}(u_2; m_2)) \) and \( g_2 = [c + \beta(\alpha_1 \text{sn}(u_1; m_1) + \alpha_2\alpha_2 \text{sn}(u_2; m_2) \text{sn}(u_3; m_3))]^2 \).

\( q_1 = \frac{g_2}{g_0} \)

The profile of the above solution for the parametric choice \( \alpha_1 = \alpha_2 = \beta = 1, \alpha_1 = 0.5, \alpha_2 = 0.8, b = 0.4, c = -4, c_1 = -1, c_2 = -2.5, d_1 = d_2 = d_3 = 0, m_1 = 0.2, m_2 = 0.3, m_3 = 0.4 \) at time \( t = 0 \) is shown in Fig. 6a. As \( m_1, m_2, m_3 \to 1 \), the above solution equation (44) degenerates into a (2,1) dromion solution given by

\[
q = \frac{-2c^b(\alpha_1\alpha_1 \text{sech}^2 u_1 + \alpha_2\alpha_2 \text{sech}^2 u_2) \text{sech}^2 u_3}{(c + \beta(\alpha_1 \text{th} u_1 + \alpha_2 \text{th} u_2) \text{th} u_3)}
\]

\( |c| > |(\alpha_1 + \alpha_2)\beta| \), where \( u_1 = a_1x + c_1t + d_1, u_2 = a_2x + c_2t + d_2 \) and \( u_3 = by + d_3 \). The dromion interaction for the parametric choice \( \alpha_1 = \alpha_2 = \beta = 1, \alpha_1 = \alpha_2 = 0.5, b = 0.4, c = -4, c_1 = -1, c_2 = -2.5, d_1 = d_2 = d_3 = 0 \) is shown in Figs. 6b–d for different time intervals.

4.1.3. Asymptotic analysis for (2,1) dromion solution corresponding to Eq. (45)

Proceeding with the analysis as was done in the previous section, we find that both the dromions are of different amplitudes and travelling with different velocities along the \( x \)-direction. At time \( t = 0 \), there is an interaction between the dromions. After interaction, we find that there is no change in amplitude or velocity of the dromions.

4.1.4. Bounded multiple solitary waves

One can also construct bounded two solitary waves as before by choosing
Fig. 6. (a) Elliptic function solution (44) for \(q(x,y,t)\), (b)-(d) (2,1) dromion solution (45) for \(q(x,y,t)\) and its interaction at time intervals (b) \(t = -4\), (c) \(t = 0\) and (d) \(t = 4\).

\[
f = \alpha \text{sn}(ax + ct + d_1; m_1),
g = \beta_1 \text{sn}(by + d_2; m_2) + \beta_2 \text{sn}(by + d_3; m_3),
\]

where \(a, \alpha, \beta_1, \beta_2, \) and \(d_1\) are arbitrary constants \((i = 1, 2; j = 1, 2, 3)\) and \(m_j\)'s are modulus parameters. Then

\[
q = \frac{\varphi_{11}}{\varphi_{12}},
\]

where

\[
\varphi_{11} = -2cax \text{cn}(u_1; m_1) \text{dn}(u_1; m_1)\left[\beta_1 b_1 \text{cn}(u_2; m_2) \text{dn}(u_2; m_2) + \beta_2 b_2 \text{cn}(u_3; m_3) \text{dn}(u_3; m_3)\right],
\]

\[
\varphi_{12} = \left(c + \alpha \text{sn}(u_1; m_1) \left[\beta_1 \text{sn}(u_2; m_2) + \beta_2 \text{sn}(u_3; m_3)\right]\right)^2,
\]

for \(v(x,y,t)\) and \(\varphi_{11}, \varphi_{12}\) again here \(|c| > |\alpha(\beta_1 + \beta_2)|\) for nonsingular solutions. The profile of the above solution for the parametric choice \(a = \beta_1 = \beta_2 = 1, a = 0.5, b_1 = 2.5, b_2 = 1.7, c = -4, c_1 = 1, c_2 = -2, d_1 = d_2 = d_3 = 0, m_1 = 0.2, m_2 = 0.3, m_3 = 0.4, t = 0\) is shown in Fig. 7a. As \(m_1, m_2, m_3 \to 1\), the above solution given by Eq. (47) degenerates into a bounded two dromion solution given by

\[
q = \frac{-2cax \text{sech}^2 u_1 (\beta_1 b_1 \text{sech}^2 u_2 + \beta_2 b_2 \text{sech}^2 u_3)}{(c + \alpha \tanh u_1 (\beta_1 \tanh u_2 + \beta_2 \tanh u_3))^2},
\]

\(|c| > |\alpha(\beta_1 + \beta_2)|\), where \(u_1 = ax + ct + d_1, u_2 = by + d_2\) and \(u_3 = by + d_3\) with the parametric choice being \(a = \beta_1 = \beta_2 = 1, a = 0.5, b_1 = 2.5, b_2 = 1.7, c = -4, c_1 = 1, c_2 = -2, d_1 = d_2 = d_3 = 0\). The (2,1) dromion is shown in Fig. 7b. The two dromions evolve quite similar to the pattern of Fig. 3b in the \(x\)-direction.

5. Generalized Nizhnik–Novikov–Veselov (NNV) equation and construction of solutions

The generalized Nizhnik–Novikov–Veselov (NNV) equation is a symmetric generalization of the K-dV equation in \((2 + 1)\) dimensions and is given by

\[
U_t + \alpha U U_x + \beta U_x U_{xx} + \gamma U_{xxx} = 0,
\]
Fig. 7. (a) Elliptic function solution (47), (b) bounded two dromion solution (48).

\[ u_x + au_{xx} + bu_{yy} + cu + du_y - 3avu_x - 3bwu_x - 3bw_y = 0, \]  
\[ u_x = v_y, \]  
\[ u_y = w_z. \]  

Here \( a, b, c \) and \( d \) are parameters. This equation which is also known to be completely integrable has been investigated and exponentially localized solutions have been generated [4,8]. We now apply the Painlevé truncation approach to obtain more general solutions. For this purpose, we again truncate the Laurent series at the constant level term to get the transformation

\[ u = u_0 + u_1 \phi^{-1} + u_2, \]  
\[ v = v_0 + v_1 \phi^{-1} + v_2, \]  
\[ w = w_0 + w_1 \phi^{-1} + w_2. \]

Again considering the vacuum solutions of the form

\[ u_2 = 0, \quad v_2 = v_2(x, t), \quad w_2 = w_2(y, t), \]

where \( v_2(x, t) \) and \( w_2(y, t) \) are arbitrary functions in the indicated variables, and collecting the coefficients of different powers of \( \phi \) as before and solving the resultant equations, we obtain the following results:

\[ u_0 = 2u_0 \phi_x^2, \quad u_1 = -2u_0 \phi_x, \]  
\[ v_0 = 2v_0 \phi_y^2, \quad v_1 = -2v_0 \phi_y, \]  
\[ w_0 = 2w_0^2, \quad w_1 = -2w_0 \phi_y. \]

Also \( v_2 \) and \( w_2 \) are related through \( \phi \) by the relation

\[ v_2 = \phi_x + a \phi_x^2 + b \phi_y^2 + c \phi_y^3 + d \phi_y^4, \]  
\[ w_2 = \phi_y + a \phi_y^2 + b \phi_x^2 + c \phi_x^3 + d \phi_x^4. \]

Making use of the expression (52) and (53) in (50) and using the resultant forms in the generalized NNV equation (49), we obtain the following system of equations which is trilinear in \( \phi \) (with the coefficient \( w_2 \) also present),

\[ \phi_{xx} (\phi_{xx} \phi_{xy} - \phi_{yx} \phi_{yx}) + \phi_x (\phi_{xx} \phi_{yx} - \phi_{yx} \phi_{xx}) = 0, \]  
\[ 6 \phi_x (\phi_{xx} \phi_{xy} - \phi_{yx} \phi_{xx} + \phi_x (\phi_{xx} \phi_{xy} - \phi_{yx} \phi_{xx}) - 3a (\phi_{xx} \phi_{yx} - \phi_{yx} \phi_{xx}) \phi_y + 3b \phi_y (\phi_{xx} \phi_{xy} - \phi_{yx} \phi_{xx}) - 3c \phi_x \phi_{xy} \phi_{xx} + 3d \phi_y \phi_{xy} \phi_{xx} \phi_{xx} - \phi_{xx} \phi_{xy} \phi_{xx} + \phi_y \phi_{xx} \phi_{xy} \phi_{xx} - \phi_{xx} \phi_{xy} \phi_{xx} \phi_{xx} - \phi_y \phi_{xx} \phi_{xy} \phi_{xx}) = 0, \]  
\[ w_2 = 0. \]

One can observe that the above set of Eqs. (54) admits a more general form of \( \phi \) involving two arbitrary functions \( f(x, t) \) and \( g(y, t) \), and four arbitrary parameters, \( c_1, c_2, c_3 \) and \( c_4 \) as

\[ \phi(x, y, t) = c_1 + c_2 f(x, t) + c_3 g(y, t) + c_4 f(x, t) g(y, t). \]
Here $v_2$ and $w_2$ take the forms (from (53))

$$v_2 = f_x + a f_{xxx} + c f_y + (c_3 + c_4 f),$$

$$w_2 = g_y + b g_{yy} + d g_y - (c_2 + c_4 g).$$

Here $c_1$, $c_2$, $c_3$, and $c_4$ are arbitrary parameters. Note here that unlike the case of $(2 + 1)$ dimensional K-dV equation only two arbitrary functions are allowed in (55).

Now substituting the above form (55) for the manifold $\phi(x, y, t)$ into the truncated Painlevé series (50) for the functions $u(x, y, t)$, $v(x, y, t)$ and $w(x, y, t)$, along with the expressions for the coefficient functions $u_0$, $v_0$, $w_0$, $u_1$, $v_1$, $w_1$, $v_2$ and $w_2$ given above, we finally obtain the solution to Eqs. (49) as

$$u(x, y, t) = \frac{2(c_2 c_3 - c_4 c_1) f g_y}{[c_1 + c_2 f(x, t) + c_3 g(y, t) + c_4 f(x, t) g(y, t)]^2}, \quad (c_2 c_3 - c_4 c_1) \neq 0,$$

$$v(x, y, t) = \frac{2(c_2 + c_4 g) f_x}{[c_1 + c_2 f(x, t) + c_3 g(y, t) + c_4 f(x, t) g(y, t)]^2} + \frac{2(c_2 + c_4 g) f_x}{c_1 + c_2 f(x, t) + c_3 g(y, t) + c_4 f(x, t) g(y, t)},$$

$$w(x, y, t) = \frac{2(c_3 + c_4 f) g_y}{[c_1 + c_2 f(x, t) + c_3 g(y, t) + c_4 f(x, t) g(y, t)]^2} + \frac{2(c_3 + c_4 f) g_y}{c_1 + c_2 f(x, t) + c_3 g(y, t) + c_4 f(x, t) g(y, t)} + \frac{g_y + b g_{yy} + d g_y - (c_2 + c_4 g)}{3 b g_y}.$$  

One can check that the above solutions of NNV equation reduce to the solutions of $(2 + 1)$ dimensional K-dV equation for the parametric choice $b = c = d = 0$ and also by considering the truncated $w$ series to be zero. One may note that the above form of solution coincides with the universal form of solutions reported by Tang et al. [5] and the special case $c_1 = d$ ($d$ is arbitrary constant), $c_2 = c_3 = 0$ was studied by Peng [8]. One can obtain periodic and localized dromion solutions here also following the procedure discussed in the earlier sections for the $(2 + 1)$ dimensional K-dV equation.

6. Discussion

In this paper, we have investigated the $(2 + 1)$ dimensional K-dV equation and obtained a four parameter solution involving three arbitrary functions by using the Painlevé truncation approach. For different choice of parameters, we have generated two broad classes of localized coherent structures and elliptic function periodic wave solutions. In particular, we have shown that the equation is trilinearizable. Even more general solutions can be generated from the trilinear equations which remains to be investigated. The existence of lower dimensional arbitrary functions helps us to construct novel localized solutions and study their interactions. We have also extended the approach to generalized NNV equation and pointed out how similar solutions as that of the $(2 + 1)$ dimensional K-dV equation can be obtained.

Acknowledgments

The work of C.S. and M.L. form part of a Department of Science and Technology, Govt. of India sponsored research project. R.R. thanks the Department of Science and Technology (DST) for sponsoring a major research project. The work of M.L. is supported by Department of Atomic Energy – Raja Ramanna Fellowship.

Appendix A. One dromion solution through Hirota bilinearization

Here we briefly point out how the $(1,1)$ dromion solution can be obtained through Hirota bilinearization method [4]. To bilinearize equation (3), we make the transformation...
\[ u = -20_\varphi (\log \varphi), \quad v = -20_x (\log \varphi), \quad (A.1) \]

which can be identified from the Painlevé analysis. The resultant bilinear form is given by
\[ (D_y D_x + D_x^2 D_y) \varphi \cdot \varphi = 0, \quad (A.2) \]
where \( D \)'s are the usual Hirota operators. To generate a \((1,1)\) dromion, one considers the ansatz
\[ \varphi = 1 + e^{\psi_1} + e^{\psi_2} + K e^{\psi_1 + \psi_2}, \quad (A.3) \]
where
\[ \psi_1 = k_1 x - k_1^2 t + \delta_1, \quad \psi_2 = l_1 y + \delta_2. \quad (A.4a, A.4b) \]
Here \( k_1, l_1, \delta_1, \delta_2 \) and \( K \) are arbitrary constants. The \((1,1)\) dromion is given by [4]
\[ u = \frac{2k_1 l_1 (1 - K)e^{\psi_1 + \psi_2}}{(1 + e^{\psi_1} + e^{\psi_2} + K e^{\psi_1 + \psi_2})^2}. \quad (A.5) \]

This is a special case of the dromion solution we have obtained in (25) with the constants \( a = \beta = 1, \quad a = \frac{b}{2}, \quad b = \frac{3}{2}, \quad c = \frac{3}{\sqrt{10}}, \quad d_1 = -\frac{b}{2}, \quad d_2 = \frac{1}{2} \log c^2 + \frac{b}{2} \). Thus the standard dromion solutions become special cases of the general solutions obtained in Section 3.

References