Chapter VI
6.1 INTRODUCTION

In this chapter, machine repair problem with multi-server Markovian queue has been analysed. The life time and repair time of machines are exponentially distributed. The values of failure and service rates are stated, according to the situation of the system. This present work is deviated from the previous chapters. Here, the steady-state probabilities for the number of failed machines, waiting for service and expected number of failed machines in the system are derived.

The applications of queueing theory have been utilized commonly in communication systems and industries. Human beings, telephone calls, flow of finished products, failed machines and so on may be considered as queueing units. In modern days, the queueing models have been analysed by assuming failed machines as the units in the queue, demanding service for repair. The behaviour of failed machines which may or may not enter the queue, depends upon the number of failed machines. The Researchers have analysed machine repair problems by using different probability models. Gross and Harris (1985) described M/M/C/m/m model with spares. Ganesan (1996) derived the transient state probabilities for general bulk service queue with dependence parameters. Jain (1998) studied M/M/R machine repair problem with spares and additional repair man. Gupta (1999) analysed N-policy queueing system with finite source and warm spares. Jain et al (2000) studied the expected number of failed machines in the system with multiple servers. Shawky (2000) analysed machine repair problem without additional repair man.

6.2 DESCRIPTION OF THE MODEL

In this chapter, machine repair problem, along with some queueing characters such as balking, spares and additional repair man is studied. There is a provision for failed machines getting replacement immediately, if spare machines are available. If all spare machines are used and a machine fails, the system becomes short. The lifetime and repair time of the machines are distributed accordingly to exponential distribution. The failed machines are repaired by a group of repair men. In busy
schedule, an additional repair man is allowed under the condition that the number of failed machines in the system is more than 'm' and he is removed when the number of the failed machines is again reduced to 'm'. After the repair is over, the repaired machines may join the spare machines, if there is no shortage in the system or may join the system itself, when there is shortage. Here it is assumed that the failure rate of machines and service rate of permanent repair men depend on the number of failed machines which are waiting for service and are denoted respectively as $\lambda_n$ and $\mu_n$. The steady-state probabilities for the number of failed machines waiting for service are derived from different situations. The expressions for the expected number of failed machines in the system are derived, based on the number of repair men and number of spare machines. The special cases are studied.

This system is illustrated by the following example: A leading industry has 'N' machines. Out of 'N' machines, 'd' machines are kept as spare machines which are used to replace failed machines. The failed machines have been serviced or repaired by 'c' repair men. The capacity of the waiting space for the failed machines which are to be repaired is considered as 'k'. Suppose the repair is going on, the number of machines waiting for service is 'k' or more, then the arriving machines may balk. In this busy schedule, even though the spare machines are used instead of failed machines, there is a need for an additional server (repair man) for servicing the failed machines, when the waiting space is full or overflows, for avoiding shortages in the production plant. After the repair is over, the machines may either go to the production plant or join spare machines. It is noted that the concept of reneging is not permitted, since the machines must be serviced for further use. But in the case of human customers or communication systems, the reneging policy is possible.

The real life illustration of this concept is the arrival and service of railway engines to the service station. The minor repairs are attended to at once but major repairs are done after providing spare machines.
For the purpose of mathematical modeling, the following notations are employed:

\( k \) – Capacity of the system
\( N \) – Population size of the machines
\( n \) – Number of machines in the system
\( d \) – Number of spare machines
\( c \) – Number of permanent repairmen in the system
\( m \) – Threshold value of the number of failed machines when additional repairman is on
\( \lambda \) – Failure rate of the machines in the system
\( \beta \) – Balking probability of the machines when all permanent repairmen are busy
\( \mu \) – Service rate of permanent repairmen
\( \mu_a \) – Service rate of additional repairman

\( P_n(t) \) – Transient state probability that there are ‘\( n \)’ failed machines in the system at time ‘\( t \)’, \( (n = 0,1,2,\ldots,d+k) \)

\( P_n \) – Steady-state probability of \( n^{th} \) state
\( P_0 \) – Steady-state probability of empty state

When all permanent repairmen are busy, the failed machines may balk with probability \( \beta \). When at least one permanent repairman is free, the balking probability is \( \beta = 1 \).

### 6.3 ANALYSIS AND QUEUEING PERFORMANCE

The above stated model is analysed based on the number of repairmen and the number of spare machines in two different cases.

**Case (i) \( c \leq d \)**

The differential-difference equations for the above said model are framed when the number of repairmen is less than or equal to the number of spare machines and are given below:
\[ P'_0(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad ; n = 0 \quad (6.3.1) \]
\[ P'_n(t) = - (\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t) \quad ; 1 \leq n < c \quad (6.3.2) \]
\[ P'_c(t) = - (\lambda_c + \mu_c) P_c(t) + \lambda_{c-1} P_{c-1}(t) + \mu_{c+1} P_{c+1}(t) \quad ; n = c \quad (6.3.3) \]
\[ P'_n(t) = - (\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t) \quad ; c + 1 \leq n \leq d \quad (6.3.4) \]
\[ P'_n(t) = - (\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t) \quad ; d + 1 \leq n < m \quad (6.3.5) \]
\[ P'_m(t) = - (\lambda_m + \mu_m) P_m(t) + \lambda_{m-1} P_{m-1}(t) + \mu_{m+1} P_{m+1}(t) \quad ; n = m \quad (6.3.6) \]
\[ P'_n(t) = - (\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t) \quad ; m + 1 \leq n < d + k \quad (6.3.7) \]
\[ P'_{d+k}(t) = - \mu_{d+k} P'_{d+k}(t) + \lambda_{d+k-1} P'_{d+k-1}(t) \quad ; n = d + k \quad (6.3.8) \]

The state dependent failure and service coefficients are defined in different situations.

\[ \lambda_n = \begin{cases} 
N \lambda & ; 0 \leq n < c \\
N \beta \lambda & ; c \leq n < d \\
(N + d - n) \beta \lambda & ; d \leq n < d + k \\
0 & ; n = d + k 
\end{cases} \quad (6.3.9) \]

and

\[ \mu_n = \begin{cases} 
n \mu & ; 0 < n \leq c \\
c \mu & ; c < n \leq m \\
c \mu + \mu_a & ; m < n \leq d + k 
\end{cases} \quad (6.3.10) \]

The equations from (6.3.1) to (6.3.8) reduce to the following system of equations after applying the coefficients given in (6.3.9) and (6.3.10) and also letting limit \( t \to \infty \).
\[-N\lambda P_0 + \mu P_1 = 0 \quad ; n = 0 \quad (6.3.11)\]

\[-(N\lambda + n\mu)P_n + N\lambda P_{n-1} + (n+1)\mu P_{n+1} = 0 \quad ; 1 \leq n < c \quad (6.3.12)\]

\[-(N\beta\lambda + c\mu)P_c + N\lambda P_{c-1} + c\mu P_{c+1} = 0 \quad ; n = c \quad (6.3.13)\]

\[-(N\beta\lambda + c\mu)P_n + N\beta\lambda P_{n-1} + c\mu P_{n+1} = 0 \quad ; c + 1 \leq n \leq d \quad (6.3.14)\]

\[-[(N + d - n)\beta\lambda + c\mu]P_n + (N + d - n + 1)\beta\lambda P_{n-1} + c\mu P_{n+1} = 0 \quad ; d + 1 \leq n < m \quad (6.3.15)\]

\[-[(N + d - m)\beta\lambda + c\mu]P_m + (N + d - m + 1)\beta\lambda P_{m-1} + (c\mu + \mu_a)P_{m+1} = 0 \quad ; n = m \quad (6.3.16)\]

\[-[(N + d - n)\beta\lambda + c\mu + \mu_a]P_n + (N + d - n + 1)\beta\lambda P_{n-1} + (c\mu + \mu_a)P_{n+1} = 0 \quad ; m + 1 \leq n < d + k \quad (6.3.17)\]

\[-(c\mu + \mu_a)P_{d+k} + (N - k + 1)\beta\lambda P_{d+k-1} = 0 \quad ; n = d + k \quad (6.3.18)\]

By summing up the equations relating to the equation (6.3.12) when plugging \(n = 1, 2, \ldots, c-1\) and by adding the equation (6.3.11), the following equation is got

\[P_n = \left(\frac{N \beta}{\mu}\right)^n P_0 \quad ; 0 \leq n \leq c \quad (6.3.19)\]

Similarly, after substituting \(n = c+1, c+2, \ldots, d\) in equation (6.3.14) and adding the relevant equations, the reduced form becomes
\[ P_n = \left( \frac{N \lambda}{\mu} \right)^n \left( \frac{\beta}{c} \right)^{n-c} P_0 \quad ; c + 1 \leq n \leq d \]  \hspace{1cm} (6.3.20)

Again, by summing up the set of equations from equation (6.3.15) by putting 
\( n = d+1, d+2, \ldots, m \) and the following steady-state probability is obtained
\[ P_n = \frac{N^d}{c!} \left( \frac{\lambda}{\mu} \right)^n \left( \frac{\beta}{c} \right)^{n-c} N_{(n-d)} P_0 \quad ; d + 1 \leq n \leq m \]  \hspace{1cm} (6.3.21)

where \( N_n = N(N-1)(N-2) \ldots (N-(n-1)) \)

Finally, adding the equations corresponding to the equation (6.3.17) for \( n = m+1, m+2, \ldots, d+k \), the following equation is got
\[ P_n = \frac{N^d \beta^{n-c} \lambda^n}{c! \mu^n (c\mu + \mu_a)^{n-m}} N_{(n-d)} P_0 \quad ; m + 1 \leq n \leq d + k \]  \hspace{1cm} (6.3.22)

The probability that there is no failed machine in the system \( (P_0) \) is estimated by using the following normalising condition.

\[ \sum_{n=0}^{d} P_n + \sum_{n=c+1}^{d} P_n + \sum_{n=d+1}^{m} P_n + \sum_{n=m+1}^{d+k} P_n = 1 \]  \hspace{1cm} (6.3.23)

Substituting the expressions from (6.3.19) to (6.3.22) in equation (6.3.23), the following equation is derived:
\[ P_0^{-1} = \sum_{n=0}^{d} \left[ \frac{N \left( \frac{\lambda}{\mu} \right)^n}{n!} + \frac{1}{c!} \left( \frac{c}{\beta} \right)^c \sum_{n=c+1}^{d} \left[ \frac{N \beta \lambda}{c \mu} \right]^n \right] + \frac{1}{c!} \left( \frac{c}{\beta} \right)^c N^d \sum_{n=d+1}^{m} \left[ \frac{\beta \lambda}{c \mu} \right]^n N_{(n-d)} \]

\[ + \frac{1}{c!} \left( \frac{c}{\beta} \right)^c N^d \sum_{n=m+1}^{d+k} \left[ \frac{\beta \lambda}{c \mu + \mu_a} \right]^n N_{(n-d)} \]  \hspace{1cm} (6.3.24)
Based on the equations from (6.3.19) to (6.3.22) and (6.3.24), the expected number of failed machines in the system is obtained as,

\[ L_s = \sum_{n=0}^{d+k} nP_n \]

\[
= \left[ \sum_{n=0}^{d+k} \left( \frac{N^{\frac{\lambda}{\mu}}}{(n-1)!} \right)^n + \frac{1}{c!} \left( \frac{c}{\beta} \right) \sum_{n=c+1}^{d} n \left[ N^\beta \lambda \right]^n \right] P_0 + \frac{1}{c!} \left( \frac{c}{\beta} \right)^c N^d \sum_{n=d+1}^{\infty} \left[ \frac{\beta \lambda}{c \mu} \right]^n N_{(n-d)}^\beta \left( c \mu + \mu_a \right) \sum_{n=m+1}^{\infty} \left[ \frac{\beta \lambda}{c \mu + \mu_a} \right]^n N_{(n-d)}^\beta \right] P_0
\]

(6.3.25)

Special case

When \( N = k \) and \( \beta = 1 \) are substituted in the expressions from (6.3.19) to (6.3.22), they become:

\[
P_n = \begin{cases} 
\frac{\left( \frac{k \lambda}{\mu} \right)^n}{n!} P_0 & \text{; } 0 \leq n \leq c \\
\frac{\left( \frac{k \lambda}{\mu} \right)^n}{c!} \frac{1}{c^{n-c}} k_{(n-d)} P_0 & \text{; } c + 1 \leq n \leq d \\
\frac{k^d \left( \frac{\lambda}{\mu} \right)^n}{c!} \frac{1}{c^{n-c}} k_{(n-d)} P_0 & \text{; } d + 1 \leq n \leq m \\
\frac{k^d \lambda^n}{c! c^{m-c} \mu^n (c \mu + \mu_a)^{n-m}} k_{(n-d)} P_0 & \text{; } m + 1 \leq n \leq d + k 
\end{cases}
\]

(6.3.26)
where

\[
P_{0}^{-1} = \sum_{n=0}^{c} \left( \frac{k\lambda}{\mu} \right)^{n} + \frac{1}{c!} \sum_{n=c+1}^{d} \left( \frac{k\lambda}{c\mu} \right)^{n} + \frac{1}{c!} c^{c} k^{d} \sum_{n=d+1}^{\infty} \left( \frac{\lambda}{c\mu} \right)^{n} k_{(n-d)}
\]

\[
+ \frac{1}{c!} c^{c} k^{d} \left( \frac{c\mu + \mu_{a}}{c\mu} \right)^{m} \sum_{n=m+1}^{\infty} \left( \frac{\lambda}{c\mu + \mu_{a}} \right)^{n} k_{(n-d)}
\]

(6.3.27)

The expected number of failed machines in the system is

\[
L_{s} = \left[ \sum_{n=0}^{c} \left( \frac{k\lambda}{\mu} \right)^{n} + \frac{1}{c!} \sum_{n=c+1}^{d} n \left( \frac{k\lambda}{c\mu} \right)^{n} + \frac{1}{c!} c^{c} k^{d} \sum_{n=d+1}^{\infty} n \left( \frac{\lambda}{c\mu} \right)^{n} k_{(n-d)}
\]

\[
+ \frac{1}{c!} c^{c} k^{d} \left( \frac{c\mu + \mu_{a}}{c\mu} \right)^{m} \sum_{n=m+1}^{\infty} n \left( \frac{\lambda}{c\mu + \mu_{a}} \right)^{n} k_{(n-d)} \right] P_{0}
\]

(6.3.28)

Case (ii) \( c > d \)

In this case, the number of repairmen is greater than the number of spare machines. Then the differential-difference equations of this model are formed as under:

\[
P_{0}(t) = -\lambda_{0} P_{0} + \mu_{1} P_{1}(t)
\]

; \( n = 0 \)

(6.3.29)

\[
P_{n}(t) = -(\lambda_{n} + \mu_{n}) P_{n}(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t)
\]

; \( 1 \leq n \leq d \)

(6.3.30)

\[
P_{n}(t) = -(\lambda_{n} + \mu_{n}) P_{n}(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t)
\]

; \( d + 1 \leq n < c \)

(6.3.31)

\[
P_{c}(t) = -(\lambda_{c} + \mu_{c}) P_{c}(t) + \lambda_{c-1} P_{c-1}(t) + \mu_{c+1} P_{c+1}(t)
\]

; \( n = c \)

(6.3.32)
\[ P'_n(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t) \quad ; c + 1 \leq n < m \quad (6.3.33) \]

\[ P'_m(t) = -(\lambda_m + \mu_m)P_m(t) + \lambda_{m-1}P_{m-1}(t) + (\mu_{m+1} + \mu_a)P_{m+1} \quad ; n = m \quad (6.3.34) \]

\[ P'_n(t) = -(\lambda_n + \mu_n + \mu_a)P_n(t) + \lambda_{n-1}P_{n-1}(t) + (\mu_{n+1} + \mu_a)P_{n+1}(t) \quad ; m + 1 \leq n \leq d + k \quad (6.3.35) \]

Here, the failure and service rates for this model are restated in different situations as follows:

\[ \lambda_n = \begin{cases} N\lambda & ; 0 \leq n < d \\ (N + d - n)\lambda & ; d \leq n < c \\ (N + d - n)\beta\lambda & ; c \leq n < d + k \\ 0 & ; n = d + k \end{cases} \quad (6.3.36) \]

and

\[ \mu_n = \begin{cases} n\mu & ; 0 < n \leq c \\ c\mu & ; c < n \leq m \\ c\mu + \mu_a & ; m < n \leq d + k \end{cases} \quad (6.3.37) \]

The steady-state equations are obtained by applying the values of \( \lambda_n \) and \( \mu_n \) from equations (6.3.29) to (6.3.35) and taking the limits as \( t \to \infty \).

\[-N\lambda P_0 + \mu P_1 = 0 \quad ; n = 0 \quad (6.3.38)\]

\[-(N\lambda + n\mu)P_n + N\lambda P_{n-1} + (n+1)\mu P_{n+1} = 0 \quad ; 1 \leq n \leq d \quad (6.3.39)\]

\[-[(N + d - n)\lambda + n\mu]P_n + (N + d - n + 1)\lambda P_{n-1} + (n+1)\mu P_{n+1} = 0 \quad ; d + 1 \leq n < c \quad (6.3.40)\]
Following the principles applied in case (i), by using the equations from (6.3.38) to (6.3.44), the required steady-state probabilities under different situations are

\[ P_n = \left[ \frac{N}{n!} \left( \frac{\lambda}{\mu} \right)^n \right] P_0 \quad ; \quad 0 \leq n \leq d \]  

\[ P_n = \frac{N^d}{d!} \left( \frac{\lambda}{\mu} \right)^n \frac{N_{(n-d)}}{c_{(n-d)}} P_0 \quad ; \quad d + 1 \leq n \leq c \]  

\[ P_n = \frac{N^d}{d!} \left( \frac{\lambda}{c} \right)^n \frac{N_{(n-d)}}{c_{(c-d)}} P_0 \quad ; \quad c + 1 \leq n \leq m \]  

\[ P_n = \frac{N^d}{d!} \left( \frac{\beta \lambda}{c \mu} \right)^n \frac{1}{\left( c \mu + \mu_0 \right)^{n-m}} \frac{N_{(k)}}{c_{(c-d)}} P_0 \quad ; \quad m + 1 \leq n \leq d + k \]
For estimating $P_0$, the following normalizing condition is used:

$$
\sum_{n=0}^{d} P_n + \sum_{n=d+1}^{c} P_n + \sum_{n=c+1}^{m} P_n + \sum_{n=m+1}^{d+k} P_n = 1
$$

(6.3.49)

Now, when the expressions from (6.3.45) to (6.3.48) are substituted in (6.3.49) and the required expression as:

$$
P_0^{-1} = \sum_{n=0}^{d} \frac{N^n}{n!} \left( \frac{\lambda}{\mu} \right)^n + \frac{N^d}{d!} \sum_{n=d+1}^{c} \left( \frac{\lambda}{\mu} \right)^n \frac{N_{(n-d)}}{c(n-d)} + \frac{N^d}{d!} \left( \frac{c}{\beta} \right)^c \frac{1}{c_{(c-d)}} \sum_{n=c+1}^{m} \left( \frac{\beta \lambda}{c \mu} \right)^n N_{(n-d)}
$$

$$
+ \frac{N^d}{d!} \left( \frac{c}{\beta} \right)^c \frac{N_{(d)}}{c_{(c-d)}} \left( \frac{c \mu + \mu_a}{c \mu} \right) \sum_{n=m+1}^{d+k} \left( \frac{\beta \lambda}{c \mu} \right)^n
$$

(6.3.50)

The expected number of failed machines in the system is derived by using the expressions from (6.3.45) to (6.3.48) and (6.3.50).

$$
L_s = \sum_{n=0}^{d+k} nP_n
$$

$$
= \left[ \sum_{n=0}^{d} \frac{N^n}{(n-1)!} \left( \frac{\lambda}{\mu} \right)^n + \frac{N^d}{d!} \sum_{n=d+1}^{c} n \left( \frac{\lambda}{\mu} \right)^n \frac{N_{(n-d)}}{c(n-d)} + \frac{N^d}{d!} \left( \frac{c}{\beta} \right)^c \frac{1}{c_{(c-d)}} \sum_{n=c+1}^{m} n \left( \frac{\beta \lambda}{c \mu} \right)^n N_{(n-d)}
$$

$$
+ \frac{N^d}{d!} \left( \frac{c}{\beta} \right)^c \frac{N_{(d)}}{c_{(c-d)}} \left( \frac{c \mu + \mu_a}{c \mu} \right) \sum_{n=m+1}^{d+k} n \left( \frac{\beta \lambda}{c \mu} \right)^n \right] P_0
$$

(6.3.51)
Special case

When \( \beta = 1 \) and \( N = k \), are applied in the expressions (6.3.45) to (6.3.48), they become:

\[
P_n = \begin{cases}
\left( \frac{k \lambda}{\mu} \right)^n \frac{P_0}{n!} & ; \quad 0 \leq n \leq d \\
\frac{k^d}{d!} \left( \frac{\lambda}{\mu} \right)^n \frac{k_{(n-d)}}{c_{(n-d)}} P_0 & ; \quad d + 1 \leq n \leq c \\
\frac{k^d}{d!} \frac{1}{c^{n-c}} \left( \frac{\lambda}{\mu} \right)^n \frac{k_{(n-d)}}{c_{(c-d)}} P_0 & ; \quad c + 1 \leq n \leq m \\
\frac{k^d}{d!} \frac{\lambda^n}{(c \mu)^m} \frac{c^c}{(c \mu + \mu_a)^n-m} \frac{k_{(k)}}{c_{(c-d)}} P_0 & ; \quad m + 1 \leq n \leq d + k
\end{cases}
\]

(6.3.52)

where

\[
P_0^{-1} = \sum_{n=0}^{d} \frac{k^n}{m!} \left( \frac{\lambda}{\mu} \right)^n \frac{1}{d!} \sum_{n=d+1}^{c} \frac{k^n}{\mu} \frac{k_{(n-d)}}{c_{(n-d)}} + \frac{k^d}{d!} \frac{c^c}{c_{(c-d)}} \sum_{n=c+1}^{m} \frac{k^n}{c_{(c-d)}} \frac{1}{c \mu + \mu_a}
\]

(6.3.53)

The expected number of failed machines in the system is

\[
L_n = \left[ \sum_{n=0}^{d} \frac{k^n}{(n-1)!} \left( \frac{\lambda}{\mu} \right)^n + \frac{k^d}{d!} \sum_{n=d+1}^{c} \frac{n}{\mu} \frac{k_{(n-d)}}{c_{(n-d)}} + \frac{k^d}{d!} \frac{c^c}{c_{(c-d)}} \sum_{n=c+1}^{m} \frac{n}{c \mu + \mu_a} \right] P_0
\]

(6.3.54)

This chapter provides a product type solution for multi-server Markovian queue with state dependent principle. This model is more general compared to other
existing models. The inclusion of additional repairman will be helpful to improve the grade of service for long queue. The performance measures of this chapter may be effectively used for improving real life systems such as manufacturing or production processes, computer and communication systems.