Chapter II
MARKOVIAN BULK SERVICE QUEUE UNDER ACCESSIBILITY RULES

2.1 INTRODUCTION

Bulk service queueing systems have been analysed in different situations by several experts right from Bailey(1954). Various bulk size rules are available in queueing literature. One of the famous bulk size rules is Neuts’ (1967) general bulk service rule which is stated as follows: If there are less than ‘a’ units waiting at the time of a departure, the server must wait until there are ‘a’ units present, whereupon it serves them together. If there are ‘a’ or more, but less than ‘b’ units waiting, all are served together. If there are ‘b’ or more units waiting, a group of ‘b’ units only are served and others must wait. This general bulk service rule is further assumed to permit the late entries to join the batch in course of ongoing service. This type of service is named Accessible batch service which is stated by Medhi (1984). If a batch being served does not utilize its full capacity for service, it may remain accessible for units arriving during the service time of the batch until its full capacity is attained or service time ends, but the total service time of the accessible batch is unaltered by the inclusion of such joining units in the course of the ongoing service.

The general bulk service rule with non-accessible batches in Markovian queues was analysed by Gohain and Borthakur(1979), Chaudhry and Templeton(1983) and Medhi(1984). The concept of accessibility in Markovian queues with bulk service was studied by Medhi(1984), Gross and Harris(1985) and Sivasamy(1990).

In this chapter, the accessibility server concept is employed in a single server Markovian queue. Here, the arrival of units follows Poisson process with arrival rate \( \lambda \). The arriving units are served in batches. The batch size, defined by Neuts (1967), \([a, b]\) is divided into two parts such as \( a \leq h \leq d - 1 \) and \( d \leq k \leq b \), which are called Accessible Batch Size(ABS) and Non-Accessible Batch Size(NABS) respectively. After finishing the service for a batch, the server may notice the queue size which falls under any one of the following three categories:

(i) \( 0 \leq r \leq a - 1 \)  
(ii) \( a \leq h \leq d - 1 \)  
(iii) \( d \leq k \leq b \)
In the first case, the server cannot begin the service and remains idle until the queue size reaches ‘a’ whereupon the server starts the service with the minimum of ‘a’ units.

In the second case, the server takes the available units, say, ‘h’ for batch service and also admits the late arrivals into the ongoing service batch till either the service period ends or the number of units in service attain the maximum accessible limit. This type of service batch is named Accessible Batch Service. Here it is noted that the service times of accessible batch follow exponential distribution with parameter $\mu_1$.

In the third case, the server takes all available units for service, if the number of units in the queue is greater than or equal to ‘d’ but less than or equal to ‘b’. It takes only ‘b’ units, if the number of units in the queue is greater than or equal to ‘b’. This type of service batch is termed Non-Accessible Batch Service. The service times of non-accessible batch follow the same exponential distribution with parameter $\mu_2$.

The expressions for the steady-state probabilities for the number of units in the queue, mean and variance of queue length are derived. The mean waiting time of the units in the queue is also obtained. In addition to that, the computational results for the mean number of units in the queue are obtained when fixing the unknown $a, d, b, \mu_1, \mu_2$ and $\theta$ with varying $\lambda$.

### 2.2 STEADY-STATE PROBABILITIES

$p_{0,n}$ : The probability that there are $n$ ($0 \leq n \leq d - 1$) units in the queue, such that:

(i) The server is idle, if there are $r$ ($0 \leq r \leq a - 1$) units and

(ii) The server is busy with AB, if there are $h$ ($a \leq h \leq d - 1$) units.

$p_{1,n}$ : The probability that there are $n$ ($n = 0, 1, 2, ..., \infty$) (excluding those in service) units when the server is busy with NAB.
Based on the above probabilities, the steady-state equations are designed as follows:

\[
\lambda P_{0,0} = \mu_1 P_{1,a} + \mu_1 \sum_{h=a}^{d-1} P_{0,h} ; \quad r = 0 \quad (2.2.1)
\]

\[
\lambda P_{0,r} = \lambda P_{0,r-1} + \mu_1 P_{1,r} ; \quad 1 \leq r \leq a - 1 \quad (2.2.2)
\]

\[
(\lambda + \mu_1 + \mu_2) P_{0,h} = \lambda P_{0,h-1} + \mu_1 P_{1,h} ; \quad a \leq h \leq d - 1 \quad (2.2.3)
\]

\[
(\lambda + \mu_1) P_{1,0} = \lambda P_{0,d-1} + \mu_2 \sum_{k=d}^{b} P_{1,k} ; \quad k = d, d+1, \ldots, b \quad (2.2.4)
\]

\[
(\lambda + \mu_1) P_{1,n} = \lambda P_{1,n-1} + \mu_2 P_{1,n+b} ; \quad n \geq 1 \quad (2.2.5)
\]

The characteristic equation of the equation (2.2.5) is written as

\[
\mu_2 z^{b+1} - (\lambda + \mu_1) z + \lambda = 0 \quad (2.2.6)
\]

According to Rouche’s theorem, the equation (2.2.6) gives only one real root that lies inside the unit circle. Let it be 'θ'. Now, the probability that there are 'n' units in the queue when the server is busy is defined as:

\[
P_{1,n} = B\theta^n ; \quad n = 0, 1, \ldots, \infty \quad (2.2.7)
\]

Here, B is the expression in terms of θ with queueing notations and is to be estimated. By summing up the equations after using the values for \( r = 1, 2, \ldots, a-1 \) in equation (2.2.2) and using equations (2.2.1) and (2.2.7) the following equation is obtained:

\[
P_{0,a-1} = \left( \frac{\mu_1}{\lambda} \right) \left[ B \sum_{r=1}^{a} \theta^r + D \right] \quad (2.2.8)
\]

where \( D = \sum_{h=a}^{d-1} P_{0,h} \)

From the above technique, apply \( h = a, a+1, \ldots, d-1 \) in equation(2.2.3) and get the equation,

\[
P_{0,d-1} = \left( \frac{1}{\lambda} \right) \left[ B \mu_1 \left\{ \sum_{r=1}^{a} \theta^r + \sum_{h=a}^{d-1} \theta^h \right\} - D\mu_2 \right] \quad (2.2.9)
\]

The equation (2.2.4) is rewritten after applying equation (2.2.7), as

\[
P_{0,d-1} = \left( \frac{B}{\lambda} \right) \left[ (\lambda + \mu_1) - \mu_2 \sum_{k=d}^{b} \theta^k \right] \quad (2.2.10)
\]

Now, using the equations (2.2.9) and (2.2.10), the expression for the concept \( D \) is obtained as follows:
The steady-state probabilities for the number of units in the queue when the server is idle are derived, by using some algebra in the equations (2.2.1), (2.2.2) and (2.2.3), as follows:

\[
P_{0,r} = \frac{B \mu_1}{\lambda (1 - \theta)} \left\{ \theta (1 - \theta^*) \right\} + \frac{D \mu_1}{\lambda} ; r = 0,1,\ldots,a-1 \quad (2.2.12)
\]
and

\[
P_{0,h} = \frac{B \mu_1}{\lambda (1 - \theta)} \left\{ \theta (1 - \theta^*) + \theta^* - \theta^{b+1} \right\} - \frac{D \mu_2}{\lambda} ; h = a,a+1,\ldots,d-1
\]

Now, the next step is to estimate the expression for \( B \) by applying the equations (2.2.7), (2.2.11), (2.2.12) and (2.2.13) in normalizing condition.

\[
B = \left[ \frac{\mu_1 \theta}{\lambda (1 - \theta)} \left\{ (a - 1)(1 - \theta) - \theta + \theta^* \right\} + \left\{ (d - a)(1 - \theta)(1 + \theta^{a-1} - \theta^*) + \theta^d - \theta^* \right\} \right]'''
\]
\[\quad + \frac{1}{(1 - \theta)} + \frac{1}{\mu_2 \lambda (1 - \theta)} \left\{ \mu_1 (\theta + \theta^* - \theta^{a+1} - \theta^d) - (1 - \theta)(\lambda + \mu_1) + \mu_2 (\theta^d - \theta^{b+1}) \right\}
\left\{ \mu_1 (a - 1) - \mu_2 (d - a) \right\}^{-1}
\]

The required steady-state probabilities, when the server is either busy or idle, are obtained by applying the equation (2.2.14) in the expressions (2.2.7), (2.2.12) and (2.2.13).

### 2.3 PERFORMANCE MEASURES

The expected number of units in the queue is stated as

\[
L_q = \sum_{r=0}^{a-1} r P_{0,r} + \sum_{h=a}^{d-1} h P_{0,h} + \sum_{n=0}^{\infty} n P_{1,n}
\]
\begin{equation}
\frac{a(a-1) \mu_1}{2 \lambda} \left\{ \frac{B \theta}{(1-\theta)} + D \right\} + \frac{1}{2 \lambda} \left\{ d^2 - d + a - a^2 \right\} \left\{ \frac{B \mu_1}{(1-\theta)} \left\{ \theta(1-\theta^a) + \theta^a \right\} - D \mu_2 \right\} \\
- \frac{B \mu_1}{\lambda(1-\theta)^2} \left\{ 2\theta - (d+2)\theta^{d+1} + d\theta^{d+2} \right\} + \frac{B \theta}{(1-\theta)^2}\right] 
\end{equation}

(Using the equations 2.2.12, 2.2.13 and 2.2.7)

(2.3.1)

The Variance for the number of units in the queue is obtained as

\[ V(n) = \left\{ \sum_{r=0}^{a-1} r^2 P_{0,r} + \sum_{h=a}^{d-1} h^2 P_{0,h} + \sum_{n=0}^{\infty} n^2 P_{1,n} \right\} - \left( L_q \right)^2 \]

\[ = \left\{ \frac{(a-1) a (2a-1) \mu_1}{6 \lambda} \right\} \left\{ \frac{B \theta}{(1-\theta)} + D \right\} + \frac{1}{6 \lambda} \left\{ (d-1) d (2d-1) - (a-1) a (2a-1) \right\} \\
\left[ \frac{\mu B}{(1-\theta)} \left\{ (1-\theta^a) + \theta^a \right\} - D \mu_2 \right] + B \left\{ 2(1-\theta)^3 \left\{ (2-\theta + (1-\theta)^{-2} + \theta (1-\theta)^{-2} \right\} + \frac{\mu_1 B \theta}{(1-\theta)} \left\{ (1-\theta)^{-1} \left\{ 2(\theta^2 - \theta^{d+2}) + 2(1-\theta) (2\theta - (d+2) \theta^{d+1}) \right\} \right\} \\
+ (1-\theta)^{-2} \left\{ (2 - (d+2)(d+1) \theta^d) + (1-\theta)^{-2} \left\{ (1-\theta^a + (1-\theta) [1-(d+1) \theta^d] \right\} \right\} \right\} \\
- \left[ \frac{a(a-1) \mu_1}{2 \lambda} \left\{ \frac{B \theta}{(1-\theta)} + D \right\} + \frac{1}{2 \lambda} \left\{ d^2 - d + a - a^2 \right\} \left\{ \frac{B \mu_1}{(1-\theta)} \left\{ (1-\theta^a) + \theta^a \right\} - D \mu_2 \right\} \\
- \frac{B \mu_1}{\lambda(1-\theta)^2} \left\{ 2\theta - (d+2)\theta^{d+1} + d\theta^{d+2} \right\} + \frac{B \theta}{(1-\theta)^2}\right]\right)^2 \right] 
\end{equation}

(2.3.2)

By applying Little's formula in the equation (2.3.1), the expected waiting time of the units in the queue is obtained as:

\[ \text{29} \]
\[ W_q = \left[ \frac{a(a-1)\mu_1}{2\lambda^2} \frac{B\theta}{(1-\theta)} + D \right] + \frac{1}{2\lambda^2} \left\{ d^2 - d + a - a^2 \right\} \frac{B\mu_1}{(1-\theta)} \left\{ (1-\theta) + \theta \right\} - D\mu_2 \]

\[ - \frac{B\mu_1}{\lambda^2(1-\theta)^2} \left\{ 2\theta - (d+2)\theta^{d+1} + d\theta^{d+2} \right\} + \frac{B\theta}{\lambda (1-\theta)^2} \]  

(2.3.3)

2.4 NUMERICAL RESULTS

The expected number of units in the queue can be computed for varying \( \lambda \) with fixed \( a, d, b, \mu_1, \mu_2 \) and \( \theta \). For this model, one can exhibit countless number of curves by taking the values of \((a,d,b)\) as \((07,10,15),(11,13,15),(15,20,25),(19,23,15),(25,29,30)\) and \( \theta=0.5 \) for the following three cases:

(i) \( \mu_1 < \mu_2 \) \( (\mu_1=8, \mu_2=10) \)
(ii) \( \mu_1 = \mu_2 \) \( (\mu_1=10, \mu_2=10) \)
(iii) \( \mu_1 > \mu_2 \) \( (\mu_1=10, \mu_2=8) \)

The mean queue length \( (L_q) \) for different values of \( \lambda \) are shown in Table 2.1 and their corresponding curves are exhibited in Figures 2.1, 2.2 and 2.3.

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Table 2.1. Mean Queue Length \( (L_q) \) Vs. \( \lambda \) for different values of \( a,d,b \) when \( \theta = 0.5 \).
Fig. 2.1. Mean queue length Vs $\lambda$ for different values of $a$, $d$ and $b$ when $\mu_1=8$, $\mu_2=10$ and $\theta=0.5$

Fig. 2.2. Mean queue length Vs $\lambda$ for different values of $a$, $d$ and $b$ when $\mu_1=10$, $\mu_2=10$ and $\theta=0.5$
• Keeping the other parameters are fixed, the above figures reveals that when \( \lambda \) increases \( L_q \) also increases.

• The different values of \((a, d, b)\) and fixed values of \(\lambda, \mu_1, \mu_2\) and \(\theta\), \(L_q\) increases.

• By considering the three cases namely \(\mu_1 < \mu_2\), \(\mu_1 > \mu_2\) and \(\mu_1 = \mu_2\), \(L_q\) is lower when \(\mu_1 < \mu_2\), higher when \(\mu_1 > \mu_2\), and when \(\mu_1 = \mu_2\) the values of \(L_q\) lies between above stated lower and higher values of \(L_q\).