Chapter-2

AN \((s,S)\) INVENTORY SYSTEM WITH NON-IDENTICALLY DISTRIBUTED INTERARRIVAL DEMAND TIMES AND RANDOM LEAD TIMES*

2.1. INTRODUCTION

Inventory systems of \((s,S)\) type had been studied quite extensively in the past. A systematic account of the probabilistic treatment in the study of inventory systems using renewal theoretic arguments has been given by Arrow, Karlin and Scarf (1958). Further details of work carried out in this field can be found in Hadley and Whitin (1963), Veinott (1966), Kaplan (1970), Gross and Harris (1971). Tijms (1972) gave a detailed analysis of \((s,S)\) inventory systems and chapter 3 of his monograph deals with its probabilistic analysis. Sivazlian (1974) has considered an \((s,S)\) inventory model in which unit demands of items occur with arbitrary interarrival times between demands and zero lead time. Srinivasan (1979) examined the same problem with random lead times. Sahin (1979) analysed the model with general interarrival demand distributions and constant lead times. In all the above situations the distribution of on hand inventory

were computed and associated optimization problems were solved.

In this chapter we consider a continuous review 
(s,S) inventory model with time between successive unit 
demands independent but not identically distributed random 
variables. Specifically $X_S, X_{S-1}, \ldots, X_1, X_0$ be the times 
between successive demands when the inventory levels are 
$S, S-1, \ldots, 1, 0$ respectively. We assume that lead 
times are independent and identically distributed (i.i.d) 
random variables and are independent of the arrivals of 
demands. It is quite natural to expect in a market that 
time gap between successive demands are non-identically 
distributed and so this model might be more realistic. 
Section 2.2 contains a complete description of the model. 
System state probabilities are derived in Section 2.3. 
The cost function of the model is formulated in 
Section 2.4. The steady state behaviour of the system 
is obtained in Section 2.5 and the last section is concerned 
with the case when lead times are zero and the associated 
optimization problem, followed by a numerical example.

The renewal theorem for independent but not 
identically distributed random variables was given by 
Smith (1964) which may be used in analysing the model 
presented here.
2.2. DESCRIPTION OF THE INVENTORY POLICY

Let $S$ be the maximum capacity of a warehouse. At time $t = 0$ the inventory level is $S$. Due to incoming demands the stock level goes on decreasing. The demands are assumed to occur for one unit at a time and the time intervals between the arrivals of two consecutive demands form a family of independent non-identically distributed random variables. As soon as the stock level drops down to $s$, the reorder level, an order for replenishment is placed for $S-s$ units. We assume that $S > 2s$ to avoid perpetual shortage. The lead time— the time interval measured from the epoch when the stock level drops to $s$ to the epoch when the quantity $S-s$ reaches the warehouse—is assumed to be distributed arbitrarily with distribution function $G(.)$ but independent of stock level and demand. Lead times are assumed to be i.i.d. random variables. The market considered here is competitive enough to rule out back-lagging of demands and the demands that emanate during the stock out period are deemed to be lost. Thus the stock level can be described by a discrete valued stochastic process $\{I(t), t \geq 0\}$ with $I(0) = S$.

Let $F_\alpha(.)$, $(\alpha = 0, 1, 2, \ldots, S)$ be the successive distribution function of the time interval $X_\alpha$ between the
arrivals of two consecutive demands, when there are \( a \) units in the inventory. For the sake of convenience, the underlying distributions are taken as absolutely continuous. The corresponding small letters denote the density functions. All the results can easily be reconstructed, however, for discrete case.

\[
I(t) \quad \text{Reorder time}
\]

\[
\square \quad \text{Replenishment time}
\]

Fig. 2.1. A typical plot of the Inventory level against time.

The following notations are used in the sequel:

- \( I(t) \) — on-hand Inventory level at time \( t \).
- \( f \ast g(x) \) — convolution \( \int_0^x f(x-y)g(y)dy \) for \( f(x), g(x) \) defined on the set of non-negative real numbers.
\( f^k(x) \) - \( k \)-fold convolution of \( f(x) \) with itself, 
\[ (f^0(x) \equiv 1) \).

\( \overline{F}(\cdot) \) - \( 1 - F(\cdot) \), the survival function

\( \hat{a}(a) \) - Laplace transform \( \int_0^\infty e^{-ax}a(x)dx \).

2.3 MAIN RESULTS

Let \( I(t) \) denotes the on hand Inventory level at arbitrary time \( t \). The principal quantity of interest is the probability mass function of the inventory level at any arbitrary time \( t \) on the time axis. i.e., \( \Pr \{ I(t) = n \} \), \( n = 0, 1, 2, \ldots, S \).

Suppose now that we consider the sequence of times at which the inventory level reaches \( s \) (the reorder level) from above. Let \( Y_1 \) denote the time elapsed from origin until the first event occurred (reaching level \( s \)). \( Y_2 \) the time elapsed between the first and second event and so on. The sequence of random variables \( \{ Y_k \} \), \( k = 1, 2, \ldots \) forms a modified renewal process [See Cox (1962)]. In each of the following expressions we will make use of the renewal density \( m(u) \) of the time points at which the inventory level reach \( s \). An explicit expression for \( m(u) \) is also given.
(i) \( \Pr \{ I(t) = 0 \} = \int_0^t m(u) \overline{G}(t-u) \int_u^t (f_s \ast \ldots \ast f_1)(v-u) \) 
\[ \ast \int_0^\infty f^m m(t-v) dv \] 
\[ du \]

(ii) For \( n=1,2,\ldots,s-1 \),

\[ \Pr \{ I(t) = n \} = \int_0^t m(u) \overline{G}(t-u) \int_u^t (f_s \ast f_{s-1} \ast \ldots \ast f_{n+1})(v-u) \]

\( \overline{F}_n(t-v) dv \) 
\[ du \]

(iii) \( \Pr \{ I(t) = s \} = \int_0^t m(u) \overline{G}(t-u) \overline{F}_s(t-u) du \)

(iv) For \( n=s+1, s+2, \ldots, S-s-1 \)

\[ \Pr \{ I(t) = n \} = (f_s \ast f_{s-1} \ast \ldots \ast \overline{F}_{n+1} \ast \overline{F}_{s-1} \ast \ldots \ast \overline{F}_n)(t) \]

\[ + \int_0^t m(u) \int_u^t \overline{F}_s(v-u) g(v-u) \int_v^t (f_s \ast f_{s-1} \ast \ldots \ast f_{n+1})(w-v) \]

\[ \overline{F}_n(t-w) dw dv du \]

\[ + \sum_{k=1}^{s-1} \int_0^t m(u) \int_u^t (f_s \ast f_{s-1} \ast \ldots \ast f_{s-k+1})(v-u) \]

\[ \int_v^t \overline{F}_{s-k}(w-v) g(w-u) \int_u^w (f_{s-k} \ast \ldots \ast f_{n+1})(x-w) \]

\[ \overline{F}_n(t-x) dx dw dv du \]
\[
+ \int_0^t m(u) \int_u^t (f_1 \ast \ldots \ast f_s)(v-u) \int_v^t \left( \sum_{m=0}^\infty f_o^m(w-v) \right) w = x \nonumber \\
+ \int_0^t F_o(x-w) g(x-u) \int_u^t (f_{S-s} \ast f_{S-s-1} \ast \ldots \ast f_{n+1})(y-x) w = x \\
F_n(t-y) dy \ dx \ dw \ dv \ du. 
\]

(v) \[ \Pr \{ I(t) = S-s \} = (\bar{F}_S \ast F_{S-1} \ast \ldots \ast F_{S-s+1} \ast F_{S-1} \ast \ldots \ast F_{S-s})(t) \]

+ \int_0^t m(u) \int_u^t \bar{F}_s(v-u) g(v-u) \int_v^t (f_s \ast f_{S-s-1} \ast \ldots \ast f_{S-s+1})(w-v) w = x \nonumber \\
\bar{F}_{S-s}(t-w) dw \ dv \ du. 
\]

+ \sum_{k=1}^{s-1} \int_0^t m(u) \int_u^t (f_s \ast f_{S-s-1} \ast \ldots \ast f_{S-s-k+1})(v-u) \int_v^t \bar{F}_{S-s-k}(w-v) g(w-u) w = x \nonumber \\
\int_w^{f_{S-s-k} \ast \ldots \ast f_{S-s+1}}(x-w) \bar{F}_{S-s}(t-x) dx \ dw \ dv \ du. 
\]

+ \int_0^t m(u) \int_u^t (f_s \ast f_{S-s-1} \ast \ldots \ast f_{S-s-k+1})(v-u) \int_v^t \left( \sum_{m=0}^\infty f_o^m(w-v) \right) w = x \nonumber \\
\int_0^t F_o(x-w) g(x-u) \bar{F}_{S-s}(t-x) dx \ dv \ du.
(vi) For \( n = S - s + 1, S - s + 2, \ldots, S - 1 \)

\[
\Pr \{ I(t) = n \} = (F_s F_{S-1} \cdots F_{n+1} - F_s F_{S-1} \cdots F_n)(t) \\
+ \int_{0}^{t} m(u) \int_{0}^{t} \bar{F}_s(v-u) g(v-u) \int_{0}^{t} (f_s f_{S-1} \cdots f_{n+1})(w-v) \\
\bar{F}_n(t-w) dw dv du
\]
Explanation of (i) - (vii):

Since \( I(o)=S \), in order to have \( I(t)=0 \) the inventory must have crossed the level \( s \) from above at least once. Let \( u \) be the last instant at which inventory level drops to \( s \). After \( u \), the replenishment of the stock does not materialise up to \( t \) and the inventory level reaches zero level at \( v \) (\( u<v\leq t \)) and there may be an infinite number of lost demands in \( (v,t] \). Using these facts we can arrive at (i).

Expressions (ii) and (iii) follow on similar lines.

To prove (iv) we recognise that \( I(t)=n \) can happen with or without crossing the level \( s \). The first event can be classified into three mutually exclusive and exhaustive set of events according as (a) after time \( u \) (the last instant at which inventory level drops to \( s \)) there is replenishment before any demand occurs and after replenishment the inventory level comes down to \( n \) at time \( t \), (b) after time \( u \) there are exactly \( k \) \((k=1,2,...,s-1)\) demands, then replenishment takes place and thereafter inventory level drops down to \( n \) at time \( t \), and (c) after time \( u \) inventory level comes down to zero level, thereafter replenishment occurs and then inventory level drops down to \( n \) at time \( t \). Expressions (v), (vi) and (vii) follow similarly.
Let $\phi_0(.)$ denotes the probability density function of $Y_1$ and $\phi(.)$ the common probability density function of the random variables $Y_2, Y_3, \ldots$. Then we have

$$\phi_0(u) = (f_s*f_{S-1}^{*} \ldots f_{S+k}^{*})(u) \quad (2.3.1)$$

and

$$\phi(u) = \int_0^{\infty} F_s(v)g(v)(f_s*f_{S-1}^{*} \ldots f_{S+k}^{*})(u-v)dv$$

$$+ \sum_{k=1}^{s-1} \int_0^{\infty} (f_s*f_{S-1}^{*} \ldots f_{S-k+1}^{*})(v) \int_0^{u} F_{S-k}(w-v)g(w) wn^{k+1}dw \, dv$$

$$+ \int_0^{\infty} (f_s*f_{S-1}^{*} \ldots f_{S+k}^{*})(v) \int_0^{u} (\sum_{m=0}^{\infty} f_{S-m}^{*}(w-v)) \cdot \int_0^{\infty} F_0(x-w)(f_s*f_{S-1}^{*} \ldots f_{S+k}^{*})(u-x)dx \, dw \, dv$$

$$\quad \quad \quad \quad (2.3.2)$$

Then the renewal density of reorder points is given by

$$m(u) = (\phi_0^{*} \sum_{n=0}^{\infty} \phi^{*n})(u) \quad (2.3.3)$$
2.4. COST FUNCTION OF THE MODEL

Having obtained an explicit expression for \( \Pr \{ I(t) = n \} \) in terms of the probability density functions of the basic random variables in question we can obtain the inventory carrying (holding) cost. If \( h \) is the holding cost per unit item per unit time, then the total inventory holding cost during the interval \((0, t)\) is

\[
H(t) = h \int_0^t I(u) \, du \quad (2.4.1)
\]

where the above integral can be interpreted in the Ito sense (see McShane (1974)). Taking expected value on both sides of (2.4.1)

we get

\[
E[H(t)] = h \sum_{n=1}^S \int_0^t \Pr \{ I(t) = n \} \, du \quad (2.4.2)
\]

Let \( K \) be the fixed order cost; \( c \) = variable procurement cost per unit and \( k \) = shortage cost per unit. The average length of time for which there is shortage is

\[
E(L - \sum_{i=1}^S X_i)^+ \quad \text{where } L \text{ is the lead time and } X^+ \text{ indicates max } (0, X).
\]

The expected number of lost demands is therefore equal to \( E(L - \sum_{i=1}^S X_i)^+ / E(X_0) \). So the expected shortage
cost per cycle (representing the length of time between two successive epochs at which the inventory level comes to reorder level) is

\[
\frac{E(L - \sum_{i=1}^{s} X_i)^+}{k E(X_o)} \quad (2.4.3)
\]

and \(K+c(S-s)\) is the fixed cost for procurement per cycle. If we multiply this by \(M(t)\), the renewal function corresponding to the renewal process \(\{Y_n\}_{n \geq 1}\), we obtain the expected procurement cost over the interval \((0,t)\). The expected shortage cost over the interval \((0,t)\) is

\[
E(L - \sum_{i=1}^{s} X_i)^+ \quad (2.4.4)
\]

Hence we have the total expected cost during the interval \((0,t)\) as

\[
C(s,S,t) = h \sum_{n=1}^{S} \int_{0}^{t} \text{Pr}\{I(u)=n\} \, du + M(t)[K+c(S-s)] + k \frac{E(L - \sum_{i=1}^{s} X_i)^+}{E(X_o)} \quad (2.4.4)
\]
2.5. **STEADY STATE BEHAVIOUR OF THE SYSTEM**

Using the asymptotic results of renewal theory, we obtain the limiting distribution of the discrete valued stochastic process $I(t)$ as follows. The limiting probability mass function $\pi(n)$ is given by

$$
\pi(n) = \frac{\int_0^S p_0(n,u)du}{\int_0^S x \phi(x) \, dx}, \quad 0 \leq n \leq S
$$

where

$$
p_0(n,t) = \lim_{\Delta \to 0} \Pr \{I(t_0+t)=n, N(t_0+t)-N(t_0)= 0/ I(t_0) = s < I(t_0-\Delta)\},
$$

$$
N(t) = \text{Sup} \{n \geq 0: Y_1+Y_2+ \ldots + Y_n \leq t\}.
$$

2.6. **THE MODEL WITH ZERO LEAD TIME**

As a particular case, if we assume that the lead time is zero in the model considered above, then $\{I(t), t \geq 0\}$ is a discrete valued continuous parameter stochastic process taking values $s+1$, $s+2$, ..., $S$. Here the sequence of random variables $\{Y_k\}$, $k=1,2,\ldots$, forms a renewal process in which
distribution of \( Y_k \) is given by

\[
Pr \{ Y \leq y \} = \int_0^y (f_S * f_{S-1} * \ldots * f_{S+1})(u) \, du
\]

Its density be denoted by \( f(\cdot) \). The probability that the \( k^{\text{th}} \) order, \( k=1,2,3,\ldots \) will be placed in the interval \( t \) and \( t+dt \) is

\[
Pr \{ t < Y_1 + Y_2 + \ldots + Y_k \leq t+dt \} = f^k(t), \quad k=1,2,3,\ldots
\]

Then the probability mass function of \( I(t) \) is:

For \( n = s+1, s+2, \ldots, S-1 \)

\[
Pr \{ I(t)=n \} = \int_0^t (f_S * f_{S-1} * \ldots * f_{n+1})(u) \, du
\]

\[
- \int_0^t (f_S * f_{S-1} * \ldots * f_n)(u) \, du \bigg] + \sum_{k=1}^\infty \int_0^t \int_0^x (f_S * f_{S-1} * \ldots * f_{n+1})(u) \, du \bigg]
\]

\[
= \int_0^x (f_S * f_{S-1} * \ldots * f_n)(u) \, du \bigg] f^k(t-x) \, dx
\]
and \( \Pr\{I(t)=S\} = \left[1-\int_0^t f_S(u)du\right] + \sum_{k=1}^\infty \int_0^t \left[1-\int_0^x f_S(u)du\right] f^k(t-x)dx \)

Let

\[
\hat{p}(n, \alpha) = \int_0^\infty e^{-\alpha t} \Pr\{I(t)=n\} dt
\]
\[
\hat{f}_n(\alpha) = \int_0^\infty e^{-\alpha t} f_n(t) dt
\]
and \( \hat{f}(\alpha) = \int_0^\infty e^{-\alpha t} f(t) dt \)

Then

For \( n=s+1, s+2, \ldots, S-1 \)

\[
\hat{p}(n, \alpha) = \frac{1}{\alpha} \left[ \hat{f}_S(\alpha) \ldots \hat{f}_{n+1}(\alpha) - \hat{f}_S(\alpha) \ldots \hat{f}_n(\alpha) \right]
\]
\[
+ \sum_{k=1}^\infty \frac{1}{\alpha} [\hat{f}_S(\alpha) \ldots \hat{f}_{n+1}(\alpha) - \hat{S}_S(\alpha) \ldots \hat{f}_n(\alpha)] [\hat{f}(\alpha)]^k
\]
\[
= \frac{1}{\alpha} \hat{f}_S(\alpha) \ldots \hat{f}_{n+1}(\alpha) [1-\hat{f}_n(\alpha)][1-\hat{f}(\alpha)]^{-1}
\]
(2.6.1)

and \( \hat{p}(S, \alpha) = \frac{1}{\alpha} [1-\hat{f}_S(\alpha)][1-\hat{f}(\alpha)]^{-1} \) (2.6.2)

**STEADY STATE DISTRIBUTION OF THE INVENTORY LEVEL**

Let \( p_n \) be the probability that exactly \( n \) units, \( n = s+1, s+2, \ldots, S \) are in the inventory in the steady state.
Then by a Tauberian theorem, [see Widder (1946)]

\[ P_n = \Pr \{ I = n \} = \lim_{t \to \infty} \Pr \{ I(t) = n \} = \lim_{\alpha \to 0} \alpha \hat{p}(n, \alpha) \]

For \( n = s+1, s+2, \ldots, S-1, \)

\[ P_n = \lim_{\alpha \to 0} \frac{\hat{f}_S(\alpha) \cdots \hat{f}_{n+1}(\alpha)[1-\hat{f}_n(\alpha)]}{[1-\hat{f}(\alpha)]} = \lim_{\alpha \to 0} \frac{\hat{f}_S(\alpha) \cdots \hat{f}_{n+1}(\alpha) \hat{f}'_n(\alpha)}{\hat{f}'(\alpha)} \quad \text{(using L'Hospital's rule)} \]

\[ = \frac{E(X_n)}{\sum_{i=s+1}^{S} E(X_i)} \]

\[ P_S = \lim_{\alpha \to 0} \frac{1-\hat{f}_S(\alpha)}{1-\hat{f}(\alpha)} = \lim_{\alpha \to 0} \frac{\hat{f}'_S(\alpha)}{\hat{f}'(\alpha)} = \frac{E(X_S)}{\sum_{i=s+1}^{S} E(X_i)} \]

Thus

\[ P_n = \frac{E(X_n)}{\sum_{i=s+1}^{S} E(X_i)} \text{, } n = s+1, s+2, \ldots, S. \quad (2.6.2) \]
Here when we assume $X_n$'s to be independent identically distributed random variables we get the results of Sivazlian (1974).

**OBJECTIVE FUNCTION AND OPTIMAL DECISION RULES FOR ZERO LEAD TIME CASE:**

If delivery of orders is instantaneous, then no shortage is allowed. Our objective function is the steady state total expected cost per unit time; we have to choose the decision variables $s$ and $S$ so as to minimize the objective function.

The expected time elapsed between two successive orders is

$$E(Y) = \sum_{i=s+1}^{S} E(X_i)$$

Therefore the expected number of orders placed per unit time is

$$\frac{1}{E(Y)} = \frac{1}{\sum_{i=s+1}^{S} E(X_i)}$$

(2.6.3)

Expected inventory level at any instant of time is

$$E(I) = \sum_{n=s+1}^{S} nP_n = \sum_{n=s+1}^{S} nE(X_n)/\sum_{i=s+1}^{S} E(X_i)$$

(2.6.4)
Total expected cost per unit time is

\[ C(s, S) = \frac{K+c(S-s)}{E(Y)} + h E(I) \]

where \( K = \) fixed order cost, \( c = \) variable procurement cost per unit, \( h = \) holding cost per unit per unit time.

Substituting for \( E(Y) \) and \( E(I) \), we have

\[ C(s, S) = \left[ K+c(S-s)+h \sum_{n=s+1}^{S} E(X_n) \right] / \sum_{n=s+1}^{S} E(X_n) \quad (2.6.5) \]

where \( s \) and \( S \) are non-negative integers and \( s < S \).

Obviously the above expression is not separable in \( s \) and \( S \). The minimization of \( C(s, S) \) can be done, knowing the first moments of the interarrival times, with the aid of a computer.

Here \( s^* = 0 \) is the optimal value. The optimum value of \( S \) is obtained by minimizing the function

\[ C_1(S) = \left[ K+cS+h \sum_{n=1}^{S} E(X_n) \right] / \sum_{n=1}^{S} E(X_n) \quad (2.6.6) \]

over the set of positive integers \( S \). We shall now give an example typical of the above case.
EXAMPLE

Let $X_n$ follow exponential distribution with parameter $\lambda_n$. Assume that $S \leq 25$. For specified values of $K, c, h$ and $\lambda_n$'s the optimal values $S^*$ and $C_1(S^*)$ obtained using a computer are given below:

<table>
<thead>
<tr>
<th>$\lambda_n$</th>
<th>$\lambda$ Values</th>
<th>$S^*$</th>
<th>$C_1(S^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n=1,2,\ldots,25$</td>
<td>1</td>
<td>23.842</td>
<td></td>
</tr>
<tr>
<td>$\lambda_n = 1/n(n=1,\ldots,25)$</td>
<td>6</td>
<td>7.000</td>
<td></td>
</tr>
<tr>
<td>$\lambda_n$'s(n=1,2,\ldots,25) are</td>
<td>14</td>
<td>17.625</td>
<td></td>
</tr>
<tr>
<td>1.2, 2.1, 1.4, 2.3, 2.5</td>
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<td> </td>
<td></td>
</tr>
<tr>
<td>4.3, 5.0, 2.3, 4.1, 4.0</td>
<td> </td>
<td> </td>
<td></td>
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<tr>
<td>4.2, 1.5, 2.4, 3.6, 4.6</td>
<td> </td>
<td> </td>
<td></td>
</tr>
<tr>
<td>3.2, 1.7, 4.9, 1.3, 2.0</td>
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<td> </td>
<td></td>
</tr>
<tr>
<td>3.9, 4.2, 1.7, 2.4, 4.9</td>
<td> </td>
<td> </td>
<td></td>
</tr>
<tr>
<td>$\lambda_n$'s(n=1,2,\ldots,25) are</td>
<td>14</td>
<td>18.841</td>
<td></td>
</tr>
<tr>
<td>1.3, 2.4, 4.2, 5.0, 3.1</td>
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<td> </td>
<td></td>
</tr>
<tr>
<td>4.0, 2.8, 4.2, 3.9, 2.1</td>
<td> </td>
<td> </td>
<td></td>
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</tr>
<tr>
<td>2.5, 2.4, 1.3, 1.2, 3.6</td>
<td> </td>
<td> </td>
<td></td>
</tr>
<tr>
<td>$\lambda_n$'s reversed in order of IIIrd row</td>
<td>14</td>
<td>19.085</td>
<td></td>
</tr>
<tr>
<td>$\lambda_n$'s reversed in order of IVth row</td>
<td>16</td>
<td>18.102</td>
<td></td>
</tr>
<tr>
<td>$\lambda_n$=n, n=1,2,\ldots,25</td>
<td>10</td>
<td>25.606</td>
<td></td>
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<tr>
<td>$\lambda_n = l/n, n=1,2,\ldots,25$</td>
<td>8</td>
<td>4.667</td>
<td></td>
</tr>
</tbody>
</table>
Remark:

The model analysed in Section 2.3 can be extended to allow vacations to the server whenever the system becomes empty. In this case also one can write expressions for the inventory level probabilities at arbitrary time points but the optimization part seems to be difficult.