5.1 INTRODUCTION

Single server vacation policy with Erlang arrival process has been employed in Chapter IV. The probability generating function of the steady-state probabilities and the average number of customers in the queue have been analysed when the server is on vacation and is available in the system.

In this chapter, a general bulk service Markovian queueing system with server's repeated vacation is considered. The system is governed by two servers who adopt their positions as idle, busy or on vacation according to the size of the queue. The closed expression for the probability generating function and expected number of customers in the queue have been derived. The expressions for the first two factorial moments for the number of customers in the queue corresponding to the status of the servers have also been derived.

Queueing systems have been characterized by the behaviour of servers who are idle, busy or on vacation according to the size of the queue. In some situations, after completion of service for existing customers, the server continues to stay in the system awaiting new arrivals. On the other hand, if there is no customer in the system, the server will leave the system for some other tasks such as tea break, a telephone call, a check up in case of a machine and so on. The duration of these secondary tasks will be referred to as vacation time. The concept and analysis of vacation queue with bulk service were summarized by Doshi (1986). A fairly large body of literature on the results of bulk size queues with single or multi-server was presented by Medhi (1984) as well as by Chaudhry and Templeton (1983). Sathiyamoorthi and Ganesan (1989) studied multi-server Erlangian queue with bulk service.

Researchers on queueing theory have studied the single or bulk service model along with vacation policy and contributed fruitful queueing performances. In a vacation queueing model with multi-server, all servers do not go on vacation at a time, but some must wait in the system and they are either idle or busy. For example,
the customers expect the server’s presence in hospitals, telegraph offices, post offices, bank counters and so on. In multi-server vacation models, the idle time, busy time and vacation time of servers are taken into account for further analysis.

5.2 DESCRIPTION OF THE MODEL

In the present queueing model, it is assumed that the arrival pattern is Poisson with parameter $\lambda$. Arriving customers are served in batches according to Neuts’ (1967) general bulk service rule. The service time of each batch is independent of customers in the batch and exponentially distributed with parameter $\mu$. This system is governed by two servers who adopt the following working principle. On completion of service, if the server finds less than ‘a’ customers in the queue and the other server is busy, he leaves the system for a random period of time called vacation which is exponentially distributed with parameter $\theta$. If one server finds the other server on vacation, he will remain in the system under the condition that only one server is allowed to go on vacation at a time. On returning from vacation, if the server finds less than ‘a’ waiting customers and the other server is idle or busy in the system he leaves for another vacation. He continues in this manner until he finds at least ‘a’ waiting customers upon returning from vacation. This process is called repeated vacation.

The queue is considered as a Markov-process on the state space $\{(j,n) | j = 0,1,2; n \geq 0\}$, where $n \geq 0$ denotes the number of waiting customers in the queue and $j$ denotes the status of the server.

The state spaces relating to the positions of servers and the waiting customers in the queue are explained in the following lines:

The state $(0, n)$ denotes that one server is idle and the other server is on vacation when $0 \leq n \leq a - 1$.

$(1, n)$ indicates that one server is busy and the other server is on vacation when $n \geq 0$.

$(2, n)$ denotes that both the servers are busy when $n \geq 0$ and

\[ P_{j,n}(t) = P\{\text{At time } t, \text{ the system is in the state } (j,n), j = 0,1,2; n \geq 0\} \]
5.3 MATHEMATICAL FORMULATION AND SOLUTIONS

Under the above assumptions, the following differential-difference equations are formed.

\[
\frac{dP_{0,0}(t)}{dt} = -\lambda P_{0,0}(t) + \mu P_{1,0}(t), \quad n = 0 \tag{5.3.1}
\]

\[
\frac{dP_{0,n}(t)}{dt} = -(\lambda + \theta)P_{0,n}(t) + \lambda P_{0,n-1}(t) + \mu P_{1,n}(t), \quad 1 \leq n \leq a - 1 \tag{5.3.2}
\]

\[
\frac{dP_{1,0}(t)}{dt} = -(\lambda + \theta)P_{1,0}(t) + \lambda P_{0,a-1}(t) + 2\mu P_{2,0}(t) + \mu \sum_{k=a}^{b} P_{1,k}(t), \quad n = 0 \tag{5.3.3}
\]

\[
\frac{dP_{1,n}(t)}{dt} = -(\lambda + \theta)P_{1,n}(t) + \lambda P_{1,n-1}(t) + \mu P_{1,n+b}(t), \quad 1 \leq n \leq a - 1 \tag{5.3.4}
\]

\[
\frac{dP_{2,0}(t)}{dt} = -(\lambda + \theta)P_{2,0}(t) + \theta \sum_{k=a}^{b} P_{1,k}(t) + 2\mu \sum_{k=a}^{b} P_{2,k}(t), \quad n = 0 \tag{5.3.5}
\]

\[
\frac{dP_{2,n}(t)}{dt} = -(\lambda + \theta)P_{2,n}(t) + \lambda P_{2,n-1}(t) + \theta P_{1,n+b}(t) + 2\mu P_{2,n+b}(t), \quad n \geq 1 \tag{5.3.6}
\]

\[
\frac{dP_{3,0}(t)}{dt} = -(\lambda + \theta)P_{3,0}(t) + \theta \sum_{k=a}^{b} P_{2,k}(t) + \mu \sum_{k=a}^{b} P_{3,k}(t), \quad n = 0 \tag{5.3.7}
\]

Now, introduce the Laplace transform as,

\[
P_{0}(s) = \int_{0}^{\infty} e^{-st} P_{0}(t) \, dt \quad \text{and} \quad s P_{0}(s) = \int_{0}^{\infty} e^{-st} \frac{dP_{0}(t)}{dt} \, dt \tag{5.3.8}
\]

Apply the Laplace transform (5.3.8) in the equations from (5.3.1) to (5.3.7), and get,

\[
(s + \lambda)P_{0,0}(s) = \mu P_{1,0}(s) \tag{5.3.9}
\]

\[
(s + \lambda + \theta)P_{0,n}(s) = \lambda P_{0,n-1}(s) + \mu P_{1,n}(s), \quad 1 \leq n \leq a - 1 \tag{5.3.10}
\]

\[
(s + \lambda + \mu)P_{1,0}(s) = \lambda P_{0,a-1}(s) + 2\mu P_{2,0}(s) + \mu \sum_{k=a}^{b} P_{1,k}(s), \quad n = 0 \tag{5.3.11}
\]

\[
(s + \lambda + \mu)P_{1,n}(s) = \lambda P_{1,n-1}(s) + 2\mu P_{2,n}(s) + \mu P_{1,n+b}(s), \quad 1 \leq n \leq a - 1 \tag{5.3.12}
\]
\[(s + \lambda + \mu + \theta)P_{1,n}(s) = \lambda P_{1,n-1}(s) + \mu P_{1,n+\theta}(s), \quad n \geq a \quad (5.3.13)\]

\[(s + \lambda + 2\mu + \theta)P_{2,0}(s) = \theta \sum_{k=a}^{b} P_{2,k}(s) + 2\mu \sum_{k=a}^{b} P_{2,k}(s) \quad (5.3.14)\]

\[(s + \lambda + 2\mu + \theta)P_{2,n}(s) = \lambda P_{2,n-1}(s) + \theta P_{1,n+\theta}(s) + 2\mu P_{2,n+\theta}(s) \quad (5.3.15)\]

Define the probability generating functions in terms of Laplace Transforms

\[P_0(s,z) = \sum_{n=0}^{\infty} z^n P_{0,n}(s) \quad \text{and} \quad P_i(s,z) = \sum_{n=0}^{\infty} z^n P_{i,n}(s), \quad i = 1,2 \quad (5.3.16)\]

The equations (5.3.12) and (5.3.13) are multiplied by \(z^{n+b}\) and summing up \(n\) such as \(n = 1, 2, \ldots, a-1\) and \(n = a, a+1, \ldots, \infty\) respectively and also equation (5.3.11) is multiplied by \(z^a\). The resultant expression is obtained as,

\[
\left[(s + \lambda + \mu + \theta) z^b - \lambda z^{b+1} - \mu \right] P_1(s,z) = \theta \sum_{n=0}^{\infty} P_{1,n}(s) z^n + 2\mu \sum_{n=0}^{\infty} P_{2,n}(s) z^n
\]

\[- \mu \sum_{n=0}^{b} P_{1,n}(s) z^n + \lambda z^b P_{0,a-1}(s) + \mu z^b \sum_{k=a}^{b} P_{1,k}(s) \]

\[= 6\mu \theta \sum_{n=0}^{\infty} P_{1,n}(s) z^n + 2\mu \sum_{k=a}^{b} P_{1,k}(s) \quad (5.3.17)\]

Similarly, the equation (5.3.15) is multiplied by \(z^{n+b}\) and adding the values \(n = 1, 2, \ldots, \infty\) and also equation (5.3.14) is multiplied by \(z^b\) through the set of expressions (5.3.16) and get,

\[
\left[(s + \lambda + 2\mu + \theta) z^b - \lambda z^{b+1} - 2\mu \right] P_2(s,z) = \theta P_1(s,z) - \theta \sum_{n=0}^{b} P_{1,n}(s) z^n - 2\mu \sum_{n=0}^{b} P_{2,n}(s) z^n
\]

\[+ \theta z^b \sum_{k=a}^{b} P_{1,k}(s) + 2\mu \sum_{k=a}^{b} P_{2,k}(s) \]

\[= \theta P_1(s,z) + \theta P_{0,0}(s) \quad (5.3.18)\]

The equation (5.3.10) is multiplied by \(z^n\) and summing over \(n\) which takes the values \(n = 1, 2, \ldots, a-1\) and adding the equation (5.3.9), the following equation is obtained,

\[(s + \lambda - \theta z + \theta) P_0(s,z) = -\lambda z P_{0,a-1}(s) z^{a-1} + \mu \sum_{n=0}^{a-1} P_{1,n}(s) z^n + \theta P_{0,0}(s) \quad (5.3.19)\]
The probability generating function of the Laplace transform for the number of customers in the queue is obtained, after using equations (5.3.17), (5.3.18) and (5.3.19)

\[
P(s, z) = (s + \lambda - \lambda z + \theta)^{-1} \left( -\lambda P_{0,a-1}(s) z^n + \mu \sum_{n=0}^{a-1} P_{1,n}(s) z^n + \theta P_{0,0}(s) \right) \\
+ \left\{ (s + \lambda + \mu + \theta) z^n - \lambda z^{n+1} - \mu \right\}^{-1} \left[ \theta z^b \sum_{n=0}^{a-1} P_{1,n}(s) z^n + 2\mu z^b \sum_{n=0}^{a-1} P_{2,n}(s) z^n \\
- \mu \sum_{b=0}^{b} P_{1,b}(s) z^n + \lambda z^b P_{0,a-1}(s) + \mu \sum_{k=a}^{b} P_{1,k} \right] \left[ 1 + \theta \left\{ (s + \lambda + 2\mu + \theta) z^b - \lambda z^{b+1} - 2\mu \right\}^{-1} \right]
\]

(5.3.20)

Assuming that \( \lim_{s \to 0} P(s, z) = P(z) \), then the expression (5.3.20) becomes

\[
P(z) = (\lambda - \lambda z + \theta)^{-1} \left( -\lambda P_{0,a-1} z^n + \mu \sum_{n=0}^{a-1} P_{1,n} z^n + \theta P_{0,0} \right) + \left\{ (\lambda + \mu + \theta) z^n - \lambda z^{n+1} - \mu \right\}^{-1} \left[ \theta z^b \sum_{n=0}^{a-1} P_{1,n} z^n + 2\mu z^b \sum_{n=0}^{a-1} P_{2,n} z^n - \mu \sum_{n=0}^{b} P_{1,n} z^n + \lambda z^b P_{0,a-1} + \mu z^b \sum_{k=a}^{b} P_{1,k} \right] \\
\left[ 1 + \theta \left\{ (\lambda + 2\mu + \theta) z^n - \lambda z^{n+1} - 2\mu \right\}^{-1} \right] + \left\{ (\lambda + 2\mu + \theta) z^n - \lambda z^{n+1} - 2\mu \right\}^{-1} \left[ \theta z^b \sum_{k=a}^{b} P_{1,k} + 2\mu z^b \sum_{k=a}^{b} P_{2,k} - \theta \sum_{n=0}^{b} P_{1,n} z^n - 2\mu \sum_{n=0}^{b} P_{2,n} z^n \right]
\]

(5.3.21)

Now, introduce some notations for further analysis

\[
\alpha_{ij} = \sum_{n=j}^{a-1} P_{i,n}, \quad i = 1, 2; \ j = 0, 1
\]

\[
\beta_{ij} = \sum_{n=1}^{i} n P_{i,n}, \quad i = 1, 2; \ j = a-1, b
\]

\[
\beta_{i,2} = \sum_{n=2}^{b} n(n-1) P_{i,n}, \quad i = 1, 2; \ j = a-1, b
\]

\[
\gamma_{i} = \sum_{k=a}^{b} P_{i,k}, \quad i = 1, 2
\]

\[
\gamma_{i,2} = \sum_{n=2}^{a-1} (b + n)(b + n - 1) P_{i,n}
\]

(5.3.22)
The steady-state probabilities \( P_n \) can be obtained by equating the coefficient of \( z^n \) in the expression (5.3.21). The expected number of customers in the queue irrespective of the nature of the servers is obtained by applying \( z = 1 \) and using the expression (5.3.22) in the first derivative of the expression (5.3.21).

\[
L_q = \frac{1}{\theta^2} \left[ \mu \alpha_{11}(\theta + 1) - \lambda \left( a \theta + \lambda \right) P_{0,a-1} + \mu P_{1,0} + \theta P_{0,0} \right]
+ \frac{(1 + \theta)}{\theta^3} \left[ \left( \theta^2 \left( b \alpha_{11} + \beta_1 \left( \frac{a-1}{a} \right) \right) + 2 \mu \theta \left( b \alpha_{21} + \beta_2 \left( \frac{a-1}{a} \right) \right) - \mu \theta \beta_1^{(b)} + \lambda b \theta P_{0,a-1} + \mu b \theta \gamma_1 \right)
+ \left( \lambda - (\mu + \theta) b \right) \left( \left( \theta - \mu \right) \alpha_{10} + 2 \mu \alpha_{20} + \lambda P_{0,a-1} \right) + \frac{1}{\theta} \left( 2 \mu b \gamma_2 + b \theta \gamma_1 - 2 \mu \beta_2^{(b)} - \theta \beta_1^{(b)} \right)
+ \frac{1}{\theta^2} \left( \lambda - (2 \mu + \theta) b \right) \left\{ 2 \mu (1 - \theta) \alpha_{20} - \left( \theta^2 - \theta + \mu \right) \alpha_{10} + \lambda P_{0,a-1} \right\} \right]
\]  
(5.3.23)

The first and second factorial moments of the number of customers in the queue are deduced from the expression (5.3.21) based on the availability of the servers in the system under the condition that servers are idle, busy or on vacation as the case may be.

**Case (i)**

One server is idle and the other server is on vacation.

\[
P_0^{(1)} = \frac{1}{\theta^2} \left[ \mu \theta \beta_1^{(a-1)} + \lambda \left( a \theta + \lambda \right) P_{0,a-1} + \lambda \theta P_{0,0} \right]
\]  
(5.3.24)

\[
P_0^{(2)} = \frac{1}{\theta} \left[ \theta^2 \left\{ \mu \beta_{1,a-1}^\lambda - \lambda a (a-1) P_{0,a-1} \right\} + 2 \lambda \theta \left\{ \lambda \beta_1^{(a-1)} - \lambda a P_{0,a-1} \right\}
+ 2 \lambda^2 \left\{ \mu \alpha_{10} - \lambda a P_{0,a-1} + \theta P_{0,0} \right\} \right]
\]  
(5.3.25)

**Case (ii)**

One server is busy and the other server is on vacation.

\[
P_1^{(1)} = \frac{1}{\theta^2} \left[ \left( \theta^2 \left( b \alpha_{11} + \beta_1 \left( \frac{a-1}{a} \right) \right) + 2 \mu \theta \left( b \alpha_{21} + \beta_2 \left( \frac{a-1}{a} \right) \right) - \mu \theta \beta_1^{(b)} + \lambda b \theta P_{0,a-1} + \mu b \theta \gamma_1 \right)
+ \left\{ \lambda - (\mu + \theta) b \right\} \left( \left( \theta - \mu \right) \alpha_{10} + 2 \mu \alpha_{20} + \lambda P_{0,a-1} \right) \right]
\]  
(5.3.26)
Case (iii)

Both servers are busy

\[
P_{2}^{(1)} = \frac{1}{\theta^3} \left[ \theta \left\{ P_{1}^{(2)} - \theta \beta_{1,2}^{(b)} - 2 \mu \beta_{2,2}^{(b)} + b(b - 1)(\theta \gamma_1 + 2 \theta \gamma_2) \right\} + 2 \theta \left\{ \lambda - (\mu + \theta) b \right\} \right] + \left\{ \theta - (\mu + \theta) b \right\} \left\{ P_{1}^{(1)} - \theta \beta_{1,1}^{(b)} - 2 \mu \beta_{2}^{(b)} + b(\theta \gamma_1 + 2 \theta \gamma_2) \right\} + \left\{ \theta - (\mu + \theta) b \right\} \left\{ P_{1}^{(1)} - \theta \alpha_{10} - 2 \mu \alpha_{20} \right\} \left\{ \theta b \left[ 2 \lambda - (\mu + \theta)(b - 1) \right] + 2 \left[ \lambda - (\mu + \theta) b \right]^2 \right\} \right] \]

(5.3.28)

\[
P_{2}^{(2)} = \frac{1}{\theta^3} \left[ \theta \left\{ P_{1}^{(2)} - \theta \beta_{1,2}^{(b)} - 2 \mu \beta_{2,2}^{(b)} + b(b - 1)(\theta \gamma_1 + 2 \theta \gamma_2) \right\} + 2 \theta \left\{ \lambda - (\mu + \theta) b \right\} \right] + \left\{ \theta - (\mu + \theta) b \right\} \left\{ P_{1}^{(1)} - \theta \beta_{1,1}^{(b)} - 2 \mu \beta_{2}^{(b)} + b(\theta \gamma_1 + 2 \theta \gamma_2) \right\} + \left\{ \theta - (\mu + \theta) b \right\} \left\{ P_{1}^{(1)} - \theta \alpha_{10} - 2 \mu \alpha_{20} \right\} \left\{ \theta b \left[ 2 \lambda - (\mu + \theta)(b - 1) \right] + 2 \left[ \lambda - (\mu + \theta) b \right]^2 \right\} \right] \]

(5.3.29)

The expressions (5.3.24), (5.3.26) and (5.3.28) are the mean number of customers in the queue when (i) one server is idle and the other is on vacation (ii) one server is busy and the other is on vacation, and (iii) both servers are busy respectively.

Also the variances of the number of customers in the queue can be easily obtained through factorial moments given from the expressions (5.3.24) to (5.3.29) for different situations of the servers.