7. Electromagnetic Spin Soliton and Logic Gate Operations in a Spin Ladder Medium

7.1. Introduction

In the earlier chapters we studied the propagation of EMW in ferro and antiferromagnetic media in the form of solitons. Also, we studied magnetization reversal which has potential technological ramifications in ultrafast magnetic recording, in a ferromagnetic film when it is exposed to a circularly polarized EM field. Very recently, magnetic compounds with magnetic spin chains arranged one next to another in the form of ladders were experimentally synthesized and found to be an interesting class of magnetic materials [45, 47, 295]. These magnetic spin ladder compounds can be visualised by assuming two one-dimensional arrangements of ions located so close to each other so that the probability for electrons to jump along the chain is of the same order of magnitude as for jumping from chain to chain. In this case, it is a good ap-
proximation to imagine the ions in a ladder geometry as stated earlier i.e. two spin chains coupled along rungs. For example, if the ions along the ladder are copper which has a spin $-\frac{1}{2}$, then we have a physical realization of spin $-\frac{1}{2}$ ladder. Ladders made from even number of legs have spin liquid ground states because of their purely short range spin correlation which is due to a finite energy gap between the ground state and the excited state called the spin-gap [296]. These even legged ladders may therefore be regarded as the realization of the singlet ground state where the total spin value is zero. Ladder with odd number of legs behave quite differently and display properties similar to single chains at low energies. Good examples for two-leg spin $-\frac{1}{2}$ ladder are vanadyl pyrophosphate $(VO)_2P_2O_7$ and some cuprates such as $SrCu_2O_3$ and example of a 3-leg ladder is $Sr_2Cu_3O_6$ [296]. These magnetic spin ladder compounds may have ferro or antiferromagnetic legs and coupling via rungs may also be ferro or antiferromagnetic in nature.

Spin ladders are considered to be important nonlinear media as ferro and antiferromagnets. Spin ladders with ferro and antiferromagnetic interactions exhibit interesting nonlinear spin excitations including solitons [297, 298]. In this chapter, we study the interaction of EM field with an anisotropic spin ladder with two ferromagnetic legs and having ferromagnetic interactions along the rungs. We study the nature of magnetization excitations effected by the external EM field and also analyse the modulation of the magnetic field component of the EMW. We then show that the magnetization is excited in the form of solitons and they undergo collision and transfer energy among the legs of the ladder leading to a switching phenomenon. Finally, we use the switching phenomenon among the interacting solitons to demonstrate the operation of the different logic gates such as AND, OR, NAND, NOR and XOR and hence a TURING machine.
7.2. Model and dynamical equations

We consider an anisotropic spin ladder with two legs characterised by ferromagnetic coupling among the spins in the legs as well as in the rungs. The spin ladder is assumed to possess an easy axis of magnetization parallel to the direction of the legs (say $x$-axis). Figure (7.1) gives a schematic representation of the two leg spin ladder. In the classical continuum limit, the magnetization dynamics in the two leg spin ladder is governed by the equation of motion [297]

\begin{align}
\frac{\partial M_1}{\partial t} &= M_1 \wedge [J_1 \nabla^2 M_1 + J_2 M_2 - 2\beta M_1^z n + 2AH], \quad (7.1a) \\
\frac{\partial M_2}{\partial t} &= M_2 \wedge [J_1 \nabla^2 M_2 + J_2 M_1 - 2\beta M_2^z n + 2AH]. \quad (7.1b)
\end{align}

\begin{align}
M_1 &= (M_1^x, M_1^y, M_1^z), \quad M_2 = (M_2^x, M_2^y, M_2^z), \quad M_1^4 = 1 = M_2^4.
\end{align}

In Eqs.(7.1), $J_1$ and $J_2$ represent exchange integrals corresponding to spin-spin coupling in the legs and in the rungs.
The variation of EM field which acts on the spin ladder system is governed by the wave equation (3.1) derived from Maxwell's equations. In the present problem, the magnetic field \( H \) in Eqs.(7.1) and (3.1) is related to the effective magnetization of the medium and the magnetic induction by the relation

\[
H = \frac{B}{\mu_0} - M_{eff}, \quad (7.2)
\]

where \( M_{eff} \) is the effective magnetization of the medium due to the magnetizations \( M_1 \) and \( M_2 \) of the two legs given by \( M_{eff} = M_1 + M_2 \). Using Eq.(7.2), Eqs.(7.1) and (3.1) can be rewritten as

\[
\begin{align*}
\frac{\partial M_1}{\partial t} & = M_1 \land \left[ J_1 \nabla^2 M_1 + (J_2 - 2A)M_2 - 2\beta M_1^2 n + \frac{2A}{\mu_0} B \right], \quad (7.3a) \\
\frac{\partial M_2}{\partial t} & = M_2 \land \left[ J_1 \nabla^2 M_2 + (J_2 - 2A)M_1 - 2\beta M_2^2 n + \frac{2A}{\mu_0} B \right], \quad (7.3b) \\
M_1^2 & = 1 = M_2^2,
\end{align*}
\]

and

\[
\nabla^2 B - \mu_0 \varepsilon \frac{\partial^2 B}{\partial t^2} = \mu_0 \left[ \nabla^2 M_{eff} - \nabla(\nabla \cdot M_{eff}) \right], \quad M_{eff} = M_1 + M_2. \quad (7.4)
\]

In Eq.(7.4), \( \varepsilon = \varepsilon_0 (1 + \chi_\varepsilon) \) is the dielectric constant of the medium. The coupled equations (7.3) and (7.4) completely describe the magnetization dynamics in a spin ladder medium when it is acted upon by an EM field which is governed by Maxwell's equations.

As the nonlinear character of Eqs.(7.3) and (7.4) in the general three dimensional case makes the analysis difficult, we consider one-dimensional coupled spin ladder chains for our study by replacing \( \nabla \) and \( \nabla^2 \) in Eqs.(7.3) and (7.4) by \( \frac{\partial}{\partial z} \) and \( \frac{\partial^2}{\partial z^2} \) respectively. Hence, the one-dimensional version of Eqs.(7.3) and
(7.4) can be written as

\[
\begin{align*}
\frac{\partial M_1}{\partial t} &= M_1 \wedge \left[ J_1 \frac{\partial M_1}{\partial x^2} + (J_2 - 2A)M_2 - 2\beta M_1 n + 2AH \right], \\
\frac{\partial M_2}{\partial t} &= M_2 \wedge \left[ J_1 \frac{\partial M_2}{\partial x^2} + (J_2 - 2A)M_1 - 2\beta M_2 n + 2AH \right], \\
M_i^2 &= 1 = M_2^2,
\end{align*}
\]

and

\[
\frac{\partial^2 B}{\partial x^2} - \mu_0 \varepsilon \frac{\partial^2 B}{\partial \tau^2} = \mu_0 \left[ \frac{\partial^2 M_{\text{eff}}^x}{\partial x^2} n - \frac{\partial^2 M_{\text{eff}}^x}{\partial x^2} \right].
\]

Now the task ahead is to solve the set of coupled equations given in Eqs. (7.5) and (7.6). We carry out this using the reductive perturbation method as done earlier. In order to separate the slowly varying amplitude and the rapidly varying part of the EMW, we introduce stretching in the form \( \xi = \varepsilon(x - vt) \) and \( \tau = \varepsilon t \) following ref. [299]. Unlike the previous problems, here \( x \) and \( t \) are stretched to the same order because we are now interested in studying the evolution of two different magnetization excitations \( M_1 \) and \( M_2 \) of same group velocity [299]. We assume that the formal solutions of the coupled equations (7.5) and (7.6) lie in the neighbourhood of uniform values of the asymptotic expansion of the fields. Hence, as done earlier, due to the anisotropic nature of the medium we expand the magnetization and the magnetic induction nonuniformly as

\[
\begin{align*}
M_j^x &= M_{(0)} + \varepsilon M_j^x(1) + \varepsilon^2 M_j^x(2) + ..., \\
M_j^y &= \varepsilon^{1/2} \left[ M_j^y(1) + \varepsilon M_j^y(2) + ... \right], \\
M_j^z &= \varepsilon^{1/2} \left[ M_j^z(1) + \varepsilon M_j^z(2) + ... \right], \quad j = 1, 2,
\end{align*}
\]
The above expansions indicate that at the zeroth order of expansion, magnetization of both the legs and the magnetic induction lie along the anisotropy axis of the medium namely $x$-direction.

Before employing the reductive perturbation method, we assume the dielectric constant of the medium to be small and hence we rescale $\varepsilon$ as $\varepsilon \to \varepsilon^2 \varepsilon$. Also, we assume that the spin-spin exchange interaction in legs is more dominant compared to coupling via rungs. Thus, we rescale $J \to \varepsilon^{-1} J$. Now, we substitute the slow variables in the component form of the one-dimensional equations (7.5) and (7.6) and then the expansions (7.7) for $M_1$, $M_2$ and $B$ and collect terms proportional to the different powers of $\varepsilon$ and solve them. From Eq.(7.6), at $O(\varepsilon^0)$, we get

\begin{align*}
B_0 &= \text{constant}, \quad (7.8a) \\
B_{1y}^y &= \mu_0 \left[ M_{1y}^y + M_{2y}^y \right], \quad (7.8b) \\
B_{1z}^z &= \mu_0 \left[ M_{1z}^z + M_{2z}^z \right], \quad (7.8c)
\end{align*}

and from the constraint on the length of the magnetization vectors we find that $M_{0z}^2 = 1$. The $y$- and $z$-component equations of Eq.(7.5) at $O(\varepsilon^0)$ reduce to Eqs.(7.8b) and (7.8c) respectively under the conditions $\frac{B_0}{M_0} = -2\mu_0$ and $J_2 = 4A$. Using the above conditions and Eqs.(7.8b) and (7.8c), we find that the $x$-component form of Eqs.(7.5) satisfies identically.
Similarly, solving Eq.(7.6) at $O(\varepsilon^1)$, we obtain

\[ B_1 = 0, \]  
\[ B_2 = \mu_0 \left[ M_{1,2}^y + M_{1,2}^z \right], \]  
\[ B_3 = \mu_0 \left[ M_{1,2}^z + M_{2,2}^z \right]. \]  

Using Eqs.(7.8), (7.9) and the conditions $B_0 = -2\mu_0$ and $J_2 = 4A$, at $O(\varepsilon^1)$ we obtain the following equations at $O(\varepsilon^1)$ from the component form of Eqs.(7.5).

\[
\left[ \frac{\partial}{\partial \tau} - v \frac{\partial}{\partial \xi} \right] M_{1,1}^y = J_1 \left[ M_{1,1}^y \frac{\partial^2 M_{1,1}^y}{\partial \xi^2} - M_{1,1}^z \frac{\partial^2 M_{1,1}^z}{\partial \xi^2} \right], \tag{7.10a}
\]

\[
\left[ \frac{\partial}{\partial \tau} - v \frac{\partial}{\partial \xi} \right] M_{1,1}^y = -J_1 M_{0} \frac{\partial^2 M_{1,1}^y}{\partial \xi^2} - 4A \left( M_{1,1}^z + M_{2,1}^z \right) M_{1,1}^z - 2\beta M_{0} M_{1,1}^z, \tag{7.10b}
\]

\[
\left[ \frac{\partial}{\partial \tau} - v \frac{\partial}{\partial \xi} \right] M_{1,1}^x = J_1 M_{0} \frac{\partial^2 M_{1,1}^x}{\partial \xi^2} + 4A \left( M_{1,1}^y + M_{2,1}^y \right) M_{1,1}^y + 2\beta M_{0} M_{1,1}^y, \tag{7.10c}
\]

\[
\left[ \frac{\partial}{\partial \tau} - v \frac{\partial}{\partial \xi} \right] M_{2,1}^x = J_1 \left[ M_{2,1}^z \frac{\partial^2 M_{2,1}^z}{\partial \xi^2} - M_{2,1}^y \frac{\partial^2 M_{2,1}^y}{\partial \xi^2} \right], \tag{7.10d}
\]

\[
\left[ \frac{\partial}{\partial \tau} - v \frac{\partial}{\partial \xi} \right] M_{2,1}^y = -J_1 M_{0} \frac{\partial^2 M_{2,1}^y}{\partial \xi^2} - 4A \left( M_{1,1}^z + M_{2,1}^z \right) M_{2,1}^z - 2\beta M_{0} M_{1,1}^z, \tag{7.10e}
\]
\[
\left[ \frac{\partial}{\partial \tau} - v \frac{\partial}{\partial \xi} \right] M^x_{\gamma(1)} = J_1 M(0) \frac{\partial^2 M^y_{\gamma(1)}}{\partial \xi^2} + 4A \left( M^x_{\gamma(1)} + M^x_{\gamma(1)} \right) M^y_{\gamma(1)} + 2\beta M(0) M^y_{\gamma(1)}. \tag{7.10f}
\]

From the conservation of the length of the magnetization vectors at \( O(\varepsilon^1) \) we get

\[
M^x_{\gamma(1)} = -\frac{1}{2M(0)} \left[ (M^y_{\gamma(1)})^2 + (M^x_{\gamma(1)})^2 \right], \tag{7.11a}
\]

\[
M^x_{\gamma(1)} = -\frac{1}{2M(0)} \left[ (M^y_{\gamma(1)})^2 + (M^x_{\gamma(1)})^2 \right]. \tag{7.11b}
\]

On subtracting Eq.(7.10c) from (7.10b) and Eq.(7.10f) from (7.10e), we obtain

\[
\left\{ \frac{1}{M(0)} \frac{\partial}{\partial \tau} - \frac{v}{M(0)} \frac{\partial}{\partial \xi} + iJ_1 \frac{\partial^2}{\partial \xi^2} + \frac{i4A}{M(0)} \left( M^y_{\gamma(1)} + M^x_{\gamma(1)} \right) + 2i\beta \right\} \left[ M^y_{\gamma(1)} - iM^x_{\gamma(1)} \right] = 0, \tag{7.12a}
\]

\[
\left\{ \frac{1}{M(0)} \frac{\partial}{\partial \tau} - \frac{v}{M(0)} \frac{\partial}{\partial \xi} + iJ_1 \frac{\partial^2}{\partial \xi^2} + \frac{i4A}{M(0)} \left( M^y_{\gamma(1)} + M^x_{\gamma(1)} \right) + 2i\beta \right\} \left[ M^y_{\gamma(1)} - iM^x_{\gamma(1)} \right] = 0. \tag{7.12b}
\]

Now, we define

\[
\psi_1 = (M^y_{\gamma(1)} - iM^x_{\gamma(1)}), \tag{7.13a}
\]

\[
\psi_2 = (M^y_{\gamma(1)} - iM^x_{\gamma(1)}). \tag{7.13b}
\]

Using Eq.(7.13) in Eqs.(7.11), we get

\[
M^x_{\gamma(1)} = \frac{-1}{2M(0)} |\psi_1|^2, \tag{7.13c}
\]

\[
M^x_{\gamma(1)} = \frac{-1}{2M(0)} |\psi_2|^2. \tag{7.13d}
\]
Using the above definitions, Eqs. (7.12) can be rewritten as

\[ i\psi_{1\tau} - J_1 M(0) \psi_{1\xi\xi} + 4A \left[ |\psi_1|^2 + |\psi_2|^2 \right] \psi_1 - i\nu \psi_{1\xi} - 2\beta \psi_1 = 0, \]  
(7.14a)

\[ i\psi_{2\tau} - J_1 M(0) \psi_{2\xi\xi} + 4A \left[ |\psi_2|^2 + |\psi_1|^2 \right] \psi_2 - i\nu \psi_{2\xi} - 2\beta \psi_2 = 0. \]  
(7.14b)

Using the transformations

\[ \psi_1 = q_1 \exp(2i\beta M(0)\tau), \]  
(7.15a)

\[ \psi_2 = q_2 \exp(2i\beta M(0)\tau), \]  
(7.15b)

and \( X = \xi + \nu \tau, \) \( T = \tau \) in Eqs. (7.14), we obtain the following coupled nonlinear Schrödinger (CNLS) equation.

\[ iq_{1\tau} - J_1 M(0) q_{1\xi\xi} + 4A \left[ |q_1|^2 + |q_2|^2 \right] q_1 = 0, \]  
(7.16a)

\[ iq_{2\tau} - J_2 M(0) q_{2\xi\xi} + 4A \left[ |q_2|^2 + |q_1|^2 \right] q_2 = 0. \]  
(7.16b)

It should be mentioned that in the above equations (7.16), the exchange integral \( J_2 \) due to interaction between spins along the rungs is involved through the relation \( J_2 = 4A. \) The CNLS Eq. (7.16) with an interchange in the variables \( X \) and \( T \) are often used to describe the propagation of two different modes of optical pulses in birefringent fibers and also single optical pulse in two mode fibers [48]. By rescaling \( T \rightarrow -J_1 M(0)T \) and redefining the term \( -\frac{2A}{J_M(0)} \) as \( -\frac{2A}{J_M(0)} = 2\gamma \) Eq. (7.16) can be rewritten as

\[ iq_{1\tau} + q_{1\xi\xi} + 2\gamma \left[ |q_1|^2 + |q_2|^2 \right] q_1 = 0, \]  
(7.17a)

\[ iq_{2\tau} + q_{2\xi\xi} + 2\gamma \left[ |q_2|^2 + |q_1|^2 \right] q_2 = 0. \]  
(7.17b)
Eq.(7.17) is the completely integrable two-coupled NLS equation also commonly known as Manakov system [167] which describes self modulation of quasi-monochromatic EMW into spatially coherent optical solitons in a birefringent dielectric medium. Eq.(7.17) is a completely integrable soliton system solvable by IST method [167]. The Lax pair for the Manakov system (7.17) can be written as

\[
L = \begin{pmatrix}
-i\lambda & q_1 & q_2 \\
-q_1^* & i\lambda & 0 \\
-q_2^* & 0 & i\lambda \\
\end{pmatrix}, 
U = \begin{pmatrix}
-2i\lambda^2 + i\gamma(|q_1|^2 + |q_2|^2) & 2\lambda q_1 + iq_1\lambda & 2\lambda q_2 + iq_2\lambda \\
-2\lambda q_1^* + iq_1\lambda & 2i\lambda^2 - i\gamma|q_1|^2 & -iq_1^* q_2 \\
-2\lambda q_2^* + iq_2\lambda & -iq_1 q_2^* & 2i\lambda^2 - i\gamma|q_2|^2 \\
\end{pmatrix},
\]

(7.18a)

where \( \lambda \) is the spectral parameter. It can also be verified that the compatibility condition \( LT - UX + [L, U] = 0 \) leads to the Manakov system (7.17). Knowing the Lax pair the multi-soliton solutions can be constructed using the IST method [167].

### 7.3. EM-spin soliton in spin ladder medium

The multisoliton solutions to Eq.(7.17) including bright and dark solitons can also be constructed using Hirota's bilinear method [158]. The coupled bright-bright one-soliton solution for Eqs.(7.17) can be written as [158]

\[
q_1 = \frac{\exp(\eta_1)}{1 + \exp(\eta_1 + \eta_1^* + W_0)},
\]

(7.19a)

\[
q_2 = \frac{\exp(\eta + N_0)}{1 + \exp(\eta_1 + \eta_1^* + W_0)},
\]

(7.19b)
where $\eta_1 = k_1 \left( X + i k_1 T + \eta_1^{(0)} \right)$ and $W_0 = \ln \left\{ \frac{A[1 + \exp(N_0 + N_0^*)]}{(k_1 + k_1^*)^2} \right\}$. Here $k_1$, $\eta_1^{(0)}$, and $N_0$ are arbitrary complex constants.

But the more general one-soliton solution can be written by choosing two more complex constants $\alpha$ and $\phi$ as [300]

\begin{align}
q_1 &= \frac{\alpha e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + R}}, \quad (7.20a) \\
q_2 &= \frac{\phi e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + R}}, \quad (7.20b)
\end{align}

where

\[ R = \ln \left\{ \frac{\gamma(|\alpha|^2 + |\phi|^2)}{(k_1 + k_1^*)^2} \right\}, \quad (7.21) \]

Here any two of the set of complex parameters $\alpha$, $\phi$ and $\eta_1^{(0)}$ are arbitrary. Using Eqs.(7.20) in Eqs.(7.10a) and (7.10d), we find that they satisfy identically leaving out the condition that $J_1 = \frac{(2k_i - \phi)}{k_i}$.

Using $q_1$ and $q_2$ in Eqs.(7.15) and in Eqs.(7.13), we obtain the magnetization components which can be written in a more convenient form as

\begin{align}
M_{1,(1)}^x &= -\frac{\alpha e^{-R}}{2 M_0} \text{sech}(\eta_{1R} + \frac{R}{2}), \quad (7.22a) \\
M_{1,(1)}^y &= \frac{\alpha e^{-R}}{2} \cos(\eta_{1I}) \text{sech}(\eta_{1R} + \frac{R}{2}), \quad (7.22b) \\
M_{1,(1)}^z &= -\frac{\alpha e^{-R}}{2} \sin(\eta_{1I}) \text{sech}(\eta_{1R} + \frac{R}{2}), \quad (7.22c) \\
M_{2,(1)}^x &= -\frac{\phi e^{-R}}{2 M_0} \text{sech}(\eta_{1R} + \frac{R}{2}), \quad (7.22d) \\
M_{2,(1)}^y &= \frac{\phi e^{-R}}{2} \cos(\eta_{1I}) \text{sech}(\eta_{1R} + \frac{R}{2}), \quad (7.22e) \\
M_{2,(1)}^z &= -\frac{\phi e^{-R}}{2} \sin(\eta_{1I}) \text{sech}(\eta_{1R} + \frac{R}{2}), \quad (7.22f)
\end{align}
Thus the magnetization excitations induced in the spin ladder medium by the EM field is in the form of solitons (EM-spin solitons). Now from Eqs.(7.8b) and (7.8c) the magnetic induction is obtained as

\[ B^y_1 = \frac{\mu_0}{2} (\alpha + \varphi) \cos(\eta_1) e^{\frac{R}{2} \text{sech}(\eta_1 R + \frac{R}{2})}, \]
\[ B^x_1 = -\frac{\mu_0}{2} (\alpha + \varphi) \sin(\eta_1) e^{\frac{R}{2} \text{sech}(\eta_1 R + \frac{R}{2})}. \]

Using Eqs.(7.22) and (7.23), in the constitutive relation (7.2), we find that the \(y\)- and \(z\)-components of the magnetic field vanishes and the \(x\)-component is found to be

\[ H^x_1 = \left( \alpha^2 + \varphi^2 \right) e^{-R \text{sech}^2(\eta_1 R + \frac{R}{2})}. \]

Eq.(7.24) shows that the EM soliton at the lowest existing order is restricted to the direction of propagation of the EMW.

The coupled bright-bright two-soliton solution of Eq.(7.17) is given by [158]

\[ q_1 = \frac{e^{\eta_1} + e^{\eta_2} + a(1, 2, 1^*) e^{\eta_1 + \eta_2 + \eta_1^*} + a(1, 2, 2^*) e^{\eta_1 + \eta_2 + \eta_2^*}}{D_{r_1}}, \]
\[ q_2 = \frac{\exp(N_0) \{ e^{\eta_1} + e^{\eta_2} + a(1, 2, 1^*) e^{\eta_1 + \eta_2 + \eta_1^*} + a(1, 2, 2^*) e^{\eta_1 + \eta_2 + \eta_2^*} \}}{D_{r_1}}, \]

where

\[ D_{r_1} = 1 + a(1, 1^*) e^{\eta_1 + \eta_1^*} + a(1, 2^*) e^{\eta_1 + \eta_2} + a(2, 1^*) e^{\eta_2 + \eta_1} + a(2, 2^*) e^{\eta_2 + \eta_2^*} + a(1, 2, 1^*, 2^*) e^{\eta_1 + \eta_2 + \eta_1^* + \eta_2^*}. \]
and

\[ a(i, j^\ast) = \frac{A[1 + \exp(N_0 + N_0^\ast)]}{(k_i + k_j^\ast)^2}, \quad a(i, j) = \frac{(k_i - k_j)^2}{A[1 + \exp(N_0 + N_0^\ast)]}, \quad (7.26a) \]

\[ a(i, j, k^\ast) = a(i, j)a(i, k^\ast)a(j, k^\ast), \quad (7.26b) \]

\[ a(i, j, k^\ast, l^\ast) = a(i, j)a(i, k^\ast)a(i, l^\ast)a(j, k^\ast)a(j, l^\ast)a(k^\ast, l^\ast). \quad (7.26c) \]

The coupled two-soliton solutions in Eq.(7.25) with ten real arbitrary parameters \( k_j, \eta_j^{(0)} \) and \( N_0 \) exhibit perfect soliton type propagation. The asymptotic analysis of the solution (7.25) shows that the amplitude of the two colliding solitons remain same before and after interaction while the phase-factor only changes. This is shown in Figure (7.2). The more general two-soliton solu-

![Figure 7.2.: Elastic two-soliton collision of \(|q_1|^2\) for the parametric values \( k_1 = 1 + i, k_2 = 2 - i, \alpha_1 = 1, \alpha_2 = 1, \varphi_1 = 1 + i \) and \( \varphi_2 = 2 + 2i \).]
tion by combining any pair of one-soliton solution with six arbitrary complex parameters \( \alpha_j, \beta_j, k_j \) is given by [300]

\[
q_1 = \frac{\alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + e^{\eta_1 + \eta_2 + \delta_1} + e^{\eta_1 + \eta_2 + \delta_2}}{Dr_2}, \quad (7.27a)
\]

\[
q_2 = \frac{\varphi_1 e^{\eta_1} + \varphi_2 e^{\eta_2} + e^{\eta_1 + \eta_2 + \delta_1} + e^{\eta_1 + \eta_2 + \delta_2}}{Dr_2}, \quad (7.27b)
\]

where

\[
Dr_2 = 1 + e^{\eta_1 + \eta_2 + R_1} + e^{\eta_1 + \eta_2 + \delta_0} + e^{\eta_1 + \eta_2 + \delta_1} + e^{\eta_1 + \eta_2 + R_2} + e^{\eta_1 + \eta_2 + \eta_3 + R_3}, \quad (7.27c)
\]

and

\[
e^{\delta_0} = \frac{\kappa_{12}}{(k_1 + k_2^*)}, \quad (7.28a)
\]

\[
e^{\delta_1} = \frac{k_1 - k_2}{(k_1 + k_2^*)(k_1^* + k_2^*)}(\alpha_1 \kappa_{21} - \alpha_2 \kappa_{11}), \quad (7.28b)
\]

\[
e^{\delta_2} = \frac{k_2 - k_1}{(k_2 + k_2^*)(k_1 + k_2^*)}(\alpha_2 \kappa_{12} - \alpha_1 \kappa_{22}), \quad (7.28c)
\]

\[
e^{\delta_1'} = \frac{k_1 - k_2}{(k_1 + k_1^*)(k_2 + k_2^*)}(\varphi_1 \kappa_{21} - \varphi_2 \kappa_{11}), \quad (7.28d)
\]

\[
e^{\delta_2'} = \frac{k_2 - k_1}{(k_2 + k_2^*)(k_1 + k_1^*)}(\varphi_2 \kappa_{12} - \varphi_1 \kappa_{22}), \quad (7.28e)
\]

\[
e^{R_1} = \frac{\kappa_{11}}{(k_1 + k_1^*)}, \quad (7.28f)
\]

\[
e^{R_2} = \frac{\kappa_{22}}{(k_2 + k_2^*)}, \quad (7.28g)
\]

\[
e^{R_3} = \frac{|k_1 - k_2|^2}{(k_1 + k_1^*)(k_2 + k_2^*)|k_1 + k_2|^2}(\kappa_{11} \kappa_{22} - \kappa_{12} \kappa_{21}), \quad (7.28h)
\]
and

\[ \kappa_{ij} = \frac{A(\alpha_i \alpha_j^* + \varphi_i \varphi_j^*)}{(k_i + k_j^*)}. \]  (7.29)

In the above all R's are real. It should be noted that the parameters \( \eta_{ij}^{(0)} \) in the previous case have been absorbed in \( \alpha_j \) and \( \beta_j \) and hence we have twelve real arbitrary parameters as compared to ten arbitrary parameters in the previous case. It can be easily checked that when \( \alpha_1 = \alpha_2 \) and \( \varphi_1 = \varphi_2 = \exp(N_0) \), Eq.(7.27) reduces to the two-soliton solution of the form (7.25). Eq.(7.27) can also be rewritten in the convenient form as

\[ q_1 = \{\alpha_2 e^{-\eta_{11} + \delta_1/2} \cosh(\eta_{1R} + \delta_1/2) + \alpha_1 (-i\eta_{21} + \delta_2/2) \cosh(\eta_{2R} + \delta_2/2)\} / D_{r3}, \]  (7.30a)

\[ q_2 = \left\{\varphi_1 e^{(-i\eta_{11} + \delta_1/2)} \cosh(\eta_{2R} + (\delta_1 - \ln(\alpha_2))/2) + \varphi_1 e^{(-i\eta_{21} + \delta_2/2)} \cosh(\eta_{2R} + (\delta_2 - \ln(\alpha_1))/2)\right\} / D_{r3}, \]  (7.30b)

where

\[ D_{r3} = \phi_1'[e^{R_3/2} \cosh(\eta_{1R} + \eta_{2R} + R_3/2) + \exp((R_1 + R_2)/2)] \cosh(\eta_{1R} - \eta_{2R} + (R_1 - R_2)/2) \phi_2'. \]  (7.30c)

Here \( \eta_{ij} = k_{jR}[\xi - 2k_{jR}T] + \eta_{ij}^{(0)} \), \( \eta_{iI} = k_{jI}[\xi + (k_{jR}^2 - k_{jI}^2)T] + \eta_{iI}^{(0)} \), \( (j = 1, 2) \), \( \phi_1' = \exp(-i(\eta_{1I} + \eta_{2I})) \) and \( \phi_2' = [\exp(-2i\eta_{1I} + \eta_{2I}) + \exp(-2i\eta_{1I} + \eta_{2I})] / 2 \), in which the suffices R and I denote the real and imaginary parts. The asymptotic analysis carried out on the expressions (7.30) shows that not only the phase-factor but also the amplitude factor also gets changed when the two-coupled one solitons move from \( T \to -\infty \) to \( T \to \infty \) [300]. Further, it is found that the amount of intensity change during
collision depends on the numerical value of the arbitrary parameters $\alpha_j$ and $\beta_j$ and the other arbitrary parameters (initial conditions). The shape changing collision among the interacting solitons occur as long as $\alpha_1 : \alpha_2 \neq \beta_1 : \beta_2$.

Using Eqs.(7.30) in Eq.(7.15) and in Eq.(7.13), we obtain the two-soliton solutions for the magnetization components. Proceeding further the $N$-soliton solutions can be found. Thus it is found that due to the EM field interaction with the spin ladder compound the magnetization excitations in both the legs evolve in the form of solitons (EM spin solitons).

### 7.4. Logic gates in a two-leg spin ladder medium

The collision properties of solitons in Manakov system (7.17) show that the colliding solitons behave like particles. They undergo elastic or inelastic (or shape changing) collision depending upon the initial conditions such as velocity, amplitude etc. Further, from the shape changing collision, it was shown the solitons which can carry information in its amplitude, width and position, its group and phase velocities and its carrier phase [301] can exchange the information in collision with another soliton while still preserving its particle identity [300].

In a different context, particle machine models based on cellular automata were studied and the notion of computation by propagating and colliding particles was introduced [302,303]. Like the Turing machine, the particle machines were able to do general computation and operate in discrete time and space. But the particle machine was an abstract model. In the same line non standard computations such as quantum computation [304], DNA computing [305] and computing based on chaos [306,307] were introduced and they have turned the standard conception of computation upside down. The present view is
that information is carried through space by particles, and computation oc-
curs when these particles collide [308]. This type of computation is based on
the communication between the interacting particles. In other words, the par-
ticles which undergo interaction should be able to change the state of one an-
other after interaction which is called a transactive collision. Many commonly
studied systems that support waves do not have this behaviour. For instance,
because of the linear superposition, colliding plane waves in a linear medium
do not interact, i.e. do not undergo any state changes and therefore cannot
have information interaction among the colliding waves. Fortunately, such a
transactive collision among the interacting solitons is observed in the Manakov
system as discussed in the previous section. This has triggered the studies on
the possibility of using soliton interactions in one-dimensional bulk medium as
a basis for a new kind of computer.

The two-soliton solution (7.30) derived for the CNLS equation (7.17) shows
that when \( \alpha_1 : \alpha_2 \neq \varphi_1 : \varphi_2 \), then the solitons undergoing collisions switch or
exchange energy between them. The parameters \( \alpha_1, \alpha_2, \varphi_1 \) and \( \varphi_2 \) controlling
the switching of energy exhibit nontrivial information transfer. Very recently,
Steiglitz and his coworkers [309] defined a polarization state \( (\varphi) \) i.e. a single
complex number to characterise a soliton in collisions [309] and showed that
the state transformations caused by soliton collisions are given by explicit lin-
ear fractional transformations (LFT). These transformations depend on the total
energies and velocities of the solitons (the complex \( k_j \)'s used in Eq.(7.30) ) which
are invariant in collisions, but can be used to tailor desired transformations.
Further, using the different properties of LFT, very recently it was shown that
by considering the colliding solitons as data and operators, one can construct
a FANOUT, NOT and NAND gates [310].

In another context, Cancellieri and his coworkers [311] numerically anal-
ysed the modified NLS equation describing the propagation of optical pulses in a five layer dielectric structure with a nonlinear core. They considered two transverse electric (TE) modes with a relative phase difference between them as inputs to the two dielectric slabs whose guiding and soliton emission characteristics are well known and the output is measured at the central region of the five layer structure. The two TE modes were assigned the Boolean variable 1 or 0 specifying the presence or absence of power by defining a threshold power level. It was found that when the phase difference between the two TE modes is zero they are attracted by the nonlinear medium, collide and a strong peak appears at the central region which is able to produce solitons in the central region. Instead, when the phase difference between them is \(\pi\) the two propagating modes tend to cancel each other in the central region so that it no longer contains power above a threshold value sufficient for the emission of soliton. Thus, if the detected amplitude in the central region is larger than the threshold, then the output is considered to be the Boolean logic 1 or 0 otherwise. Using the above idea Cancellieri and his coworkers [311] were able to construct logic gates such as OR, AND, NAND and XOR.

Now, following the power measuring technique of Cancellieri and his coworkers [311], we demonstrate the operation of different logic gates using the shape changing collision property of the EM-spin solitons in a spin ladder medium by numerically plotting the general two solitons for the magnetization \(M_{1,1}^z\) and \(M_{2,1}^z\). Here we consider the two colliding EM-spin solitons of \(M_{1,1}^z\) (which is more generally equivalent to two one-solitons of \(M_{1,1}^z\) and \(M_{2,1}^z\)) as the two inputs and the soliton resulting after interaction as output. Further, we have shown that the EM-spin soliton interaction in a contour diagram which gives a clear picture about the intensity of the interacting solitons before and after collision. The starting point of the two solitons (say \(T \to -\infty\)) is considered
as the two input ports and the ending point after interaction (say $T \rightarrow \infty$) as the output port. A threshold value on the intensity (or power) of the solitons is fixed at both the input and output ports. We call the two inputs (solitons before collision) as $S_1$ and $S_2$ and the output (after interaction) as $S_3$. If the detected amplitude at both the input and output ports are larger than the threshold level, we set $S_1 = S_2 = S_3 = 1$ or 0 otherwise. This threshold condition is a critical one which is to be fixed because if the lower level is very small then it will correspond to a 0 state even when there is no excitation in the medium and if it is too large, then we need solitons of higher amplitudes or power. Hence, by carefully choosing the threshold level, we demonstrate the different logic operations in a spin ladder medium by changing the values of the amplitudes and velocities of the interacting EM-spin solitons. In our study we have chosen different threshold values for different gates. In Table (7.1) we give the different values of the arbitrary parameters used for demonstrating different logic gate operations. Based on the above idea now, we explain the construction of OR, AND, NAND and NOR gates using the shape changing collisions of the EM-spin solitons.

<table>
<thead>
<tr>
<th></th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\varphi_1$</th>
<th>$\varphi_2$</th>
</tr>
</thead>
<tbody>
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<td>0.468+i</td>
<td>0.46-i</td>
<td>0</td>
<td>0.56</td>
<td>0.95</td>
<td>0.76</td>
</tr>
<tr>
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<td>1.08</td>
<td>0.86-i</td>
<td>2.0</td>
<td>0.156</td>
<td>1.209</td>
<td>1.209</td>
</tr>
<tr>
<td>Figures 7.3e,f</td>
<td>1</td>
<td>0.46-i</td>
<td>0</td>
<td>0.56</td>
<td>0.95</td>
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<tr>
<td>Figures 7.4a,b</td>
<td>2-i</td>
<td>1</td>
<td>0.4 + 0.8i</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Figures 7.5a,b</td>
<td>1+i</td>
<td>2-i</td>
<td>0.75i</td>
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<td>1.157 -i</td>
<td>2.563</td>
<td>2.563</td>
<td>0.8i</td>
<td>0.8-3.55 i</td>
</tr>
</tbody>
</table>

Table 7.1.: The different values of the velocities $k_1$, $k_2$, and amplitudes $\alpha_1$, $\alpha_2$, $\varphi_1$, $\varphi_2$ of the two colliding solitons $S_1$ and $S_2$ are given for different logic gate operations.
OR gate

For an OR gate operation, the boolean table shows an output logic 1 for two

Figure 7.3.: The OR gate operation for different parametric values given in Table 7.1

conditions i.e only if both the inputs or any one of the inputs are high i.e Boolean 1. This can be realised by setting the amplitudes of both S1 and S2 or either S1 or S2 as high. For example, to demonstrate the boolean table for OR
gate, we set \( S_1 = 1 \) and \( S_2 = 0 \) and from Figure (7.3a), we find that the output \( S_3 = 1 \). This is also demonstrated by the contour diagram given in Figure (7.3b) which shows that the intensity of the soliton \( S_2 \) after interaction diminishes. Similarly the other form of the second condition of the OR truth table can be realised through Figures (7.3c) and (7.3d) where \( S_1 = 0, S_2 = 1 \) and the output \( S_3 = 1 \). The case that for a high output when both the inputs are high is demonstrated in Figures (7.3e) and (7.3f). The contour diagram (7.3f) shows that the intensity of both the interacting solitons \( S_1 \) and \( S_2 \) adds up after collision. The other condition of the OR truth table which shows that the output is zero when there is no input is an obvious one. The different values of the parameters used for Figures (7.3a-f) are given in Table (7.1).

**AND gate**

The truth table for AND gate states that the output remains zero, even if one of the inputs is zero i.e. Boolean 0. For example if we consider the inputs \( S_1 = 1 \)
(Boolean 1) and $S2 = 0$ (Boolean 0), then according to the AND truth table the output $S3 = 0$ (Boolean 0). This is realised in Figure (7.4a). The corresponding contour plot Figure (7.4b) shows that the interacting soliton having high intensity $S1$ exchanges its intensity with $S2$ during collision and comes out such that both $S1$ and $S2$ have intensity less than the threshold value so that the output $S3 = 0$ i.e Boolean 0. The other condition of the AND truth table which states that the output remains high only if both the inputs are high can be realised from Figures (7.3e) and (7.3f). The different values of the parameters used for Figures (7.4a,b) are given in Table (7.1).

NOR gate

One of the important conditions in the case of NOR gate which is considered to be the universal gate is when both the inputs are high i.e $S1 = S2 = 1$ (Boolean 1), then the output should be zero indicating that $S3 = 0$ (Boolean 0). This is demonstrated in Figure (7.5a) while the contour diagram Figure (7.5b) shows that both $S1$ and $S2$ exchange their amplitudes during collision such that the intensity of the outgoing soliton $S3$ is very small below the threshold value so that the output $S3 = 0$ giving Boolean 0 even when both the inputs are high i.e. Boolean 1. The other condition of the NOR truth table namely the output remains zero even when any one of the inputs is high is realised from Figures (7.4a) and from the contour diagram (7.4b). The third condition in the NOR truth table that the output remains high i.e. Boolean 1 ($S3 = 1$), even when both the inputs are low i.e Boolean 0 ($S1 = S2 = 0$) has been demonstrated in Figure (7.5c). The contour plot Figure (7.5d) shows that the amplitudes of both the interacting solitons $S1$ and $S2$ which are below the threshold value adds up to give an output $S3$ above the threshold level showing Boolean 1. The different values of the parameters for Figures (7.5a-d) are given in Table (7.1).
NAND gate

The NAND gate which is another universal gate is based on the truth table which shows that the output is Boolean 1 (S3 = 1) when any one of the inputs is Boolean 1 i.e S1 = 1 or S2 = 1. This can be realised from Figures (7.3a) as well as from (7.3b) and (7.3c) and (7.3d). The next condition of the NAND gate namely the output remains high i.e S3 = 1 (Boolean 1) when both the inputs are low i.e. S1 = S2 = 0 (Boolean 0) can be again realised from Figures (7.5c) and (7.5d). The final condition in the NAND truth table that is the output becomes
Boolean 0 \((S3 = 0)\) when both the inputs have a Boolean 1 \((S1 = S2 = 1)\) is demonstrated in Figures (7.5a) and (7.5b).

**XOR gate**

The truth table for an XOR gate shows that the output is Boolean 1 \((S3 = 1)\) only if one of the inputs is 1 \((S1 = S2 = 1)\). This can be realised from Figures (7.3a), (7.3b) and (7.3c), (7.3d). Also, the other condition of the XOR truth table that the output is Boolean 0 \((S3 = 0)\), even when both the inputs are Boolean 1 \((S1 = S2 = 1)\) is demonstrated in Figures (7.5c) and (7.5d).

As it has been analytically proved that the total energy of the system during shape changing collision of the two solitons in the two-CNLS equations is conserved [300], here in our present problem also the energy is conserved. Thus the possibility of all logic operations using the interaction between EM-spin solitons of the two legs of the spin ladder medium is shown explicitly for different amplitudes and velocities of the interacting solitons which can be achieved with the help of the propagating EMW. Our idea of construction of logic gates in a magnetic medium resembles closely with a recent experimental investigation by Cowburn [59] in which networks of interacting magnetic dots at nanoscale level are used to perform logic operations and to propagate information at room temperature. In the above case, the logic states are signalled by the magnetization direction of the single-domain magnetic dots which couple to their neighbours through magnetostatic interactions. Magnetic solitons carry information through the networks, and an applied oscillating magnetic field feeds energy into the system and serves as a clock. These networks form a magnetic microchip and they offer a several thousand fold increase in integration density and a hundred fold reduction in power dissipation over current magnetoelectronic technology.
7.5. Conclusions

In this chapter, we studied magnetization excitations in an anisotropic two leg spin ladder medium when acted upon by EM field. Following the same procedure used in the case of ferromagnetic medium we deduced the coupled Maxwell and LL-type equations to the well known completely integrable coupled NLS equation of Manakov type. Using the one-soliton solutions and a more general two-soliton solutions available for the above equations we constructed the EM-spin solitons and further obtained the one and two solitons for the magnetic induction and magnetic field. It was noticed that the EM soliton is restricted to the direction of propagation of the EMW. Then we described the operations of all the basic logic gates such as AND, OR, XOR and the universal gates like NOR and NAND using the shape changing EM-spin solitons. This is achieved by fixing a threshold level for the amplitudes of the EM spin solitons before and after interaction. The Boolean 1 and 0 are assigned if the amplitude is higher and below the threshold values respectively. We were able to construct all the logic gates and hence a TURING machine. These logic components are expected to find useful applications in circuit design where for example the series circuit is operated by AND operation and the parallel circuit by OR operation. The final goal will be the realization of magneto-optical logic gates in the magnetic micro chips where the transfer of the data rate are faster than the conventional electronic instruments.