4. Parallel Propagation of EMW in an Anisotropic Ferromagnetic Medium

4.1. Introduction

When EM field interacts with the magnetization of an isotropic ferromagnetic medium we found in the previous chapter that the magnetization excitations induced in the medium are governed by soliton and further the EM field is modulated in the form of soliton. However, when free charges are placed in the medium, both the EM and EM-spin solitons undergo damping and distortion. In many ferromagnetic materials the spin-orbit coupling of the ferromagnetic ion in the medium may interact with the surrounding crystal field due to electron cloud thus creating an easy axis or easy-plane-type anisotropy. Typical example for an easy axis type is $CoCl_2.2NC_5H_5$. Easy plane type anisotropy exists in ferromagnets such as $CsNiF_3$.

Further, in sec.(1.7.1) it was pointed out that in the case of a treated ferromagnetic material the constitutive relationship between magnetic induction ($B$) and magnetic field ($H$), is nonlinear because of hysteresis. Therefore, in the present chapter we study the propagation of EMW in anisotropic untreated
and treated ferromagnetic media. Here we consider an easy axis anisotropy in the ferromagnetic medium and analyse the modulation of the magnetic field component of EMW by choosing the direction of propagation to be parallel to the anisotropic axis.

The above studies pertain to propagation of single EM pulse through the ferromagnetic medium. However, in real communications a whole lot of signals have to be sent simultaneously. Therefore, we study the co-propagation of $n$-EM signals through the anisotropic ferromagnetic medium and establish the lossless propagation of the $n$-EM signals which is also shown numerically. Finally towards the end of this chapter, we study the propagation of EMW in anisotropic biquadratic and bilinear inhomogeneous ferromagnetic media in short.

### 4.2. Model and dynamical equations

The dynamics of magnetization in a classical continuum anisotropic Heisenberg ferromagnet in the presence of an external magnetic field is governed by the LL equation given as

$$\frac{\partial M}{\partial t} = M \wedge [J\nabla^2 M - 2\beta M^2 n + 2AH], \quad M^2 = 1. \quad (4.1)$$

In the present problem we choose the crystal field anisotropy along $x$-direction so that it is the easy axis of magnetization (i.e.) $n = (1,0,0)$. In the absence of external field Eq.(4.1) represents the dynamics of a pure anisotropic spin chain and was solved in one dimension by Nakamura and Sasada [188] as discussed in sec.(3.4) and the elementary magnetization excitations are found to be governed by envelope solitons in addition to linear magnons and other
localized excitations such as kinks. However from a different point of view Mikeska [197, 198, 250] approximated the evolution of the one dimensional ferromagnetic system with easy plane anisotropy to the integrable sine-Gordon model in the limit $\frac{2AH}{J} \ll 1$, which prompted theoretical [36] and several experimental investigations [179] on $CsNiF_3$.

The modulation of EMW as it propagates in an anisotropic charge free ferromagnetic medium is governed by Eq.(2.36). Thus the coupled system of equations (4.1) and (2.36) completely describe the modulation of EM field when it interacts with the magnetization of the ferromagnetic medium and also the magnetization dynamics of the medium. As the nonlinear nature of Eq.(4.1) makes the analysis very difficult, we consider quasi one dimensional ferromagnetic medium and hence for our study we solve the one dimensional version of Eqs.(4.1) and (2.36) as done in the isotropic case. In this case the LL equation for anisotropic medium takes the form

$$\frac{\partial M}{\partial t} = M \wedge \left[ J \frac{\partial^2 M}{\partial z^2} - 2\beta M^2 n + 2AH \right], \quad M^2 = 1. \quad (4.2)$$

Similarly the one dimensional version of the Maxwell's equation is given by Eq.(3.1).

4.3. Analysis of dispersion relation

Having constructed the dynamical equations now we analyse the characteristics of the wave that propagates in the anisotropic medium. The anisotropy of the medium facilitates us to study the EMW propagation in a direction parallel to the anisotropy axis. As we have chosen the anisotropy axis along $x$-direction, we choose the angle of incidence of the EMW $\theta = 0$ in Eq.(3.4) (with the undis-
turbed state in the present case as \( \mathbf{M} = M_0(\cos \theta, \sin \theta, 0) \), so that the propagating EMW is parallel to the anisotropy axis. We solve the coupled equations (4.2) and (3.1) by considering a plane wave in the form \( \exp[i(\kappa x - \omega t)] \). The dispersion relation determining the nature of the wave that can propagate in an anisotropic ferromagnetic medium can be derived as

\[
\omega^2 (\omega^2 - c^2 \kappa^2)^2 - 4A^2 M_0^2 [(\alpha'' + 2\beta + 1)\omega^2 - (\alpha'' + 2\beta)c^2 \kappa^2]^2 = 0.
\] (4.3)

Now, we shall expand the phase velocity \( \frac{\omega}{\kappa} \) in power series in \( k \). For \( k \ll 1 \), using the method of successive approximations, we find the following dispersion relations from Eq.(4.3).

\[
\frac{\omega}{\kappa} = \left[ \frac{\alpha'' + 2\beta}{1 + \alpha'' + 2\beta} \right]^{\frac{1}{2}} c + \frac{\alpha'' + 2\beta}{2(1 + \alpha'' + 2\beta)^2} k\lambda + O(k^2 \lambda^2),
\] (4.4a)

\[
\frac{\omega}{\kappa} = \frac{1 + \alpha'' + 2\beta}{(\alpha'' + 2\beta)^2} \frac{c}{(1 + \alpha'' + 2\beta)^2} k\lambda + O(k^3 \lambda^3),
\] (4.4b)

where we have considered the waves propagating parallel to \( x \)-direction for the mode (4.4). Since we are interested in finite waves with phase velocity \( \frac{\omega}{\kappa} \) as \( k\lambda \to 0 \), we consider exclusively the waves corresponding to Eq.(4.4a). Thus from Eq.(4.4a) we can write

\[
\omega = v \kappa + \mu_2 \kappa^2,
\] (4.5a)

and hence

\[
\kappa x - \omega t = \kappa(x - vt) + \mu_2 \kappa^2 t,
\] (4.5b)

where \( v = \frac{(\alpha'' + 2\beta)}{1 + \alpha'' + 2\beta} \) and \( \mu_2 = \frac{(\alpha'' + 2\beta)c}{2(1 + \alpha'' + 2\beta)^2} \). This suggests us to introduce the wave variable \( \xi = \kappa x - \omega t \) and stretch it along with the time variable \( t \) using \( k\lambda \) as the
small parameter $\varepsilon$ as

$$\xi = \varepsilon(x - vt), \quad (4.6a)$$

$$\tau = \varepsilon^2 t, \quad (4.6b)$$

where $v = \left(\frac{(\alpha' + 2\beta)}{(1 + \alpha' + 2\beta)}\right)^{\frac{1}{2}} c$. The dispersion relation Eq.(4.4a) means that the effect of dispersion and the anisotropy or both are at $O(\kappa l)$.

### 4.4. Magnetic and EM soliton in an anisotropic ferromagnetic medium

Knowing the form of waves that can propagate in the ferromagnetic medium corresponding to the dispersion relation Eq.(4.4a), we now try to investigate the modulation of the EMW by solving the set of coupled dynamical equations (3.1) and (4.2). Before proceeding further, we consider the component form of the one dimensional version of Eqs.(3.1) and (4.2) after using the constitutive relation (1.12). The component forms of Eqs.(3.1) are already given by Eq.(3.19) after using the constitutive relation (1.12) while that of Eq.(4.2) can be written as

$$\frac{\partial M^x}{\partial t} = J \left[ M^y \frac{\partial^2 M^z}{\partial x^2} - M^z \frac{\partial^2 M^y}{\partial x^2} \right] + \frac{2A}{\mu_0} [M^y B^z - M^z B^y], \quad (4.7a)$$

$$\frac{\partial M^y}{\partial t} = J \left[ M^x \frac{\partial^2 M^z}{\partial x^2} - M^z \frac{\partial^2 M^x}{\partial x^2} \right] + \frac{2A}{\mu_0} [M^x B^z - M^z B^x] - 2\beta M^z M^x, \quad (4.7b)$$

$$\frac{\partial M^z}{\partial t} = J \left[ M^y \frac{\partial^2 M^x}{\partial x^2} - M^x \frac{\partial^2 M^y}{\partial x^2} \right] + \frac{2A}{\mu_0} [M^x B^y - M^y B^x] + 2\beta M^y M^x. \quad (4.7c)$$

Before actually using the reductive perturbation method we introduce the slow variables given in Eq.(4.6) in Eqs.(3.19) and (4.2). Further we assume that the
ferromagnetic medium under study possesses a strong exchange interaction and weak anisotropy which is implemented by rescaling $J$ and $\beta$ as $J \to \epsilon^{-1}J$ and $\beta \to \epsilon^{\frac{\beta}{2}}$. Due to the anisotropic nature of the medium with the anisotropic axis along the direction of propagation ($x$-direction), we expand the components of magnetization and magnetic induction nonuniformly by writing

$$M^x = M_0 + \epsilon M_1^x + \epsilon^2 M_2^x + \ldots,$$

$$M^y = \epsilon^{\frac{1}{2}} [M_1^y + \epsilon M_2^y + \ldots],$$

$$M^z = \epsilon^{\frac{1}{2}} [M_1^z + \epsilon M_2^z + \ldots],$$

and

$$B^x = B_0 + \epsilon B_1^x + \epsilon^2 B_2^x + \ldots,$$

$$B^y = \epsilon^{\frac{1}{2}} [B_1^y + \epsilon B_2^y + \ldots],$$

$$B^z = \epsilon^{\frac{1}{2}} [B_1^z + \epsilon B_2^z + \ldots].$$

It may be noted that along the direction of propagation of the EMW (i.e.) along $x$-direction the magnetization and the magnetic induction have been expanded about $M_0$ and $B_0$ respectively where $M_0$ is the saturated magnetization of the medium and $B_0$ is the corresponding magnetic induction. The above expansions further indicate that at the zeroth order of expansion, both the magnetization and the magnetic induction lie along the anisotropy axis of the medium namely along the $x$-direction which is also the direction of propagation and at the first order of perturbation they turn around to the $y$-$z$ plane.

We now substitute the asymptotic expansions of $M$ and $B$ as given in Eqs.(4.8) and (4.9) in Eqs.(3.19) and (4.7). After using the stretched variables $\xi$ and $\tau$, we
collect the coefficients of different powers of \( \epsilon \). At \( O(\epsilon^0) \) from Eq.(3.19) we have

\[
\frac{\partial^2 B_0}{\partial \xi^2} = 0, \tag{4.10a}
\]

\[
\frac{\partial^2 B^y_1}{\partial \xi^2} = \frac{1}{\epsilon_0 (c^2 - v^2)} \frac{\partial^2 M^y_1}{\partial \xi^2}, \tag{4.10b}
\]

\[
\frac{\partial^2 B^z_1}{\partial \xi^2} = \frac{1}{\epsilon_0 (c^2 - v^2)} \frac{\partial^2 M^z_1}{\partial \xi^2}, \tag{4.10c}
\]

and from Eq.(4.7) we have

\[
M^y_1 B^y_1 - M^z_1 B^z_1 = 0, \tag{4.11a}
\]

\[
M^z_1 B_0 - M_0 B^z_1 = 0, \tag{4.11b}
\]

\[
M^y_1 B_0 - M_0 B^y_1 = 0. \tag{4.11c}
\]

On solving Eq.(4.10a), we find that the magnetizing field \( B_0 \) is a constant. Similarly on solving Eqs.(4.10b), (4.10c), (4.11b) and (4.11c), we obtain

\[
B^y_1 = k M^y_1, \tag{4.12a}
\]

\[
B^z_1 = k M^z_1, \tag{4.12b}
\]

provided \( k \equiv \frac{B_0}{M_0} = \left[ \epsilon_0 (c^2 - v^2) \right]^{-1} \). Using Eqs.(4.12), we find that Eq.(4.11a) satisfies identically. Further, from the constraint on the conservation of length of magnetization vector \( M^2 = 1 \), we get

\[
M_0^2 = 1. \tag{4.13}
\]

At the next order i.e. at \( O(\epsilon^1) \), we obtain the following relation from the con-
M\textsuperscript{2} = 1.

\begin{equation}
M_1^z = \frac{1}{2M_0} \left[ (M_1^y)^2 + (M_1^x)^2 \right].
\end{equation}

Also at \(O(\varepsilon^1)\) we get the following equations from Eqs.(3.19)

\begin{align}
\frac{\partial^2 B^y_1}{\partial \xi^2} &= -2vk\varepsilon_0 \frac{\partial^2 B_0}{\partial \xi \partial \tau}, \\
\frac{\partial^2 B^y_2}{\partial \xi^2} - k \frac{\partial^2 M^y_2}{\partial \xi^2} &= - \frac{2v}{(c^2 - v^2)} \frac{\partial^2 B^y_1}{\partial \xi \partial \tau}, \\
\frac{\partial^2 B^z_2}{\partial \xi^2} - k \frac{\partial^2 M^z_2}{\partial \xi^2} &= - \frac{2v}{(c^2 - v^2)} \frac{\partial^2 B^z_1}{\partial \xi \partial \tau},
\end{align}

and from Eqs.(4.8), we obtain

\begin{align}
\frac{\partial M^x_1}{\partial \xi} &= -\frac{J}{v} \left[ M^y_1 \frac{\partial^2 M^y_1}{\partial \xi^2} - M^z_1 \frac{\partial^2 M^z_1}{\partial \xi^2} \right] - \frac{2A}{\mu_0 v} \left[ M^y_1 B^z_2 + M^z_1 B^z_1 - M^z_1 B^z_2 - M^y_1 B^y_1 \right], \\
\frac{\partial M^y_1}{\partial \xi} &= \frac{JM_0}{v} \frac{\partial^2 M^y_1}{\partial \xi^2} - \frac{2A}{\mu_0 v} \left[ M^y_1 B^z_1 + M^z_1 B^z_0 - M_0 B^z_2 - M^z_1 B^z_1 \right] - \beta M_0 M^z_1, \\
\frac{\partial M^z_1}{\partial \xi} &= -\frac{JM_0}{v} \frac{\partial^2 M^z_1}{\partial \xi^2} - \frac{2A}{\mu_0 v} \left[ M^z_1 B^y_1 + M_0 B^y_2 - M^y_1 B^y_1 - M^z_1 B^z_0 \right] + \beta M_0 M^y_1.
\end{align}

Since \(B_0\) is a constant, from Eq.(4.15a), we get

\begin{equation}
B^z_1 = 0.
\end{equation}

Also, using Eqs.(4.12), we can rewrite Eqs.(4.15b) and (4.15c) as

\begin{align}
\frac{\partial}{\partial \xi} \left[ B^y_2 - kM^y_2 \right] &= -2vk\varepsilon_0 \left( \frac{\partial B^y_1}{\partial \tau} \right), \\
\frac{\partial}{\partial \xi} \left[ B^z_2 - kM^z_2 \right] &= -2vk\varepsilon_0 \left( \frac{\partial B^z_1}{\partial \tau} \right),
\end{align}
Similarly on using Eqs.(4.12), Eq.(4.16), can be written as

$$\frac{\partial M_i^y}{\partial \xi} = \frac{-J}{v} \left[ M_i^x \frac{\partial^2 M_i^x}{\partial \xi^2} - M_i^x \frac{\partial^2 M_i^y}{\partial \xi^2} \right] - \frac{2AM_0}{v\mu_0} [M_i^y(B_2^z - kM_2^z)] - M_i^z(B_2^y - kM_2^z)], \quad (4.19a)$$

$$\frac{\partial M_i^y}{\partial \xi} = \frac{JM_0}{v} \frac{\partial^2 M_i^x}{\partial \xi^2} - \frac{2A}{v\mu_0} [M_i^xB_1^x - M_0(B_2^z - kM_2^z)] + \frac{\beta M_0}{v} M_i^x, \quad (4.19b)$$

$$\frac{\partial M_i^x}{\partial \xi} = -\frac{JM_0}{v} \frac{\partial^2 M_i^x}{\partial \xi^2} - \frac{2A}{v\mu_0} [M_i^yB_1^y - M_i^yB_1^y + M_0(B_2^y - kM_2^y)] - \frac{\beta M_0}{v} M_i^y. \quad (4.19c)$$

On using Eqs.(4.17) and (4.18), we find that Eqs.(4.19) can be further reduced to

$$\frac{\partial M_i^x}{\partial \xi} = \left[ \frac{-J}{v} M_1 \wedge \left( \frac{\partial^2 M_1}{\partial \xi^2} - \frac{4\epsilon_0 k^2}{J\mu_0} \int_{-\infty}^{\xi} \frac{\partial M_1}{\partial \tau} d\xi' \right) \right], \quad (4.20a)$$

$$\frac{\partial M_i^y}{\partial \xi} = \frac{JM_0}{v} \frac{\partial^2 M_i^x}{\partial \xi^2} + \frac{2Ak}{v\mu_0} \left[ M_i^xM_i^x - 2\epsilon_0 M_0 k v \int_{-\infty}^{\xi} \frac{\partial M_i^x}{\partial \tau} d\xi' \right] + \frac{\beta M_0}{v} M_i^x, \quad (4.20b)$$

$$\frac{\partial M_i^z}{\partial \xi} = -\frac{JM_0}{v} \frac{\partial^2 M_i^y}{\partial \xi^2} - \frac{2Ak}{v\mu_0} \left[ M_i^yM_i^y - 2\epsilon_0 M_0 k v \int_{-\infty}^{\xi} \frac{\partial M_i^y}{\partial \tau} d\xi' \right] - \frac{\beta M_0}{v} M_i^y. \quad (4.20c)$$

By subtracting Eq.(4.20c) from Eq.(4.20b), we obtain the following equation

$$\frac{\partial}{\partial \xi} (M_i^y - iM_i^x) = \frac{JM_0}{v} \frac{\partial^2}{\partial \xi^2} (M_i^x - iM_i^y) + \frac{2Ak}{v\mu_0} [M_i^x(M_i^x - iM_i^y)]$$

$$- 2\epsilon_0 M_0 k v \int_{-\infty}^{\xi} \frac{\partial}{\partial \tau} (M_i^x - iM_i^y) d\xi' + \frac{\beta M_0}{v} (M_i^z - iM_i^y). \quad (4.21)$$

Similarly on adding Eqs.(4.20b) and (4.20c) we get the complex conjugate of Eq.(4.21). Since we study the propagation of EMW in an anisotropic ferromagnetic medium, we shall consider the direction of propagation parallel to the anisotropy axis. In order to identify Eq.(4.21) with a standard nonlinear
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We define a new complex field \[ \psi = (M_1^y - iM_1^z). \] (4.23)

Using this new field \( \psi \) in Eq.(4.14) i.e in equation obtained from the physical constraint on length of the magnetization vector is conserved, we find that
\[ M_1^z = \frac{-1}{2M_0} |\psi|^2. \] (4.24)

Further the above definition is in accordance with the nonuniform expansions (4.8). On substituting Eqs.(4.23) in Eq.(4.21) and differentiating once with respect to \( \xi \), we obtain
\[ i \frac{\partial \psi}{\partial \tau} - i \frac{\partial^3 \psi}{\partial \xi^3} + \frac{v}{J M_0} \frac{\partial^2 \psi}{\partial \xi^2} + i \gamma \frac{\partial}{\partial \xi} [\psi |^2 \psi] - \frac{i \beta}{J} \frac{\partial \psi}{\partial \xi} = 0, \] (4.25)

where \( \gamma = \frac{A_{kb}}{J M_0 \mu_0} \) and \( \tau \) is rescaled as \( \frac{4 A_{tub}}{J \mu_0} \tau \). We assume that the group velocity of the wave \( v \) is small and under this assumption Eq.(4.25) reduces to
\[ \frac{\partial \psi}{\partial \tau} - \frac{\partial^3 \psi}{\partial \xi^3} + \gamma \frac{\partial}{\partial \xi} [\psi |^2 \psi] - \frac{\beta}{J} \frac{\partial \psi}{\partial \xi} = 0 \] (4.26)

On using the transformation \( X = \xi + \frac{\theta}{\gamma} \tau \), Eq.(4.26) can be written as
\[ \psi_t - \psi_{XXX} + \gamma [\psi |^2 \psi]_x = 0. \] (4.27)

Here the suffixes \( X \) and \( \tau \) represent partial derivatives. It may be verified that on using the definitions (4.23), Eq.(4.20a) can also be rewritten as Eq.(4.27). Eq.(4.27) is the well known complex modified KdV equation. On further using
the transformation

\[ \psi(Z, T) = q(Z, T) \exp \left[ -i \left( \frac{k^n}{6} Z - \frac{k^n \gamma}{116} T \right) \right], \]  

(4.28a)

where

\[ Z = X + \frac{k^n \gamma}{12} \tau, \]  

(4.28b)

\[ T = \tau \]  

(4.28c)

we can rewrite Eq.(4.27) as

\[ iqT - \frac{k^n}{2} qzz + \beta |q|^2 q - iqzzz + i\gamma (|q|^2 q)_z = 0, \]  

(4.29)

where \( k^n = \frac{6\delta}{\gamma} \) and Eq.(4.29) represents the dynamics of complex pulse envelope amplitude of the light wave in monomode optical fibers in subpicosecond-femtosecond domain region. Mihalache and his coworkers have shown that Eq.(4.29) and hence Eq.(4.28) fails to pass the Painlevé test [252]. But still, the generalised form of Eq.(4.29) as a set of two coupled higher-order NLS (HNLS) has been solved for N-solitons [253], using Hirota’s bilinearization procedure. Using the same procedure we can find the bright and dark soliton solutions for the different sign of the GVD term \( k^n \) of the complex modified KdV equation (4.27) following ref. [253].

### 4.4.1. Bright EM and EM-spin solitons in an anisotropic medium

The sign of \( k^n \) varies according to the sign of \( \gamma \), which in turn depends on the sign of the saturation magnetization \( M_0 \) as all other terms in the expression for \( \gamma \) are fixed. The constraint on the length of the magnetization vector gives the
result that $M_0^2 = 1$ (Eq.(4.13)). If the propagating wave is in the anomalous GVD region then the corresponding saturation magnetization $M_0 = 1$ because of the relation $H_0 = (1 - \frac{1}{(s^2 \cdot v^2)})M_0$. then we get bright solitons while the EMW wave is in the normal GVD region $M_0 = -1$ dark soliton exists.

Now, on using the transformation

$$\psi = \frac{g(X, \tau)}{f(X, \tau)},$$  \hspace{1cm} (4.30)

where $g$ is a complex function and $f$ is a real function and using Hirota bilinear operators given in Appendix A Eq.(4.27) can be written as [253]

$$f^2 \left[ \{D_\tau - D_\chi^3\}g \cdot f \right] + \left[ D_\chi^3 \cdot f + \gamma gg^* \right] (3D_\chi g \cdot f) + \gamma gfD_\chi g^* = 0. \hspace{1cm} (4.31)$$

Eq.(4.31) can be decoupled as

$$A_1 g \cdot f = 0, \hspace{1cm} (4.32a)$$

$$A_2 f \cdot f = -\gamma gg^*, \hspace{1cm} (4.32b)$$

$$A_3 g \cdot g^* = 0, \hspace{1cm} (4.32c)$$

where the operators $A_1$, $A_2$ and $A_3$ are given by

$$A_1 = (D_\tau + D_\chi^3), \hspace{1cm} (4.33a)$$

$$A_2 = D_\chi^2, \hspace{1cm} (4.33b)$$

$$A_3 = D_\chi. \hspace{1cm} (4.33c)$$

Next we proceed in the standard way to construct the bright soliton solutions.
4.4.2. One-soliton solution

To find one-soliton solution, we assume $g$ and $f$ in the form given in Eq.(3.53) and substitute in Eqs.(4.33) and then collect terms with like powers of $\chi$ we obtain

\begin{align*}
\chi^1 & : \mathcal{A}_1 g_1 \cdot 1 = 0, \quad \mathcal{A}_2 (1 \cdot f_2 + f_2 \cdot 1) = -\gamma g_1 g_1^* \tag{4.34a} \\
\chi^2 & : \mathcal{A}_3 g_1 \cdot g_1^* = 0, \tag{4.34b} \\
\chi^3 & : \mathcal{A}_4 g_1 \cdot f_2 = 0, \tag{4.34c} \\
\chi^4 & : \mathcal{A}_5 (f_2 \cdot f_2) = 0. \tag{4.34d}
\end{align*}

The solution which is consistent with Eqs.(4.34) for the present case is given as

\begin{align*}
g_1 &= \exp(k_1 + \eta_0), \tag{4.35a} \\
f_2 &= \frac{C_1}{k_1^2} \exp(2k_1), \tag{4.35b}
\end{align*}

where

\begin{align*}
k_1 &= k_1 (X + k_1^2 \tau), \tag{4.36a} \\
C_1 &= -\frac{\gamma}{8} \exp(\eta_0 + \eta_0^*), \tag{4.36b}
\end{align*}

where $\eta_0$ is a complex constant while all other parameters are real. Using Eq.(4.35) and (3.53) in (4.30), after absorbing $\chi$, the bright one-soliton solution is found as [253]

\[ \psi = k_1 \rho \text{sech} \left[ k_1 (X + k_1^2 \tau) + \delta_0 \right]. \tag{4.37} \]
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Here

\[ \rho_1 = \left( \frac{\exp(2\eta_0)}{4C_1} \right)^{\frac{1}{2}}, \quad (4.38a) \]
\[ \delta_0 = \frac{1}{2} \ln \left( \frac{C_1}{k_1^2} \right). \quad (4.38b) \]

Using Eq.(4.37) in the relations (4.23), and also in Eqs.(4.12) we obtain one-soliton for the components of the magnetization of the ferromagnetic medium and the magnetic induction of the EMW as

\[ M_1^x = k_1^2 |\rho_1|^2 \text{sech}^2 \left[ k_1 (X + k_1^2 \tau) + \delta_0 \right], \quad (4.39a) \]
\[ M_1^y = k^{-1} B_1^y = k_1 (Re \rho_1) \text{sech} \left[ k_1 (X + k_1^2 \tau) + \delta_0 \right], \quad (4.39b) \]
\[ M_1^z = k^{-1} B_1^z = k_1 (Im \rho_1) \text{sech} \left[ k_1 (X + k_1^2 \tau) + \delta_0 \right]. \quad (4.39c) \]

Here \( Re \rho_1 \) and \( Im \rho_1 \) represent the real and imaginary parts of \( \rho_1 \) respectively.\n
Knowing \( B \) and \( M \), the one-soliton corresponding to magnetic field \( H \) of the EMW can be obtained using the linear constitutive relation between \( H, B \) and \( M \) given in Eq.(1.12) as

\[ H_1^x = -k_1^2 |\rho_1|^2 \text{sech}^2 \left[ k_1 (X + k_1^2 \tau) + \delta_0 \right], \quad (4.40a) \]
\[ H_1^y = \left[ \frac{k}{\mu_0} - 1 \right] k_1 (Re \rho_1) \text{sech} \left[ k_1 (X + k_1^2 \tau) + \delta_0 \right], \quad (4.40b) \]
\[ H_1^z = \left[ \frac{k}{\mu_0} - 1 \right] k_1 (Im \rho_1) \text{sech} \left[ k_1 (X + k_1^2 \tau) + \delta_0 \right]. \quad (4.40c) \]

The above results show that the excitation of magnetization, the magnetic induction and hence the magnetic field of the EMW are highly localized and appear in the form of solitons. We find that \( B_1^x \) and also the higher orders of \( B^x \) namely \( B_2^x, B_3^x, \ldots \) vanish due to the linear relation between the components of \( B_1 \) and \( M_1 \), which is actually the solutions of the LL equation at the zeroth
order of perturbation. Hence, the magnetic induction of the EMW is restricted to the plane normal to the direction of propagation (i.e. $y-z$ plane) as the EMW propagates in the charge-free ferromagnetic medium. However, the soliton-like excitations of the magnetization of the medium is not restricted to any particular plane.

### 4.4.3. Two-soliton solutions

Next, to find the 2-soliton solutions we assume $g$ and $f$ as given in Eqs.(3.63) and proceed as before to obtain [253]

\[ g_1 = \exp(k_1 + \eta_0) + \exp(k_2 + \eta_0), \]  
\[ g_3 = \frac{C_1(k_1 - k_2)^2}{(k_1 + k_2)^2} \left[ \exp \left( \frac{2k_1 + k_1 + \eta_0}{k_1^2} \right) + \exp \left( \frac{k_1 + 2k_2 + \eta_0}{k_1^2} \right) \right], \]  
\[ f_2 = C_1 \left[ \frac{\exp(2k_1)}{k_1^2} + \frac{8\exp(k_1 + k_2)}{(k_1 + k_2)^2} + \frac{\exp(2k_2)}{k_2^2} \right], \]  
\[ f_4 = \frac{C_1^2(k_1 - k_2)^4}{k_1^2k_2^2(k_1 + k_2)^4} \exp(2(k_1 + k_2)), \]

where $\eta_j = k_j(X + k_j^2r)$, $j = 1,2.$ and $k_1$ and $k_2$ are real parameters. Using Eq.(4.41) in Eq.(3.63), we find that the 2-soliton solutions of Eq.(4.27) is written as

\[ \psi = \frac{2\rho_1(k_1 + k_2)[k_2\cosh(k_1 + \delta_2) + k_1\cosh(k_2 + \delta_2)]}{|k_1 - k_2| \left[ \cosh(k_1 + k_2 + \delta_4) + \frac{(k_1+k_2)^2}{(k_1-k_2)^2}\cosh(k_1 - k_2 + \delta_0 - \delta_1) - C_2 \right]}, \]

where

\[ \delta_1 = \frac{1}{2} \ln \left( \frac{C_1}{k_2^2} \right), \]
4.4 Magnetic and EM soliton in an anisotropic ferromagnetic medium

\[ \delta_2(3) = \frac{1}{2} \ln \left[ \frac{C_1(k_1 - k_2)^2}{S_2(3)(k_1 + k_2)^2} \right], \]  

(4.42c)

\[ \delta_4 = \frac{1}{2} \ln \left[ \frac{C_2^2(k_1 - k_2)^4}{k_1^2k_2^2(k_1 + k_2)^4} \right], \]  

(4.42d)

\[ C_2 = \frac{8k_1k_2}{(k_1 - k_2)^2}. \]  

(4.42e)

Similarly using Eqs. (4.42a) in Eq. (4.23) and (4.12), we can find 2-soliton form of the components of magnetization and using the constitutive relation, we obtain the magnetic field components at the lowest order of expansion.

4.4.4. Multisoliton solutions

Further, to find the N-soliton solutions, we can generalize the expressions for \( g \) and \( f \) as [253]

\[ g(X, \tau) = \sum_{\mu=0,1} R_1(\mu) \exp \left[ \sum_{j=1}^{2N} \mu_j \eta_j + \sum_{j=1}^{2N} \mu_i \mu_j \varphi_j \right], \]  

(4.43a)

\[ f(X, \tau) = \sum_{\mu=0,1} R_2(\mu) \exp \left[ \sum_{j=1}^{2N} \mu_j \eta_j + \sum_{j=1}^{2N} \mu_i \mu_j \varphi_j \right]. \]  

(4.43b)

Here \( \eta_j = k_j (X + k_j^2\tau), \ j = 1, 2, ... N \)

\[ \exp(\varphi_{ij}) = \begin{cases} \frac{4C_1}{(k_i + k_j)^3}, & \text{for } i = 1, 2, ... N \text{ and } j = N + 1, ... 2N, \\ \frac{(k_i - k_j)^2}{4C_1}, & \text{for } i = 1, 2, ... N \text{ and } j = 1, 2, ... N, \\ i = N + 1, N + 2, ... 2N \text{ and } j = N + 1, ... 2N, \end{cases} \]  

(4.44a)
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\[
R_1(\mu) = \begin{cases} 
\exp(\gamma_0) & \text{when } 1 + \sum_{i=1}^{N} \mu_{i+N} = \sum_{i=1}^{N} \mu_i, \\
0 & \text{otherwise},
\end{cases} \tag{4.44b}
\]

\[
R_2(\mu) = \begin{cases} 
1 & \text{when } 1 + \sum_{i=1}^{N} \mu_{i+N} = \sum_{i=1}^{N} \mu_i, \\
0 & \text{otherwise},
\end{cases} \tag{4.44c}
\]

### 4.4.5. Dark soliton solution

Similarly, in order to construct the exact dark soliton solution, Eq.(4.31) can be decoupled into different set of bilinear equations as [253]

\[
B_1 g \cdot f = 0, \tag{4.45a}
\]
\[
B_2 f \cdot f = -\gamma gg^*, \tag{4.45b}
\]
\[
\mathcal{A}_3 g \cdot g^* = 0, \tag{4.45c}
\]

where the operators \( B_1 \) and \( B_2 \) are given as

\[
B_1 = (D_r - D_x^3 + 3\lambda D_x), \tag{4.46a}
\]
\[
B_2 = D_x^2 - \lambda. \tag{4.46b}
\]

Here \( \lambda \) is a constant. In order to find the dark one-soliton we assume [253]

\[
g = \gamma_1 (1 + \chi g_1), \tag{4.47a}
\]
\[
f = 1 + \chi f_1, \tag{4.47b}
\]
where $\gamma_1$ is a complex constant. Using Eq.(4.47) in Eq.(4.45) and collecting the coefficients of different powers of $\chi$, we get

\begin{align*}
\chi^0 : & \quad |\gamma_1|^2 = \frac{\lambda}{\gamma}, \quad B_1(1 \cdot f_1 + g_1 \cdot 1) = 0, \quad B_2(1 \cdot f_1 + f_1 \cdot 1) = -2\gamma|\gamma_1|^2 g_1 \quad (4.48a) \\
\chi^1 : & \quad A_3(g_1 \cdot 1) = 0 \quad B_1(g_1 \cdot f_1) = 0, \quad (4.48b) \\
\chi^2 : & \quad B_2 f_1 \cdot f_1 = -\gamma|\gamma_1|^2 g_1^2. \quad (4.48c)
\end{align*}

We find that the system of equations (4.48) satisfies the following forms of $g_1$ and $f_1$

$$g_1 = -f_1 = -\exp \left[ m_1(X - \frac{m_1^2}{2} \tau) + \xi^{(0)}_1 \right], \quad (4.49)$$

where $m_1^2 = \gamma|\gamma_1|^2$, $m_1$ and $\xi^{(0)}_1$ are real constants. Using Eqs.(4.49) in Eqs.(4.47) and then in the bilinear transformation (4.30), we obtain the dark-one soliton for $\psi$ as

$$\psi = \gamma_1 \exp(\pm i\pi) \tanh \left[ \frac{1}{2} \{ m_1(X - \frac{1}{2} m_1^2 \tau) + \xi^{(0)}_1 \} \right]. \quad (4.50)$$

Using Eq.(4.50) in the relations (4.23), and in Eqs.(4.12), we can find the dark soliton form of the magnetization and the magnetic induction as

\begin{align*}
M_1^x &= \gamma_1^2 \tanh^2 \left[ \frac{1}{2} \{ m_1(X - \frac{1}{2} m_1^2 \tau) + \xi^{(0)}_1 \} \right], \quad (4.51a) \\
M_1^y &= k^{-1} B_1^y = -\frac{\gamma_1}{2} \exp(i\pi) \mp \exp(-i\pi) \tanh \left[ \frac{1}{2} \{ m_1(X - \frac{1}{2} m_1^2 \tau) + \xi^{(0)}_1 \} \right], \quad (4.51b) \\
M_1^z &= k^{-1} B_1^z = \frac{i\gamma_1}{2} \exp(i\pi) \pm \exp(-i\pi) \tanh \left[ \frac{1}{2} \{ m_1(X - \frac{1}{2} m_1^2 \tau) + \xi^{(0)}_1 \} \right], \quad (4.51c)
\end{align*}

Similarly by using Eqs.(4.51) in the relation (1.12) we can calculate the mag-
netic field components as

\[ H_i^x = \gamma_i^2 \tanh^2 \left[ \frac{1}{2} \left( m - \frac{1}{2} \gamma \tau \right) + \xi_1^{(0)} \right] , \]  

(4.52a)

\[ H_i^y \equiv k^{-1} B_i^y = -\frac{\gamma_i}{2} \left( k - \frac{1}{\mu_0} \right) \left( \exp(i\pi) \mp \exp(-i\pi) \right) \tanh \left[ \frac{1}{2} \left( m - \frac{1}{2} \gamma \tau \right) + \xi_1^{(0)} \right] , \]  

(4.52b)

\[ H_i^z \equiv k^{-1} B_i^z = \frac{i\gamma_i}{2} \left( k - \frac{1}{\mu_0} \right) \left( \exp(i\pi) \mp \exp(-i\pi) \right) \tanh \left[ \frac{1}{2} \left( m - \frac{1}{2} \gamma \tau \right) + \xi_1^{(0)} \right] , \]  

(4.52c)

We find that the dark solitons of the magnetic induction is restricted to the \( y - z \) plane while that of the magnetic field and magnetization do not suffer any restrictions.

**4.5. EM soliton in treated anisotropic medium**

As discussed in sec.(1.7.1) ferromagnetic materials when treated exhibit non-linear relationship between the magnetic induction and the magnetic field in the form of hysteresis. It was pointed out that \( B \) is a multivalued function of \( H \) and the function \( f(H) \) depends on the history of preparation of the ferromagnetic material. So, in this section we study the propagation of EMW in an anisotropic treated ferromagnetic medium that possesses a nonlinear constitutive relation connecting the magnetic field, the magnetic induction and the
magnetization of the ferromagnetic medium which is written as

\[ H = \frac{1}{\mu_0} [B + B_{NL}] - M. \]  \tag{4.53}

where \( B_{NL} \) is the nonlinear component of the magnetic induction. Assuming that the magnetization \( M \) of the ferromagnetic medium is very large compared to the magnetic field \( H \) of the electromagnetic field in practice, we treat \( B \) as proportional to \( M \). Therefore, for our analysis we choose the nonlinear part of the magnetic induction \( B_{NL} \) in terms of the magnetization of the ferromagnetic medium. Specifically, we write \( B_{NL} \) in the form

\[ B_{NL} = -\vartheta \mu_0 \int (\nabla \cdot M) M d\vec{x}, \] \tag{4.54}

where \( \vartheta \) is the nonlinearity parameter. As it appears in Eq.(4.54), the nonlinear part of the magnetic induction \( B_{NL} \) is characterized by an effective field density which appears in the form of the product of the divergence of the magnetization in a small volume of the ferromagnetic material and the magnetization in the same volume \( \int (\nabla \cdot M) M d\vec{x} \). Using Eqs.(4.53) and (4.54) in Eq.(4.1), we obtain

\[ \frac{\partial M}{\partial t} = M \wedge \left[ J \nabla^2 M - 2\beta M^2 n + \frac{2A}{\mu_0} \left( B - \omega \mu_0 \int (\nabla \cdot M) M d\vec{x} \right) \right], \quad M^2 = 1 \] \tag{4.55}

Similarly Eq.(3.1) after using Eq.(4.54) takes the form

\[ \left[ \epsilon^2 \nabla^2 - \frac{\partial^2}{\partial t^2} \right] B = \frac{1}{\epsilon_0} \left[ \nabla^2 M - \nabla(\nabla \cdot M) + \frac{1}{\mu_0} \nabla \wedge (\nabla \wedge B_{NL}) \right]. \] \tag{4.56}

Here \( B_{NL} \) assumes the form given in Eq.(4.54). Thus, the set of coupled equations (4.55) and (4.56) with \( B_{NL} \) as given in Eq. (4.54) completely describe the propagation of EMW in an anisotropic charge-free nonlinear ferromagnetic
medium.

### 4.5.1. Magnetization excitations and EMW modulation in a treated ferromagnetic medium

Now, we try to solve the set of coupled equations (4.55) and (4.56) with $B_{NL}$ as given in Eq.(4.54). As in the previous case, in order to reduce the mathematical complexity we consider the one dimensional form of Eqs.(4.55) and (4.56). We also stretch the wave and time variables to the same extent as in the case of the untreated medium (as given in Eqs.(4.6)), and also rescale and redefine the parameters $J$ and $\beta$ as done earlier ($J \rightarrow \varepsilon^{-1} J$ and $\beta \rightarrow \varepsilon \beta$). Then we expand the components of magnetization and magnetic induction nonuniformly as given in Eqs.(4.8) and (4.9) and substitute in the one dimensional form of Eqs.(4.55) and (4.56). We then collect the coefficients of different powers of $\varepsilon$ and solve the resultant equations. At $O(\varepsilon^0)$ we get the same solutions (4.12) as in the untreated case and at $O(\varepsilon^1)$. From Eq.(4.56), we obtain $B_i^z = 0$ and the other equations read as

$$
\frac{\partial^2 B_y^y}{\partial \xi^2} - k \frac{\partial^2 M_y^y}{\partial \xi^2} = \frac{2v}{(c^2 - v^2)} \frac{\partial^2 B_y^y}{\partial \xi \partial \tau} + \phi k \frac{\partial}{\partial \xi} \left[ M_y^y \frac{\partial M_y^y}{\partial \xi} \right], \quad (4.57a)
$$

$$
\frac{\partial^2 B_z^z}{\partial \xi^2} - k \frac{\partial^2 M_z^z}{\partial \xi^2} = \frac{2v}{(c^2 - v^2)} \frac{\partial^2 B_z^z}{\partial \xi \partial \tau} + \phi k \frac{\partial}{\partial \xi} \left[ M_z^z \frac{\partial M_z^z}{\partial \xi} \right]. \quad (4.57b)
$$

Eqs.(4.57) on simplification reduces to

$$
\frac{\partial}{\partial \xi} \left[ B_y^y - k M_y^y \right] = k \left( \phi M_z^z \frac{\partial M_z^z}{\partial \xi} - 2v \varepsilon_0 \frac{\partial B_z^z}{\partial \tau} \right), \quad (4.58a)
$$

$$
\frac{\partial}{\partial \xi} \left[ B_z^z - k M_z^z \right] = k \left( \phi M_z^z \frac{\partial M_z^z}{\partial \xi} - 2v \varepsilon_0 \frac{\partial B_z^z}{\partial \tau} \right). \quad (4.58b)
$$
Similarly the equations obtained from Eqs. (4.55) at $0(\epsilon^1)$ are similar to Eqs. (4.19), which after using Eqs. (4.12) and (4.58) can be written as

\[
\frac{\partial M_f}{\partial \xi} = \frac{-J}{v} M_f \frac{\partial^2 M_f}{\partial \xi^2} - M_f \frac{\partial^2 M_f}{\partial \xi^2} + \frac{4A_k^2 \epsilon_0}{\mu_0} \left[ M_f \int_{-\infty}^{\xi} \frac{\partial M_f}{\partial \tau} d\xi' - M_f \int_{-\infty}^{\xi} \frac{\partial M_f}{\partial \tau} d\xi' \right] - \frac{2A \delta k}{\mu_0 v} \left[ M_f \int_{-\infty}^{\xi} M_1 \frac{\partial M_f}{\partial \xi} d\xi' - M_1 \int_{-\infty}^{\xi} M_f \frac{\partial M_f}{\partial \xi} d\xi' \right], \tag{4.59a}
\]

\[
\frac{\partial M_l}{\partial \xi} = \frac{J M_0 \delta^2 M_f}{v \delta \xi^2} + \frac{2A_k}{v \mu_0} \left[ M_f M_1 - 2 \epsilon_0 M_0 v \int_{-\infty}^{\xi} \frac{\partial M_f}{\partial \tau} d\xi' \right] + \frac{2A \delta M_0 k}{v \mu_0} \left[ \int_{-\infty}^{\xi} M_l \frac{\partial M_f}{\partial \xi} d\xi' \right] + \frac{\beta M_0}{v} M_l, \tag{4.59b}
\]

\[
\frac{\partial M_l}{\partial \xi} = \frac{-J M_0 \delta^2 M_f}{v \delta \xi^2} + \frac{2A_k}{v \mu_0} \left[ M_f M_l - 2 \epsilon_0 M_0 v \int_{-\infty}^{\xi} \frac{\partial M_l}{\partial \tau} d\xi' \right] - \frac{2A \delta M_0 k}{v \mu_0} \left[ \int_{-\infty}^{\xi} M_l \frac{\partial M_f}{\partial \xi} d\xi' \right] - \frac{\beta M_0}{v} M_l. \tag{4.59c}
\]

As before, by combining Eqs. (4.59b) and (4.59c) and using the new field $\psi$ as defined in Eqs. (4.23), we rewrite them as

\[
i \frac{\partial \psi}{\partial \tau} - i \frac{\partial^3 \psi}{\partial \xi^3} + \frac{\psi}{JM_0 \delta \xi^2} + i \gamma \frac{\partial}{\partial \xi} \left[ \psi \left| \psi \right|^2 \right] + i \gamma' \frac{\partial}{\partial \xi} \left[ \left| \psi \right|^2 \right] - i \frac{\beta}{J} \frac{\partial \psi}{\partial \xi} = 0. \tag{4.60}
\]

where $\gamma = \frac{A_k}{JM_0 \mu_0}$, $\gamma' = \frac{A_0 k}{J_0 \mu_0}$ and $\tau$ is rescaled as $\frac{4A_0 k^2}{J_0 \mu_0} \tau$. Using the same assumption that the group velocity of the wave $v$ is small Eq. (4.60) reduces to

\[
i \frac{\partial \psi}{\partial \tau} - i \frac{\partial^3 \psi}{\partial \xi^3} + i \gamma \frac{\partial}{\partial \xi} \left[ \psi \left| \psi \right|^2 \right] + i \gamma' \frac{\partial}{\partial \xi} \left[ \left| \psi \right|^2 \right] - i \frac{\beta}{J} \frac{\partial \psi}{\partial \xi} = 0. \tag{4.61}
\]
Now making the transformation \( \psi(\zeta, \xi) = q(\zeta, \xi) \exp \left[ \frac{i}{8} (-\zeta + \xi) \right] \), where \( \zeta = \xi + \frac{i}{18} \) and \( \xi = \tau \), Eq.(4.61) can be written as

\[
i q_t + \frac{1}{2} q_{\zeta \zeta} + i \lambda q_{\zeta} + \beta q + \gamma' q^2 q + i \lambda q_{\zeta \zeta} + \beta q + \gamma' q^2 q + \gamma(1 q^2)q = 0,
\]

(4.62)

where \( \lambda = \frac{\beta}{J - 1/36} \), \( \beta = \frac{\beta}{6J - 1/216} \), \( \gamma = 1 \), \( \gamma' = 6 \gamma \) and \( \beta = \gamma[1/M_0 + \vartheta] \). In Eq.(4.62) the terms proportional to \( \lambda \) and \( \beta \) can be transformed using a simple transformation. Using Painlevé analysis it is established that the Eq.(4.62) is integrable and possess soliton solution only for the following parametric restrictions \( \gamma' = 1 \) and \( \gamma : \beta : \gamma = 1 : 6 : 3 \). Hence by choosing \( \lambda = 6J_0 M_0^2, \beta = J_6, \vartheta = \frac{-1}{2M_0} \) we can achieve the above parametric restrictions. Under the above conditions Eq.(4.62) can be rewritten as

\[
i q_t + \frac{1}{2} q_{\zeta \zeta} + q^2 q + i \left[ q_{\zeta \zeta} + 6 q^2 q + 3(1 q^2)q \right] = 0.
\]

(4.63)

Eq.(4.63) is a well known integrable nonlinear evolution equation proposed by Sasa and Satsuma [254] while studying the integrability aspects of generalized nonlinear Schrödinger equations and N-soliton solutions for Eq.(4.63) were obtained by them using IST method. Eq.(4.63) also describes the propagation of optical pulses in the femtosecond regime through monomode optical fibers when the higher order effect of the medium namely the Raman process is taken into account [255]. The Lax pair for this equation is given as [254]

\[
L = \begin{pmatrix}
-i\lambda & 0 & q \\
0 & 1\lambda & q^* \\
-q^* & -q & -1\lambda
\end{pmatrix}
\]

(4.64a)
4.6 EM soliton in a treated ferromagnetic medium

where λ is the isospectral parameter. Due to the complicated nature of IST, we use Hirota’s direct method to find the soliton solution of Eq.(4.63).

4.6. EM soliton in a treated ferromagnetic medium

Having derived the single equation (4.63) determining the modulation of the EMW and the magnetization excitations induced in the medium, now we solve the equation for soliton solutions. Using the transformation (4.30), where \( f = f(\zeta, \bar{\theta}) \) and \( g = g(\zeta, \bar{\theta}) \), Eq.(4.63) can be decoupled into the bilinear equations (4.32), where the form of bilinear operators are

\[
\mathcal{A}_1 = (iD_t + \frac{1}{2}D^2 \zeta - D^3 \zeta), \\
\mathcal{A}_2 = D^2 \zeta, \\
\mathcal{A}_3 = D \zeta.
\]
On solving the above bilinear equations using the same power series solutions for $g$ and $f$ as in Sec. (4.4.1), we obtain the one soliton as

\[ q(\zeta, \xi) = \frac{k_1}{2} \text{sech} \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right] \exp \left[ \frac{i k_1^2 \xi}{2} \right]. \] (4.66)

Here $k_1$ is real parameter. Using this in Eqs. (4.23) we can obtain the magnetization and the magnetic induction components at the lowest order of expansion as

\[ M_1^x = \frac{k_1^3}{4} \text{sech}^2 \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right], \] (4.67a)

\[ M_1^y = k^{-1} B_1^y = \frac{k_1}{4} \text{sech} \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right] \cos \left[ \frac{k_1^2 \xi}{2} \right], \] (4.67b)

\[ M_1^z = k^{-1} B_1^z = -\frac{k_1}{4} \text{sech} \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right] \sin \left[ \frac{k_1^2 \xi}{2} \right]. \] (4.67c)

Using Eqs. (4.66) in Eq. (4.54), we can find $B_{NL}$ whose component forms are written as

\[ B_{1, NL}^x = \frac{-\vartheta k_1^4 \mu_0}{32} \text{sech}^4 \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right], \] (4.68a)

\[ B_{1, NL}^y = \frac{\vartheta \mu_0}{12 k_1} \text{sech}^3 \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right] \cos \left[ \frac{k_1^2 \xi}{2} \right], \] (4.68b)

\[ B_{1, NL}^z = \frac{-\vartheta \mu_0}{12 k_1} \text{sech}^3 \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right] \sin \left[ \frac{k_1^2 \xi}{2} \right]. \] (4.68c)

Using this in Eq. (4.53), we obtain the magnetic field components as

\[ H_1^x = \frac{k_1^2}{4} \text{sech}^2 \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right] \left\{ \frac{\vartheta k_1^2}{8} \text{sech}^2 \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right] - 1 \right\}, \] (4.69a)

\[ H_1^y = \text{sech} \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right] \cos \left[ \frac{k_1^2 \xi}{2} \right] \left\{ G + \frac{\vartheta}{12 k_1} \text{sech}^2 \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right] \right\}, \] (4.69b)

\[ H_1^z = -\text{sech} \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right] \sin \left[ \frac{k_1^2 \xi}{2} \right] \left\{ G + \frac{\vartheta}{12 k_1} \text{sech}^2 \left[ k_1 (k_1 \zeta + k_1^2 \xi) \right] \right\}, \] (4.69c)
where \( G = [\eta_1/4][k/\mu_0 - 1] \). Thus, like the untreated anisotropic medium, the excitations of the components of magnetization, magnetic field and magnetic induction are highly localized and appear in the form of solitons as the EM propagates through the nonlinear ferromagnetic medium. Again in the present case we find that \( B_1^x, B_2^x, B_3^x, \ldots \) vanish. Thus, the soliton excitation of the magnetic induction of the EMW is restricted to the plane normal to the direction of propagation (i.e. y-z plane) as the EMW propagates in the charge-free nonlinear ferromagnetic medium. However, the soliton excitations of the magnetization of the medium and the magnetic field are not restricted to any particular plane.

Proceeding further, as carried out in sec.(3.5.1.B), we can find the two soliton solution of Eq.(4.63) as

\[
q(\zeta, \tilde{t}) = \frac{\sqrt{\eta^+}}{\eta^-} \left\{ k_1 \cosh[k_2 \zeta + k_3 \tilde{t}] \exp\left(\frac{ik_4 \tilde{t}}{2}\right) + k_2 \cosh[k_1 \zeta + k_3 \tilde{t}] \exp\left(\frac{-ik_4 \tilde{t}}{2}\right) \right\},
\]

where \( \eta^+ = k_1 + k_2 \) and \( \eta^- = k_1 - k_2 \). Generalizing the expressions for \( g \) and \( f \), the N-soliton solutions are found to be similar to ones given in Eqs.(4.43) where now \( \eta_j = X_j^x + \eta_j^{(0)} \), \( X_j = \frac{\eta_j^g}{2} + \eta_j^\mu \), \( R_1(\mu) = R_2(\mu) = 1 \) and

\[
\exp(\varphi_{ij}) = \begin{cases} 
\frac{2}{(\eta_i + \eta_j)^2}, & \text{for } i = 1, 2, \ldots N \text{ and } j = N + 1, \ldots 2N, \\
\ln\left(\frac{\eta_i - \eta_j}{2}\right)^2, & \text{for } i = 1, 2, \ldots N \text{ and } j = 1, 2, \ldots N, \\
\ln\left(\frac{\eta_i - \eta_j}{2}\right)^2, & \text{for } i = N + 1, N + 2, \ldots 2N \text{ and } j = N + 1, \ldots 2N.
\end{cases}
\]  

4.7 Simultaneous propagation of many EM signals

In long distance optical communication systems the bit rate of transmission is limited by the dispersive character of the medium which causes the pulse to spread out and eventually overlap to such an extent that all the information is
lost. Dispersion of a pulse is an important consideration in the design of communication systems, since it plays a major role in determining the maximum transmission distance between a transmitter and a receiver and information transmission rate. However, as pointed out in sec.(2.4), for the pulses in the optical frequency range this limitation is overcome by the nonlinear Kerr effect of the medium so that the pulse propagates in the form of soliton. In the previous sections it was shown that ferromagnetic medium can also help the propagation of EMW in the microwave frequency range. We found that the nonlinear effects in the medium modulate the amplitude of the propagating EMW in the form of soliton. But the above studies pertain to the propagation of single EM pulse through the ferromagnetic medium. However, in real communication, a whole lot of signals have to be sent simultaneously, without overlapping for error free communication. Therefore, it is important to study the simultaneous propagation of many number of signals say \( n \)-signals of different amplitudes through a ferromagnetic medium. In this section we study the amplitude modulation of the \( n \) copropagating signals in a charge-free anisotropic ferromagnetic medium.

Let us consider the propagation of \( n \) signals with the magnetic field components \( H_1, H_2, ..., H_n \). The effective field created by the copropagating signals that interact with the medium is given by \( H_e = (H_e^x, H_e^y, H_e^z) = \sum_{j=1}^{n} H_j (= H_1 + H_2 + ... + H_n) \). Here we have considered the linear superposition of the magnetic field components of the propagating EM signals and we assume that there is no phase matching. Further, in the case of communication systems as the length of the medium is very large compared to the other dimensions, we can study the propagation in the one-dimensional medium. Then in the absence of static and moving charges, the propagation of the EM signals in a
one dimensional magnetic medium is given by the Maxwell's equations.

\[
\frac{\partial^2 H_e}{\partial x^2} - \varepsilon \frac{\partial^2 H^x_e}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} (H_e + M),
\]

where \( H_e \) is the effective magnetic field created by the \( n \)-copropagating signals. Here we have chosen the direction of propagation to be along \( x \)-axis. The magnetization dynamics of the anisotropic medium in the presence of \( n \)-signals is governed by the LL equation given as

\[
\frac{\partial M}{\partial t} = M \wedge \left[ J \frac{\partial^2 M}{\partial x^2} + 2A H_e \right], \quad M^2 = 1.
\]

Here again the external magnetic field is replaced by the effective field \( H_e \) due to the propagation of \( n \)-signals simultaneously in the medium. Also, the anisotropy is assumed to be in \( x \)-direction.

4.7.1. Modulation of \( n \)-EM signals

We now study the nonlinear amplitude modulation of the slowly varying EM plane wave in the ferromagnetic medium reductive perturbation method. For this, we introduce slow variables given in Eqs.(4.6). As we have considered an anisotropic ferromagnetic medium with easy axis of magnetization along the direction of propagation (i.e. along \( x \)-direction), we nonuniformly expand the magnetic field components of the EM signals about uniform magnetic fields \( H_{j,(0)} \) by assuming that the fields are dominant along the the \( x \)-direction. Thus
we write

\[ H_e^x = H(0) + \varepsilon H_{e,(1)}^x + \varepsilon^2 H_{e,(2)}^x + \ldots, \]  

\[ H_e^y = \varepsilon^\frac{1}{2} \left[ H_{e,(1)}^y + \varepsilon H_{e,(2)}^y + \ldots \right], \]  

\[ H_e^z = \varepsilon^\frac{1}{2} \left[ H_{e,(1)}^z + \varepsilon H_{e,(2)}^z + \ldots \right]. \]

The magnetization of the ferromagnetic medium is also expanded in the same way as in Eqs. (4.8). Before proceeding further we assume that in accordance with natural ferromagnetic materials the ferromagnetic medium possesses strong exchange interaction and weak anisotropy. This is implemented in the problem by rescaling \( J \) and \( \beta \) as \( J \rightarrow \varepsilon^{-1}J \) and \( \beta \rightarrow \varepsilon^\frac{\beta}{2} \) respectively. Now we substitute the above expansions of \( H_e \) and \( M \) into the component form of Eqs. (2.36) and (4.1), collect the terms proportional to different powers of \( \varepsilon \) and solve them.

On solving the equations at \( O(\varepsilon^0) \), we arrive at the relations \( M_{(1)}^y = \frac{1}{k} H_{e,(1)}^y \), \( M_{(0)}^z = \frac{1}{k} H_{e,(1)}^z \) and \( M_{(0)}^2 = 1 \) from the the equation for the constraint on the length of magnetization vector \( M^2 = 1 \). Here \( k = \frac{\varepsilon \beta}{\sqrt{\varepsilon^2 - \nu^2}} = \frac{1}{M_{(0)}} H_{e,(0)}. \) At \( O(\varepsilon^1) \), after using the results of \( O(\varepsilon^0) \) we obtain \( M_{(1)}^z = -H_{e,(1)}^z = \frac{1}{M_{(0)}} \left[ (H_{e,(1)}^x)^2 + (H_{e,(1)}^y)^2 \right] \) and

\[
\frac{\partial H_{e,(1)}^x}{\partial \xi} = \frac{J}{k^2 \nu} \left[ H_{e,(1)}^x \wedge \left( \frac{\partial^2 H_{e,(1)}^x}{\partial \xi^2} - \frac{4Ak(1+k)}{J} \int_{-\infty}^{\xi} \frac{\partial H_{e,(1)}^x}{\partial \tau} d\xi' \right) \right], \]  

\[
\frac{\partial H_{e,(1)}^y}{\partial \xi} = \frac{JM_{(0)}^2}{\nu} \left[ \frac{\partial^2 H_{e,(1)}^y}{\partial \xi^2} - \frac{2A(1+k)}{JM_{(0)}} \left( H_{e,(1)}^z H_{e,(1)}^x + \frac{2k}{v} \int_{-\infty}^{\xi} \frac{\partial H_{e,(1)}^x}{\partial \tau} d\xi' \right) \right] + \frac{2\beta M_{(0)}^2}{v} H_{e,(1)}^z, \]
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\[
\frac{\partial H_{e,(1)}^z}{\partial \xi} = -\frac{JM(0)}{v} \left[ \frac{\partial^2 H_{e,(1)}^x}{\partial \xi^2} + \frac{2A(1+k)}{JM(0)} \left( H_{e,(1)}^x H_{e,(1)}^y + \frac{2k}{v} \int_{-\infty}^{\xi} \frac{\partial H_{e,(1)}^y}{\partial \tau} d\xi' \right) \right] - \frac{2\beta M(0)}{v} H_{e,(1)}^y.
\]

(4.75c)

Now Eqs. (4.75b) and (4.75c) can be combined to give

\[
J \left[ \frac{\partial^2}{\partial \xi^2} \left[ H_{e,(1)}^z + iH_{e,(1)}^y \right] - \frac{2A(1+k)}{JM(0)} \left( H_{e,(1)}^x \left[ H_{e,(1)}^x + iH_{e,(1)}^y \right] + \frac{2k}{v} \int_{-\infty}^{\xi} \partial_x \left[ H_{e,(1)}^z + iH_{e,(1)}^y \right] d\xi' \right) \right]
- \frac{v}{M(0)} \partial_x \left[ H_{e,(1)}^y - iH_{e,(1)}^z \right] + 2\beta \left[ H_{e,(1)}^z + iH_{e,(1)}^y \right] = 0.
\]

(4.76)

We now define a new complex field \( \hat{\psi}_e \) in terms of the components of the effective magnetic field as [256]

\[
\hat{\psi}_e = H_{e,(1)}^y - iH_{e,(1)}^z.
\]

(4.77a)

Using this in the relation for conservation of length of the magnetization vector

\[
H_{e,(1)}^x = \frac{1}{2M(0)k^2} \left[ (H_{e,(1)}^y)^2 + (H_{e,(1)}^z)^2 \right]
\]

we find

\[
| \hat{\psi}_e |^2 = 2M(0)k^2 H_{e,(1)}^x
\]

(4.77b)

When the group velocity of the wave \( v \) is very small, using Eqs. (4.77a) and (4.77b) and the Galilean-like transformation \( X = \xi + \frac{2\beta}{J} \tau \), and after some algebra Eq. (4.76) can be written as

\[
\hat{\psi}_{e\tau} - \hat{\psi}_{eXXX} + \gamma [ | \hat{\psi}_e |^2 \hat{\psi}_e ] X = 0,
\]

(4.78)

where \( \gamma = \frac{A(1+k)^2}{JM(0)^2} \) and while writing Eq. (4.78) we have rescaled \( \tau \) in terms of \( \frac{M(0)}{4Ak(1+k)} \). It may be verified that on using the definitions (4.77a) and (4.77b), Eq. (4.75a) will also lead to Eq. (4.78) after following the same procedure.
Eq.(4.78) is in the form of the complex modified Korteweg-de Vries (CMKDV) equation for the effective field $\psi_e$ similar to the single pulse case Eq.(4.27). While writing Eq.(4.78), it is assumed that all the signals are propagating with a single frequency. However, in practice the n-signals may propagate with different frequencies $\omega_1, \omega_2, ..., \omega_n$ and in order to take care of this we write

$$\psi_e = \sum_{j=1}^{n} \psi_j e^{i\omega_j \tau}, \tag{4.79}$$

where $\omega_j$'s are the different frequencies ($\omega_1, \omega_2, ..., \omega_n$) of the n-signals. In this case the constraint on the conservation of length of the magnetization vector $M^2 = 1$ at $O(\varepsilon^1)$ leads to

$$\sum_{j=1}^{n} H_{j,1}(\varepsilon) e^{i\omega_j \tau} = \frac{1}{2M_0 k^2} \left[ \sum_{j=1}^{n} |\psi_j|^2 + \sum_{j \neq k} \psi_j \psi_k^* e^{i(\omega_j - \omega_k) \tau} \right]. \tag{4.80}$$

It may be noted that the constraint on the length of the magnetization vector generates terms corresponding to oscillations at frequencies $[(\omega_j - \omega_k)]$ which result from mixing of the different signals. If the new frequency terms are to build up significantly it is necessary to satisfy the phase matching condition (i.e.) frequencies of the different signals must be equal [48]. However, in practice unless special arrangements are made the condition is not generally satisfied [48]. Therefore assuming that phase matching does not occur in our case, we drop the terms proportional to mixing of waves. Under this assumption Eq.(4.76) after suitable exponential transformation can be rewritten as a set of n-coupled equations given by

$$\psi_{j\tau} - \psi_{jXX} + \gamma \left[ \sum_{l=1}^{n} |\psi_l|^2 \psi_j \right]_X = 0. \tag{4.81}$$
Eq. (4.81) represents the dynamics of the magnetic field components of the n-EM signals with different frequencies while propagating through the anisotropic charge-free ferromagnetic medium at the first order of expansion which is in the form of n-coupled CMKDV equation. On making the transformation 
\[ \psi_j = q_j \exp\left[\frac{\pi}{6}(\zeta - \frac{\pi}{18})\right], \]
where \( \zeta = X + \frac{\pi}{6} \) Eq. (4.81) can be rewritten as n-coupled higher order nonlinear Schrödinger equations
\[
\left[i \partial_t + \frac{1}{2} \partial^2_{\zeta} - i \partial^3_{\zeta}\right] q_j - \gamma \sum_{l=1}^{n} \left[|q_l|^2 q_j - 6 i \partial_{\zeta} (|q_l|^2 q_j)\right] = 0. \tag{4.82}
\]
When \( n=1 \), this equation also represents the propagation of light wave in a monomode optical fiber in the subpicosecond-femtosecond domain [257]. Though it failed to pass the Painlevé test [252] even for the \( n=1 \) case, it has been solved for N-solitons (for \( n=1 \)) [257] using Hirota's bilinearization procedure. Also bright N-solitons and dark one-soliton solutions have been constructed for Eq. (4.81) in a different context for \( n=2 \) [253]. Here we give the bright and dark one-soliton solutions.

**Bright one-soliton**: The bright one soliton solution takes the form
\[
\psi_j = P \kappa_j \text{sech} \left[P (X + P^2 \tau) + \delta\right], \quad j = 1, 2, ..., n, \tag{4.83}
\]
where \( \kappa_j = \left[\frac{\exp(2 \eta_j)}{4 c_1}\right]^\frac{1}{2}, \delta = \frac{1}{2} \ln\left(\frac{\delta_j}{\gamma}\right) \) and \( c_1 = \frac{\gamma}{8} \sum_{l=1}^{n} \exp(\eta_l + \eta_l^*) \). Here \( \eta_j \)'s are complex constants and \( P \) is a real constant that are used in the Hirota's bilinearization technique. Using Eq. (4.83) in Eqs. (4.77a) and (4.77b) through Eq. (4.79) the bright one-soliton for the magnetic field components of the copropagating n-EM signals can be obtained. The result will contain superposition of n-solitons with different amplitudes proportional to \( k_j \)'s corresponding to the sum of n-EM-solitons \( H_j \)'s, \( j = 1, 2, ..., n \). Now assuming that these EM-solitons do not mix.
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with each other, we can write

\[ H_{j,x}^{(1)} = \frac{P^2}{2M(0)c^2} |\kappa_j|^2 \text{sech}^2 \left[ P \left( X + P^2 \tau \right) + \delta \right] e^{-i\omega_1 t}, \quad (4.84a) \]

\[ H_{j,y}^{(1)} = P \text{Re}(\kappa_j) \text{sech} \left[ P \left( X + P^2 \tau \right) + \delta \right], \quad (4.84b) \]

\[ H_{j,z}^{(1)} = -P \text{Im}(\kappa_j) \text{sech} \left[ P \left( X + P^2 \tau \right) + \delta \right]. \quad (4.84c) \]

Here \( \text{Re}(\kappa_j) \) and \( \text{Im}(\kappa_j) \) represent the real and imaginary parts of \( \kappa_j \) respectively.

Dark one-soliton: The dark one soliton solution takes the form [253]

\[ \psi_j = -\rho_j \tanh \left\{ \frac{1}{2} \left[ Q \left( X - \frac{1}{2} Q^2 \tau \right) + \delta_1 \right] \right\}, \quad j = 1, 2, ..., n, \quad (4.85) \]

where \( \rho_j \)'s are complex constants, \( Q = \sqrt{2\alpha \sum_j |\rho_j|^2} \) and \( \delta_1 \) are real constants.

Using Eq.(4.85) in Eqs.(4.77a) and (4.77b) as done in the bright-soliton case the dark one-soliton for the magnetic field components of the copropagating n-EM signals can be obtained as

\[ H_{j,x}^{(1)} = \frac{1}{2M(0)c^2} |\rho_j|^2 \tanh^2 \left\{ \frac{1}{2} \left[ Q \left( X - \frac{1}{2} Q^2 \tau \right) + \delta_1 \right] \right\}, \quad (4.86a) \]

\[ H_{j,y}^{(1)} = \text{Re}(\rho_j) \tanh \left\{ \frac{1}{2} \left[ Q \left( X - \frac{1}{2} Q^2 \tau \right) + \delta_1 \right] \right\}, \quad (4.86b) \]

\[ H_{j,z}^{(1)} = \text{Im}(\rho_j) \tanh \left\{ \frac{1}{2} \left[ Q \left( X - \frac{1}{2} Q^2 \tau \right) + \delta_1 \right] \right\}. \quad (4.86c) \]

Using the bright and dark soliton solutions in the relations

\[ M_x^{(1)} = -(H_{1,x}^{(1)} + H_{2,x}^{(1)} + ... + H_{n,x}^{(1)}), \quad M_y^{(1)} = \frac{1}{k} \left[ H_{1,y}^{(1)} + H_{2,y}^{(1)} + ... + H_{n,y}^{(1)} \right] \quad \text{and} \quad M_z^{(1)} = \frac{1}{k} \left[ H_{1,z}^{(1)} + H_{2,z}^{(1)} + ... + H_{n,z}^{(1)} \right] \]

which we obtained while solving the equations we can calculate the components of magnetization of the ferromagnetic medium and using this in the relation connecting the effective magnetic field, the effective magnetic induction and the magnetization \( (B_e = \mu_0[H_e + M]) \) we can calculate the magnetic induction.
of the copropagating EM signals. From Eqs.(4.84) and (4.86) we find that the magnetic field component of the n-copropagating EM signals propagating in the anisotropic ferromagnetic material medium free from charges have been modulated in the form of solitons as they interact with the magnetization of the medium which also exhibits soliton excitations. Further it can be noted from the relations below Eq.(4.86) that the value of the magnetization components after interaction with the EM signals become the sum of the respective magnetic field components of the propagating signals.

4.7.2. Interaction of EM signals

![Figure 4.1: Interaction of three electromagnetic solitons](image)

Figure 4.1.: Interaction of three electromagnetic solitons $|H_{1,(1)}^z|^2(H1)$, $|H_{2,(1)}^z|^2(H2)$, $|H_{3,(1)}^z|^2(H3)$ from Eq.(4.84a) originating from different points during propagation.
In order to demonstrate the lossless propagation of the n-EM signals in the anisotropic charge free ferromagnetic medium when there is no wave mixing we have plotted as an example the interaction of three copropagating EM signals (x-component of the propagating magnetic fields $|H^x_{1,1}|^2$ (H1), $|H^x_{2,1}|^2$ (H2) and $|H^x_{3,1}|^2$ (H3)) each of them starting the propagation at different positions viz -8.2, -19.5, 14.2 units along the space-axis with different amplitudes viz. $|\kappa_1|^2 = \sqrt{30}$, $|\kappa_2|^2 = \sqrt{90}$, $|\kappa_3|^2 = \sqrt{60}$, and all of them propagating with the same uniform velocity $P = 1.2321$. Here the phase constant $\delta$ is chosen to be $\delta = 0.4310$ for all the three copropagating waves. It shows that as the EM signals start to propagate in a ferromagnetic material medium the magnetic field component of the EM signals are getting modulated in the form of solitons (bright solitons) and they propagate with the same velocity and phase (Eq.(4.84a)). As the magnetic field components of the different signals ($H^x_{3,1}$) interact the amplitudes of the signals superimpose (add) at the point of interaction and after interaction they come out and propagate without any change in their original amplitude. In a similar way, it can be shown that the y and z components of the magnetic field namely $H^y_{3,1}$ and $H^z_{3,1}$ also propagate without any loss. This can be proved for n-signals and thus clearly explains the lossless copropagation of n-EM signals in a charge free anisotropic ferromagnetic material medium when there is no wave mixing.

4.8. EMW propagation in a biquadratic ferromagnetic medium

As pointed out in sec.(1.7.1), magnetic compounds with higher order magnetic interactions such as biquadratic exchange interaction exhibits interest-
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ing physical phenomena. The magnetization dynamics in a one-dimensional anisotropic Heisenberg ferromagnet with biquadratic exchange interaction in the presence of an external magnetic field in the classical continuum limit is given by the LL equation

\[
\frac{\partial M}{\partial t} = \mathbf{M} \times \left\{ J \left[ \frac{\partial^2 M}{\partial x^2} + \hat{b}_1 \frac{\partial^4 M}{\partial x^4} + \hat{b}_2 \left[ \left( \mathbf{M} \cdot \left( \frac{\partial^2 M}{\partial x^2} \right) \right) \frac{\partial^2 M}{\partial x^2} + \frac{2}{3} \left( \mathbf{M} \cdot \left( \frac{\partial^3 M}{\partial x^3} \right) \right) \frac{\partial \mathbf{M}}{\partial x} \right] \right] \right\} - 2 \beta M^2 \mathbf{n} + \frac{2A}{\mu_0} \mathbf{H}, \quad M^2 = 1,
\] (4.87)

where \( J \) is the bilinear exchange interaction parameter and \( \hat{b}_1 = \frac{a^2}{12} \) and \( \hat{b}_2 = \frac{J a^2}{(1+2J)} \) with \( J \), the biquadratic exchange parameter and \( a \) the lattice parameter in the ferromagnetic spin system. \( \mathbf{H} \) is the magnetic field component of the EM field. The last two terms in the right hand side of the LL equation (4.87) correspond to the anisotropy and interaction with the magnetic field component of the EM field with the usual meaning for the coefficients. Pure magnetization excitations in biquadratic ferromagnetic spin systems have been studied in detail and soliton excitations were identified in the isotropic case and in the absence of external magnetic field when \( \hat{b}_2 = \frac{a^2}{2} \hat{b}_1 \) by mapping the spin chain onto a moving helical space curve [see refs. [258, 259]]. However, for our problem we consider the more general case.

The propagation of EMW in the above one-dimensional anisotropic biquadratic ferromagnetic medium is governed by the wave equation (3.1). Using the constitutive relation (1.12) connecting the fields \( \mathbf{H}, \mathbf{B} \) and \( \mathbf{M} \) and replacing the permittivity of free space \( \epsilon_0 \) by the dielectric constant of the medium \( \epsilon \) which is given by the relation \( \epsilon = \epsilon_0 (1 + \chi_e) \), we can rewrite Eq.(3.1) as

\[
\frac{\partial^2 \mathbf{B}}{\partial x^2} - \mu_0 \epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = \mu_0 \left[ \frac{\partial^2 \mathbf{M}}{\partial x^2} - \mathbf{n} \frac{\partial^2 M^x}{\partial x^2} \right].
\] (4.88)
Now, the set of coupled equations (4.87) and (4.88) describe the propagation of EMW in an anisotropic biquadratic ferromagnetic medium.

We solve the set of coupled equations (4.87) and (4.88) using reductive perturbation method as done earlier. Unlike the previous cases as the medium here is characterised by higher order interactions we stretch the spatial variable $x$ as $\xi = \epsilon(x - vt)$ and time $t$ as $\tau = \epsilon t$ in a different way. We assume the dielectric constant of the medium to be small and therefore rescale $\epsilon$ as $\epsilon \rightarrow \epsilon^2 \epsilon$. Further, we assume that the bilinear and biquadratic exchange interactions are stronger than the other interactions and hence rescale $J$ as $J \rightarrow \epsilon^{-1} J$ and $\hat{b}_1$ and $\hat{b}_2$ as $\hat{b}_1 \rightarrow \epsilon^{-3} \hat{b}_1$ and $\hat{b}_2 \rightarrow \epsilon^{-3} \hat{b}_2$. Since the medium is anisotropic in nature we expand the components of magnetization $M$ and magnetic induction $B$ nonuniformly as given in Eqs.(4.8) and (4.9). Using the constitutive relation (1.12), the stretched variables $\xi$ and $\tau$ and the nonuniform expansions (Eqs.(4.8) and (4.9)) in the component form of Eqs.(4.87) and (4.88) we solve the resultant equations at different orders of $\epsilon$ as done earlier. On solving Eq.(4.88) at $O(\epsilon^0)$, we obtain

$$B_1^y = \mu_0 M_1^y,$$  \hspace{1cm} (4.89a)

$$B_1^x = \mu_0 M_1^x.$$  \hspace{1cm} (4.89b)

From Eq.(4.87) also we obtain the same set of equations (4.89) at $O(\epsilon^0)$ when $\frac{B_y}{M_0} = \mu_0$. At $O(\epsilon^1)$, from Eq.(4.88) we obtain

$$B_1^x = 0,$$  \hspace{1cm} (4.90a)

$$B_2^y = \mu_0 M_2^y,$$  \hspace{1cm} (4.90b)

$$B_2^x = \mu_0 M_2^x.$$  \hspace{1cm} (4.90c)
From Eq. (4.87) at $O(\epsilon^1)$, we obtain the following equations.

\[
\left[ \frac{\partial}{\partial \tau} - v \frac{\partial}{\partial \xi} \right] M_1^x = J \left[ M_1^y \left( \frac{\partial^2 M_1^x}{\partial \xi^2} + b_1 \frac{\partial^4 M_1^x}{\partial \xi^4} \right) - M_1^x \left( \frac{\partial^2 M_1^y}{\partial \xi^2} + b_1 \frac{\partial^4 M_1^y}{\partial \xi^4} \right) \right], \quad (4.91a)
\]

\[
\left[ \frac{\partial}{\partial \tau} - v \frac{\partial}{\partial \xi} \right] M_1^y = J \left[ -M_0 \left( \frac{\partial^2 M_1^x}{\partial \xi^2} + b_1 \frac{\partial^4 M_1^x}{\partial \xi^4} \right) \right] - b_2 \left[ M_0 \frac{\partial^2 M_1^x}{\partial \xi^2} + M_1^x \frac{\partial^2 M_1^y}{\partial \xi^2} + M_1^x \frac{\partial^2 M_1^x}{\partial \xi^2} \right] \frac{\partial^2 M_1^y}{\partial \xi^2}, \quad (4.91b)
\]

\[
\left[ \frac{\partial}{\partial \tau} - v \frac{\partial}{\partial \xi} \right] M_1^z = J \left[ M_0 \left( \frac{\partial^2 M_1^x}{\partial \xi^2} + b_1 \frac{\partial^4 M_1^x}{\partial \xi^4} \right) \right] + b_2 \left[ M_0 \frac{\partial^2 M_1^x}{\partial \xi^2} + M_1^x \frac{\partial^2 M_1^y}{\partial \xi^2} + M_1^x \frac{\partial^2 M_1^z}{\partial \xi^2} \right] \frac{\partial^2 M_1^y}{\partial \xi^2}, \quad (4.91c)
\]

While writing Eqs. (4.91) we have used the results from the lower order.

Eqs. (4.91b) and (4.91c) can be combined using Eqs. (4.23) and (4.24) (Note that here the constraint on the length of the magnetization vector results in Eqs. (4.23) and (4.24)), which after using the transformations $X = \xi - \frac{v}{M_0} \tau$, $T = -M_0 \tau$ and $\psi = q \exp(i \beta \tau)$ can be written in a compact form as

\[
 iq_T + J q_{xx} + A |q|^2 q = -J \left\{ b_1 q_{xxxx} + b_2 M_0 (M_0 + 1/2) \left[ q_{xx}(|q|^2)_{xx} + \frac{2}{3} q_x(|q|^2)_{xxx} \right] \right\},
\]

\[(4.92)\]

Eq. (4.92) is a perturbed NLS equation wherein the perturbation in the r.h.s. of Eq. (4.92) is due to the biquadratic exchange interaction. When $b_1 = b_2 = 0$, Eq. (4.92) reduces to the completely integrable cubic NLS equation. After suitable rescaling and redefinition of parameters the one-soliton solution of the
cubic NLS equation obtained through IST method can be written as [84]

\[ q = 2b \sech [2b(X + 4aT + \delta)] \exp \left[ -2i \left( aX + 2(a^2 - b^2)T + a\delta_2 \right) \right] \] \hspace{1cm} (4.93)

where, \( a, b \) are constants related to the spectral parameter and \( \delta_1 \) and \( \delta_2 \) are phase constants. Using Eq.(4.93), we can evaluate the components of magnetization of the medium through the relations (4.23) and (4.24) and the components of magnetic induction using Eqs.(4.89) at the lowest existing order. The results read

\[ M^x_1 = \frac{4b^2}{M_0} \sech^2 \left[ 2b(X + 4aT + \delta_1) \right] , \hspace{1cm} (4.94a) \]

\[ M^y_1 \equiv \mu_o^{-1} B^y_1 = 2b \sech \left[ 2b(X + 4aT + \delta_1) \right] \cos \left[ 2aX + 4(a^2 - b^2)T + 2a\delta_2 \right] , \hspace{1cm} (4.94b) \]

\[ M^z_1 \equiv \mu_o^{-1} B^z_1 = 2b \sech \left[ 2b(X + 4aT + \delta_1) \right] \sin \left[ 2aX + 4(a^2 - b^2)T + 2a\delta_2 \right] \hspace{1cm} (4.94c) \]

Thus the magnetization of the medium is excited in the form of solitons by the applied EM field, so also the magnetic induction. Then using the constitutive relation (1.12) we can evaluate the magnetic field components of the EM field which show that only the x-component of the magnetic field exists at the lowest existing order and it is given by

\[ H^x_1 = \frac{4b^2}{M_0} \sech^2 \left[ 2b(X + 4aT + \delta_1) \right] . \hspace{1cm} (4.95) \]

The result shows that when EMW propagates through anisotropic biquadratic ferromagnetic medium, the EMW which can propagate without loss in the form of solitons in the case of anisotropic bilinear medium is spoiled by the presence of biquadratic exchange interaction.
4.9. EMW propagation in a site-dependent ferromagnetic medium

In certain magnetic materials the exchange interaction among the nearest neighbours can be site-dependent [192,260] which generally occurs if (a) the distance between the neighbouring atoms varies along the chains (for example charge transfer complexes, organo- metallic insulators) and (b) the wave function itself varies from site to site although the atoms themselves may be equally spaced (for example magnetic insulator placed in a weak, static inhomogeneous electric field and by the deliberate introduction or imperfections of impurities in the vicinity of a bond so as to alter the electronic wave functions without causing appreciable lattice distortion [192,260]). The dynamics of magnetization in a one-dimensional anisotropic ferromagnet with site-dependent exchange interaction in the presence of an external magnetic field in the classical continuum limit is given by the LL equation

$$\frac{\partial M}{\partial t} = M \wedge \left\{ J \left[ \frac{\partial^2 M}{\partial x^2} + \frac{\partial F}{\partial x} \frac{\partial M}{\partial x} \right] - 2\beta M^2 n + 2A H \right\}, \quad M^2 = 1. \quad (4.96)$$

Here the function $F(x)$ represents the inhomogeneity in the exchange interaction. Eq.(4.96), through space curve mapping is found to be equivalent to an integro-differential NLS equation which for a linear homogeneity of the form $F(x) = ax + b$ is found to be completely integrable and the spin excitations are governed by solitons [261]. In our problem we assume that the field $H$ in Eq.(4.96) represents the magnetic field component of the EM field governed by Eq.(4.88) and the term proportional to $\beta$ corresponds to the anisotropy with the usual meaning. Using the same procedure followed in the previous cases, we solve the set of coupled equations (4.96) and (4.88).
As in the biquadratic case, we rescale $\epsilon$ and $J$ as $\epsilon \rightarrow \epsilon^2 \epsilon$ and $J \rightarrow \epsilon^{-1} J$. Also, we stretch the space and time variables as $\xi = \epsilon(x - vt)$ and $\tau = \epsilon t$. We expand the magnetization $M$ and the magnetic induction $B$ nonuniformly after using the constitutive relation (1.12) and substitute in the component equations of (4.96) and (4.88). We then collect terms proportional to different powers of $\epsilon$ and solve them. At $O(\epsilon^0)$, we obtain the same results as in the biquadratic case. At $O(\epsilon^1)$ we obtain equations exactly the same as Eqs.(4.90) and from Eq.(4.96) the following set of equations.

\begin{equation}
\begin{aligned}
\left[ \frac{\partial}{\partial \tau} - \nu \frac{\partial}{\partial \xi} \right] M_t^x &= JF \left[ M_t^x \frac{\partial^2 M_t^z}{\partial \xi^2} - M_t^z \frac{\partial^2 M_t^x}{\partial \xi^2} \right], \\
\left[ \frac{\partial}{\partial \tau} - \nu \frac{\partial}{\partial \xi} \right] M_t^y &= -JM_0 \left[ F \frac{\partial^2 M_t^z}{\partial \xi^2} + \frac{\partial F}{\partial \xi} \frac{\partial M_t^z}{\partial \xi} \right] - 2AM_t^x M_t^z - \beta M_0 M_t^x, \\
\left[ \frac{\partial}{\partial \tau} - \nu \frac{\partial}{\partial \xi} \right] M_t^z &= JM_0 \left[ F \frac{\partial^2 M_t^y}{\partial \xi^2} + \frac{\partial F}{\partial \xi} \frac{\partial M_t^y}{\partial \xi} \right] + 2AM_t^x M_t^z - \beta M_0 M_t^x.
\end{aligned}
\end{equation}

While writing Eqs.(4.97) we have used the results from the lower orders. Using Eqs.(4.23) and (4.24, we can combine Eqs.(4.97b) and (4.97c) and upon making the transformations $X = \xi - \frac{\nu}{M_0} \tau$, $T = \tau$ and $\psi = \exp(i\beta \tau)$ we finally obtain

\begin{equation}
iq \tau + JF q xx + A|q|^2 q = -JM_0 \frac{\partial F}{\partial x} q x.
\end{equation}

Now choosing $F$ as $F = c - \frac{N(X)}{N}$, where $c$ is a constant, Eq.(4.98) can be rewritten as

\begin{equation}
iq \tau + Jc q xx + A|q|^2 q = J \left[ N(X) q xx + M_0 \frac{\partial N(X)}{\partial x} q x \right].
\end{equation}

Eq.(4.99), when the inhomogeneity is absent reduces to the completely integrable NLS equation whose one-soliton solution is given by Eq.(4.93) from which the solitons for the magnetization, the magnetic induction and the mag-
netic field can be evaluated. Here also the magnetic field is restricted to the direction of propagation. Thus, from Eq.(4.99), we observe that when inhomogeneity is introduced in the anisotropic bilinear ferromagnetic medium the lossless propagation of EMW in the form of soliton is disturbed.

4.10. Conclusions

We investigated the nature of propagation of EMW in a charge free untreated and treated anisotropic ferromagnetic media and the nature of nonlinear excitations in the magnetization of ferromagnetic media and the modulation that takes place on the magnetic induction and in the magnetic field component of the EMW. For this, through a reductive perturbation method we solved the component forms of the coupled LL and Maxwell equations. We found that in the case of an untreated anisotropic ferromagnetic medium the magnetic field component of the EMW induces soliton excitations in the magnetization of the ferromagnetic medium which in turn modulates the magnetic induction and hence the magnetic field component of EMW in the form of solitons. In the case of a treated ferromagnetic medium, it is found that a specific form of nonlinearity of the medium as given in Eq.(4.54) balances the dispersion of EMW so that the magnetic induction and the magnetic field component of the EMW propagate in the form of solitons. The excitations of magnetization in the medium have also been found to be governed by soliton modes. Further, in both the cases the soliton excitations of the magnetic induction is restricted to a plane normal to the direction of propagation of the EMW. Further, we have studied the co-propagation of n-EM waves of different frequencies and phases through an untreated anisotropic charge free ferromagnetic medium using the same procedure and found that the magnetic field components of all co-propagating
signals are modulated in the form of soliton. The lossless propagation of n-EM waves are also numerically established. Further, we have studied the propagation of EMW in anisotropic biquadratic ferromagnetic medium and in inhomogeneous bilinear ferromagnetic medium. The results show that the biquadratic exchange interaction and the inhomogeneity in the medium disturb the lossless propagation of EMW in the form of solitons.