Chapter-3

AN \((s,S)\) INVENTORY SYSTEM WITH STATE DEPENDENT DEMANDS

3.1 INTRODUCTION

Conventional inventory models assume demand and inventory level as independent quantities. In this chapter we consider a continuous review \((s,S)\) inventory model in which quantity demanded by each arriving unit is a positive integer valued random variable that depends on the present inventory level. The time durations between successive demands are i.i.d random variables with finite expectations. It is assumed that quantity demanded will not exceed what is available. In situations like famine etc. the Government directs the shopkeepers to exhibit the quantity of items available with them and its price. Customers rationally buy items depending on its availability. Sometimes customers may be motivated to procure with the ease of availability. This kind of behaviour may be approximated by a stock dependent demand pattern.

Gupta and Vrat (1986) suggested an EOQ model through cost minimization technique to take care of stock dependent consumption rate. This could not take care of stock-dependent demand rate except where the demand rate is dependent on replenishment size. Mandal and Phaujdar (1989)
proposed an EOQ model with instantaneous replenishment, without shortages and demand rate depending upon the current stock level, which is assumed to be linearly increasing with stock status.

Two models are treated in this chapter. In section 3.2 we consider the model with zero lead time. Using renewal theoretic arguments, the system state probability distribution at arbitrary time and also the limiting distribution are obtained. The results are illustrated by a numerical example and a method of finding optimal decision rules is briefly discussed. Section 3.3 is concerned with the model with random lead time. In this case inventory level probability distribution at arbitrary time is derived by applying the techniques of semi-regenerative process. The computation of limiting distribution is also indicated.

We introduce the following notations used in this chapter.

\[ I(t) \quad - \quad \text{Inventory level at time } t \quad (\geq 0) \]

\[ F(.) \quad - \quad \text{Distribution function of time between two successive demands (interarrival time distribution).} \]

\[ f(.) \quad - \quad \text{Density function of } F(.). \]
\( F^{*n}(t) \) - n-fold convolution of \( F \) with itself, 
\( n=1,2, \ldots, \) with \( F^{*0}(t) \equiv 1. \)

\( G(.) \) - Lead time distribution function

\( g(.) \) - Density function of \( G(.) \)

\( k(u) \) - \[ \sum_{n=0}^{\infty} i^{*n}(u) \]

\( P(n,t) \) - Probability that \( I(t)=n, n=1,2, \ldots, S. \)

\( \hat{P}(n,u) \) - Laplace transform of \( P(n,t), n=1,2, \ldots, S. \)

\( q_{ij} \) - Probability that \( j \) units are demanded when the inventory level is \( i \)

\( p_{ij} \) - Probability that at a demand epoch there were \( i \) units and due to the demand the level is brought to level \( j \).

\( \mathbb{N} \) - \( S-s \)

\( \mathcal{E} \) - \( \{C,1,2, \ldots, s \} \)

\( \mathbb{E} \) - \( \{C,1,2, \ldots, S \} \)

\( F \) - \( \{s+1, s+2, \ldots, S \} \)
3.2 MODEL-I: ZERO LEAD TIME CASE

Here we assume that lead time is zero and no shortage is permitted. As soon as the inventory level falls to \( s \) or below an order is placed to bring back the inventory to \( S \). If \( X_n \) denotes the inventory level after the \( n \)th demand, then \( \{X_n\} \) forms a Markov chain with state space \( F = \{s+1, s+2, \ldots, S\} \) and its transition probabilities are given by

\[
\begin{align*}
\varphi_{i,j,n}^p &= \begin{cases} 
    \sum_{i_1, i_2, \ldots, i_{n-1}} p_{i_1} p_{i_2} \cdots p_{i_{n-1} j} ; & (i > i_1 > i_2 > \ldots > i_{n-1} > j) \\
    \sum_{i_1, i_2, \ldots, i_{n-1}} p_{i_1} p_{i_2} \cdots p_{i_{n-1} j} ; & (i > s, j \leq s) 
  \end{cases} 
\end{align*}
\]

First of all, we shall obtain the distribution of a cycle which is defined as the time duration between two successive \( S \) to \( S \) transitions. We assume that \( X_0 = I(0) = S \).
Let $Z$ be the length of a cycle. Then

$$h(z) = \Pr\{ z < Z \leq z+dz \}$$

$$= \sum_{n=1}^{M} \Pr\{ z < Y_1+Y_2+ \ldots + Y_n \leq z+dz \} \phi_{S,S}^n$$

where $Y_1, Y_2, \ldots$ are i.i.d random variables with distribution $F(.)$ and $\phi_{S,S}^n$ is the probability that starting from $S$, the inventory level reaches back to $S$ at the $n$th transition for the first time.

$$= \sum_{i_1, i_2, \ldots, i_{n-1}} \Pr\{ i_1 + i_2 + \ldots + i_{n-1} < Y_1 + \ldots + Y_n \} \phi_{S,S}^n$$

where $i_1, i_2, \ldots, i_{n-1} \in F$ and $S > i_1 > i_2 > \ldots > i_{n-1} > s$.

Thus

$$h(z) = \sum_{n=1}^{M} f^{*n}(z) \phi_{S,S}^n$$

(3.2.1)

Let $Z_1, Z_2, \ldots$ be the lengths of successive cycles. The distribution of $Z_i$'s are i.i.d with p.d.f. $h(.)$. Then $\{Z_i\}$ forms a renewal process and the corresponding renewal density is given by

$$m(u) = \sum_{r=1}^{\infty} h^{*r}(u)$$

(3.2.2)
Now we can find out the probability distribution of the system size. We have

\[ P(S,t) = 1 - F(t) + \int_0^t m(u) \left[ 1 - F(t-u) \right] du \]  \hspace{1cm} (3.2.3)

and for \( s+1 \leq i < S \)

\[ P(i,t) = \sum_{j=1}^{S-i} \left[ F^j(t) - F^{j+1}(t) \right] \phi^j_{S,i} \]

\[ + \int_0^t m(u) \sum_{j=1}^{S-i} \left[ F^j(t-u) - F^{j+1}(t-u) \right] \phi^j_{S,i} du \]  \hspace{1cm} (3.2.4)

where \( \phi^j_{S,i} \) is the probability of first visit to \( i \) in \( j \) transitions, starting from \( S \), without visiting the state \( S \) in between.

LIMITING PROBABILITY DISTRIBUTION

Taking Laplace transforms of both sides of (3.2.3) we get

\[ \hat{P}(S, \alpha) = \frac{1}{\alpha} \left[ 1 - \hat{f}(\alpha) \right] + \hat{m}(\alpha) \frac{1}{\alpha} \left[ 1 - \hat{f}(\alpha) \right] \]

But \( \hat{m}(\alpha) = \sum_{r=1}^{\infty} \left[ \hat{h}(\alpha) \right]^r = \frac{\hat{h}(\alpha)}{1 - \hat{h}(\alpha)} \) (since \( \hat{h}(\alpha) < 1 \))
where \( \hat{h}(a) = \sum_{n=1}^{M} \hat{f}(a)^n \varnothing^n_{S,S} \)

Therefore

\[
\hat{P}(S,a) = \frac{1}{a}[1 - \hat{f}(a)] + \left[ 1 + \frac{\sum_{n=1}^{M} \hat{f}(a)^n \varnothing^n_{S,S}}{1 - \sum_{n=1}^{M} \hat{f}(a)^n \varnothing^n_{S,S}} \right] \quad (3.2.5)
\]

Similarly, taking Laplace transforms of both sides of (3.2.4), we get

\[
\hat{P}(i,a) = \sum_{j=1}^{S-i} \left[ \frac{1}{a} \hat{f}(a)^j - \frac{1}{a} \hat{f}(a)^{j+1} \right] \varnothing^j_{S,i} \\
+ \sum_{j=1}^{S-i} \hat{m}(a) \left[ \frac{1}{a} \hat{f}(a)^j - \frac{1}{a} \hat{f}(a)^{j+1} \right] \varnothing^j_{S,i} \\
= \sum_{j=1}^{S-i} \frac{1}{a} \hat{f}(a)^j \left[ 1 - \hat{f}(a) \right] \varnothing^j_{S,i} \\
+ \frac{\sum_{n=1}^{M} \hat{f}(a)^n \varnothing^n_{S,S}}{1 - \sum_{n=1}^{M} \hat{f}(a)^n \varnothing^n_{S,S}} \quad , \quad (3.2.6)
\]

\( i = s+1, s+2, \ldots, S-1. \)
Let $P_n$ be the probability that the inventory level is $n$ ($n = s+1, s+2, \ldots, S$) in the steady state. Then,

$$P_n = \lim_{t \to \infty} Pr \{I(t) = n\} = \lim_{\alpha \to 0} \alpha \hat{P}(n, \alpha)$$

It is easy to verify from (3.2.5) that

$$P_S = \lim_{\alpha \to 0} \frac{[1 - \hat{f}(\alpha)] M \phi_S^n, S}{1 - \sum_{n=1}^{M} [\hat{f}(\alpha)]^n \phi_S^n, S}$$

To obtain the limiting value of this indeterminate expression, we apply L'Hospital's rule once, yielding

$$P_S = \frac{\sum_{n=1}^{M} \phi_S^n, S}{\sum_{n=1}^{M} n \phi_S^n, S}$$  \hspace{1cm} (3.2.7)$$

Similarly, we get

$$P_i = \frac{\sum_{j=1}^{S-i} \phi_i^n, S \phi_j^n, S \sum_{n=1}^{M} \phi_S^n, S}{\sum_{n=1}^{M} n \phi_S^n, S}$$ \hspace{1cm}, \ i = s+1, s+2, \ldots, S-1$$  \hspace{1cm} (3.2.8)
Thus, in the steady state, the inventory level is distributed as given in (3.2.7) and (3.2.8) and is independent of the distribution of the interarrival time between demands. One can easily see that this result reduces to Sivazlian (1974) when unit quantities are demanded. Further this reduces to Sahin (1983) and Ramanarayanan and Jacob (1987) with zero lead time when quantity demanded are i.i.d random variables.

Example

Suppose that $s=0$, $s=5$. Then $F = \{1, 2, 3, 4, 5\}$

Let $Q = (q_{ij})_{i,j \in F} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & 0 \\
1/4 & 1/4 & 1/4 & 1/4 & 0 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
\end{bmatrix}$

Then $P = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
1/2 & 0 & 0 & 0 & 1/2 \\
1/3 & 1/3 & 0 & 0 & 1/3 \\
1/4 & 1/4 & 1/4 & 0 & 1/4 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
\end{bmatrix}$
\[ \phi^k_{S_i}, \ i \in F \text{ and } k = 1, 2, \ldots, 5 \] are obtained as given in the following table.

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/5</td>
<td>13/60</td>
<td>3/40</td>
<td>1/120</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1/5</td>
<td>7/60</td>
<td>1/60</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1/5</td>
<td>1/20</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1/5</td>
<td>5/12</td>
<td>7/24</td>
<td>1/12</td>
<td>1/120</td>
</tr>
</tbody>
</table>

The steady-state probabilities are calculated using (3.2.7) and (3.2.8) as

<table>
<thead>
<tr>
<th>n</th>
<th>( p_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.218978134</td>
</tr>
<tr>
<td>2</td>
<td>0.145985422</td>
</tr>
<tr>
<td>3</td>
<td>0.109489067</td>
</tr>
<tr>
<td>4</td>
<td>0.087591253</td>
</tr>
<tr>
<td>5</td>
<td>0.437956268</td>
</tr>
<tr>
<td>( \Sigma )</td>
<td>1.000000144</td>
</tr>
</tbody>
</table>
JOINT DISTRIBUTION OF THE QUANTITY ORDERED AND THE LENGTH OF A CYCLE

Let Q denotes the quantity ordered in a cycle whose length is Z. Then the joint distribution of Q and Z is

\[ \eta(n,z) = \Pr\{Q=n, z < Z \leq z+dz\} \]

\[ = \sum_{k=1}^{M} \Pr\{Q=n, z < Z \leq z+dz | k \text{ demands}\} \frac{\Pr\{k \text{ demands}\}}{\Pr\{k \text{ demands}\}} \]

\[ = \sum_{k=1}^{M} \sum_{S \leftarrow i_1 \leftarrow i_2 \leftarrow \ldots \leftarrow i_{k-1} \leftarrow S} p_{S i_1 i_2} \cdots \prod_{j=1}^{k-2} q_{i_{j-1} i_j} \prod_{j=1}^{k-1} (S-n) f^k(z) \]

Now the expected value of the quantity ordered per unit time can be calculated as

\[ E[Q/Z] = \sum_{n=M}^{S} \int_{0}^{\infty} \frac{\eta(n,z)}{z} dz \]  

(3.2.9)

Also

\[ E[Z] = \sum_{n=1}^{M} n E(Y) \phi_n^{SS} \]  

(3.2.10)
OBJECTIVE FUNCTION AND OPTIMAL DECISION RULE

We assume that the procurement cost consists of a fixed cost $K$ and variable cost $c$ per unit. The holding cost is $h$ per unit per unit time. Our objective function here is the steady state expected total cost per unit time; the decision variables $s$ and $S$ are to be selected so as to minimize the objective function.

Expected inventory at any given time is

$$E(I) = \frac{S}{s+1} \sum_{i=s+1}^{S} p_i,$$

where $p_i$'s are given by (3.2.7) and (3.2.8).

The total expected cost per unit time is

$$C(s,S) = \frac{K}{E(Z)} + c E[Q/Z] + h E(I) \quad (3.2.11)$$

The value of $s$ and $S$ which minimize the above expression are the optimal values (for a given $(q_{ij})$).

Remark

The model with zero lead time and quantity demanded not restricted to be at most what is available can be analysed in a similar fashion if we assume that the replenishment is done in such a way as to bring the inventory on hand back to $S$ after meeting the demand that has just taken place.
3.3 MODEL-II: RANDOM LEAD TIME CASE

In this section we consider the model with random lead times. The quantities demanded depend on the inventory level at the demand epoch. Not more than what is available will be demanded (will be sold). Lead times are i.i.d random variables with distribution function \( G(.) \) and density \( g(.) \). No backlogging is allowed. As soon as the inventory level falls to \( s \) or below due to a demand, an order is placed and the quantity ordered for is to bring back the level to \( S \) (i.e., if inventory on hand is \( i \) \( \leq s \) at the time of ordering, then the quantity ordered is \( S-i \). The demands that arise during a dry period are lost (in fact, by our assumption no item will be demanded by the arrivals during dry period).

Let \( Y_0,Y_1,Y_2,\ldots,Y_n \ldots \) be the successive inventory levels at which orders are placed and \( 0 = T_0 < T_1 < T_2 < \ldots < T_n < \ldots \) be the corresponding ordering epochs. Then \( \{(Y_n,T_n), n = 0,1,2,\ldots \} \) constitutes a Markov renewal process on the state space \( E = \{0,1,2,\ldots,s\} \). The semi-Markov kernel of this process is

\[
Q(i,j,t) = P[Y_{n+1}=j, T_{n+1}-T_n \leq t / Y_n=i]
\]
and is given by

\[ Q(i,j,t) = \int_{v}^{t} \int_{u}^{t} \int_{o}^{u} \sum_{n=1}^{i} f^{n}(u) \varphi_{i,n}^{n,k(v-u)} g(w) \]

\[ \sum_{m=1}^{S-i-j} \left[ \frac{F^{m}(t-v)-F^{m+1}(t-v)}{1-F^{m}(w-v)} \right] \varphi_{S-i,j}^{m} \int_{v}^{t} \int_{u}^{t} \int_{o}^{u} \sum_{n=1}^{i} f^{n}(u) \varphi_{i,k}^{n} g(v) \]

\[ \sum_{m=1}^{S-i+k-j} \left[ \frac{F^{m}(t-v)-F^{m+1}(t-v)}{1-F^{m}(v-u)} \right] \varphi_{S-i+k,j}^{m} \int_{v}^{t} \int_{u}^{t} \int_{o}^{u} \]

\[ i,j \in E \quad (3.3.1) \]

In the above expression for \( Q(i,j,t) \), the first term on the right deals with the case of arrivals (demands) taking place during dry period and second one considers the case of no dry period in between two consecutive replenishments. Let an order placing point be taken as time origin and suppose at such a point the level falls to \( i (\leq s) \), another \( n \) (\( n=1,2,...,i \)) demands bring it to level zero at time \( u \), (if at time 0 the level has not already become zero due to the demand) then follows a dry period with a number of arrivals, this is represented by \( k(v-u) \), demand quantity by those arrivals is zero by our
assumption— or we may call these unmet demands, the last such taking place at time \(v\). The replenishment takes place at time \(w(>v)\). Now the inventory level is \(S-i\). The next demand (the one after time \(v\)) takes place after time \(w\) and in the interval \((w,t)\) there are exactly \(m\) demands, taking away a total of \(S-i-j\) units to bring the level to \(j(\leq s)\) thereby resulting in the next order placing. This explains the first term on the right in (3.3.1). The second term is similarly explained except that in this case there is no dry period.

The Markov renewal functions of the process is given by

\[
R(i,j,t) = \sum_{n=0}^{\infty} Q^n(i,j,t), \quad i,j \in E
\]  

(3.3.2)

where

\[
Q^n(i,j,t) = \Pr [ Y_n=j, T_n \leq t / Y_0=i ]
\]

\[
= \sum_{k \in E} \int_{0}^{t} Q(i,k,du)Q^{n-1}(k,j,t-u)
\]

for \(n \geq 1, \ t \geq 0\)

and

\[
Q^0(i,j,t) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for all } t \geq 0
\]
TIME DEPENDENT PROBABILITY DISTRIBUTION OF INVENTORY LEVEL.

Assume that at time zero an order has just been placed with inventory level at \( i \) (\( i \leq s \)) and the demand process starts. Define

\[
P(i,j,t) = \Pr \left[ I(t) = j \mid I(0) = Y_0 = i \right],
\]

\[i \in E, \; j \in E^c.
\]

Since the stochastic process \( \{I(t), \; t \geq 0\} \) is a semi-regenerative process with the embedded Markov renewal process \( (Y,T) = \{(Y_n,T_n), \; n=0,1,2,\ldots\} \), the function \( P(i,j,t) \) satisfies the following Markov renewal equation. [see Cinlar (1975)].

\[
P(i,j,t) = K(i,j,t) + \int_0^t \sum_{r \in E} Q(i,r,du) P(r,j,t-u), \quad (3.3.3)
\]

\[i \in E \text{ and } j \in E^c
\]

where

\[
K(i,j,t) = \Pr[I(t) = j, \; T_1 > t/I(0) = i], \; i \in E, \; j \in E^c
\]

For any \( t \geq 0 \), the function \( K(i,j,t) \) is given by

1) \( i \in E \) and \( i < j \leq s \)

\[
K(i,j,t) = 0
\]
ii) for \( i \in E, j < i \)

\[
K(i,j,t) = \mathbb{G}(t) \sum_{m=1}^{i-j} [F^m(t) - F^{(m+1)}(t)] \varnothing_{i,j}^m
\]

iii) for \( i \in E, s < j \leq S-i \)

\[
K(i,j,t) = \sum_{k=1}^{S-i+k-j} \int_{o}^{t} \int_{0}^{s} \int_{n=1}^{i-k} f^n(u) \varnothing_{i,k}^n g(v) \\
\sum_{m=1}^{S-i-j} \int_{m}^{S-i-j} \int_{o}^{t} \int_{0}^{s} \int_{n=1}^{i-k} f^n(u) \varnothing_{i,k}^n g(v)
\]

and finally

iv) for \( i \in E, S-i < j \leq S \)

\[
K(i,j,t) = \int_{o}^{t} \int_{0}^{s} \int_{n=1}^{i-k} f^n(u) \varnothing_{i,k}^n g(v) \\
\sum_{m=1}^{S-i+k-j} \int_{m}^{S-i+k-j} \int_{o}^{t} \int_{0}^{s} \int_{n=1}^{i-k} f^n(u) \varnothing_{i,k}^n g(v)
\]
Let $\hat{P}_\alpha$, $\hat{K}_\alpha$ and $\hat{Q}_\alpha$ denote matrices whose $(i,j)$th elements are $\hat{P}(i,j,\alpha)$, $\hat{K}(i,j,\alpha)$ and $\hat{Q}(i,j,\alpha)$ respectively, where

$$\hat{Q}(i,j,\alpha) = \int_0^\infty \exp(-\alpha t) Q(i,j,dt)$$

Then the Laplace transform of the set of Markov renewal equations can be expressed as

$$\hat{P}_\alpha = \hat{K}_\alpha + \hat{Q}_\alpha \hat{P}_\alpha$$

which in turn yields

$$\hat{P}_\alpha = (I-\hat{Q}_\alpha)^{-1} \hat{K}_\alpha = \hat{R}_\alpha \hat{K}_\alpha \quad (3.3.4)$$

where $\hat{R}_\alpha$ is the matrix of Laplace transform of Markov renewal functions of the Markov renewal process $(Y,T)$ which exists for $\alpha > 0$ [see Cinlar (1975)].

**LIMITING DISTRIBUTION OF THE INVENTORY LEVEL**

In order to obtain the limiting distribution of the stock level, consider the Markov chain $Y = \{Y_n, n \geq 0\}$ associated with the Markov renewal process $(Y,T)$. The transition probability matrix $Q' = (Q'(i,j))$ of order $s$
is given by

\[ Q'(i,j) = \lim_{t \to \infty} Q(i,j,t) \]

If the chain Y is irreducible, it possesses a unique stationary distribution \( \pi = (\pi_0, \pi_1, \ldots, \pi_s) \) which satisfies \( \pi Q' = \pi \) and \( \sum \pi_j = 1 \).

Let \( \mathbf{P} = (P_0, P_1, P_2, \ldots, P_S) \) denotes the steady state probability vector of the inventory level where \( P_j = \lim_{t \to \infty} P(i,j,t) \) is the limiting distribution of the inventory level. Now making use of the result given in Cinlar (1975) and assuming that \((Y,T)\) is irreducible recurrent aperiodic we have

\[ P_j = \sum_{k \in E} \pi_k \int_0^\infty K(k,j,t) dt / \sum_{k \in E} \pi_k m_k \]  \( (3.3.5) \)

where \( m_k \) is the mean sojourn time in state \( k \), given by

\[ m_k = \int_0^\infty [1 - \sum_j Q(k,j,t)] dt. \]