5.1 INTRODUCTION

In the previous chapter we discussed state dependent demand process. In this chapter we shall introduce another type of dependence in the basic process. The interarrival times of demands are i.i.d random variables with distribution function $G(.)$ that is absolutely continuous with density $g(.)$. Each arrival demands exactly one unit. Initially the inventory level is $s$ resulting in an order placing. Order is placed for $M$ units and let $S = M + s$. Lead times are i.i.d random variables which are independent of the quantity ordered for and inventory level, having absolutely continuous distribution function $F(.)$ and its density be $f(.)$. We fix a $c (> s)$. The ordering levels other than the initial one are determined as follows.

Suppose the number of demands during a lead time is $J$ then the next ordering level is $I = \min(J, c)$. Thus the ordering level can be $0, 1, 2, \ldots, c$. The order quantity will be such as to bring back the inventory level to $S$ at the ordering
epoch. Thus S-I units are ordered if I is the ordering level. No backlog is permitted. We discuss the time dependent probability distribution of the inventory level in Section 5.2. Correlation between the number of demands during a lead time and the length of the next inventory dry period is obtained in Section 5.3. Some illustrations are also given. This model has been discussed earlier by Ramanarayanan and Jacob (1986). However their method has a drawback that computation is hard and further passage to the limit is rather difficult.

In the sequel we use the following notations.

$G(\cdot), g(\cdot)$ — Cumulative distribution function (c.d.f) and probability density function (p.d.f), respectively, of the interarrival time between demands.

$F(\cdot), f(\cdot)$ — c.d.f and p.d.f, respectively, of the lead times.

denotes convolution

$f_n^*(x) = n$-fold convolution of $f(x)$ with itself

$(f^0(x) \equiv 1)$.

$E = \{0,1,2,\ldots,s, \ldots c\}$

$R^+ = \text{Set of non-negative real numbers}$

$N = \text{Set of natural numbers}$
5.2 ANALYSIS OF THE MODEL

Let \( T_0 = 0, T_1, T_2, ..., T_n, ... \) be the epochs at which the initial, first, ..., \( n^{th} \) orders are placed for replenishment and \( X_0(=s), X_1, X_2, ..., X_n, ... \) be the corresponding ordering levels. Assume that \( Y_0, Y_1, Y_2, ..., Y_n, ... \) be respectively the number of demands during the lead times these start at \( T_0, T_1, T_2, ..., T_n, ... \). Then \( \{(X_n, Y_n), n=0,1,2,\ldots\} \) constitutes a Markov chain on the set \( E \times \mathbb{N}^0 \).

Now define \( Z_n = (X_n, Y_n) \). The process \( \{(Z_n, T_n), n \in \mathbb{N}^0\} \) constitutes a Markov renewal process with the underlying semi-Markov process \( \{Z_t, t \in \mathbb{R}^+\} \) where

\[
Z_t = (X_n, Y_n) \text{ for } T_n \leq t < T_{n+1}
\]

The semi-Markov kernel is given by

\[
Q((i,I),(j,J),t) = P \left\{ \begin{array}{l}
(X_{n+1}, Y_{n+1}) = (j,J), T_{n+1} - T_n \leq t/ \\
(X_n, Y_n) = (i,I)
\end{array} \right\}
\]

\( i,j \in E, I,J \in \mathbb{N}^0 \)
Now,

\[ Q((i,I),(j,J),t) = Q_1((i,I),(j,J),t) + Q_2((i,I),(j,J),t) \]  

(5.2.1)

where \( Q_1(\ldots,t) \) and \( Q_2(\ldots,t) \) correspond to, respectively, transition from \((i,I)\) to \((j,J)\) in time \(t\) without and with a dry period in between. Note that if \( I \leq c \) then \( j = I \) and for \( I > c \), \( j = c \). Further, if \( i > I \), then

\[ Q((i,I),(j,J),t) = Q_1((i,I),(j,J),t) \]

and if \( i \leq I \), then

\[ Q((i,I),(j,J),t) = Q_2((i,I),(j,J),t) \]

where \( j = \min \{c,I\} \).

Now \( Q_1((i,I),(j,J),t) \) is given by

\[ Q_1((i,I),(I,J),t) = \int_{0}^{t} \int_{0}^{u} \int_{v}^{w} g^*(u) f(v) \frac{q^*(S-2I)(w-u)}{1-G(v-u)} \]

\[ g^J(x-w) \left[ 1-F(x-w) \right] dx \, dw \, dv \, du \]

which is valid for \( i > I \).
For $c > I > 1$, we have

$$Q_2((i,I),(I,J),t) = \int \int \int \int g^*(u) f(v) \frac{g^{*(S-i-I)(w-u)}}{1-G(v-u)} \times g^*(x-w)[1-F(x-w)] dx \, dw \, dv \, du$$

Finally for $I \geq c$, we have

$$Q_2((i,I), (c,J), t) = \int \int \int \int g^*(u) f(v) \frac{g^{*(S-i-c)(w-u)}}{1-G(v-u)} \times g^*(x-w)[1-F(x-w)] dx \, dw \, dv \, du.$$

Now we are in a position to find the system size probability distribution at arbitrary time point $t$. For this purpose consider the Markov renewal function $R((s,I), (j,J), t)$ of the process under consideration.

This is given by

$$R((s,I), (j,J), t) = \sum_{m=0}^{\infty} Q^m((s,I), (j,J), t) \quad (5.2.2)$$

Note that $R((s,I), (j,J), t)$ represents the number of visits to $(j,J)$ from $(s,I)$ in $(0,t]$. Let its density be represented by $r((s,I), (j,J), t)$. 
Let $P^{(s,I)}_{(n,j)}(t) = \Pr \{ \text{Inventory level at time } t \text{ is } n \\
\text{and the last reorder level is } j \\
given that initially the system was in state } (s,I) \}$.

Then for $c < n \leq S$,

$$P^{(s,I)}_{(n,j)}(t) = \int \int \int \sum_{k=0}^{j-1} r((s,I),(j,k),u) g^k(w-u) f(v-u)$$

$$\int \int \int \left[ \frac{G^*(S-k-n)(t-w) - G^*(S-k-n+1)(t-w)}{1 - G(v-w)} \right] dv \, dw \, du$$

$$+ \int \int \int \sum_{k>j} r((s,I),(j,J),u) g^k(w-u) f(v-u)$$

$$\int \int \int \left[ \frac{G^*(S-j-n)(t-w) - G^*(S-j-n+1)(t-w)}{1 - G(v-w)} \right] dv \, dw \, du$$

For $j < n \leq c$,

$$P^{(s,I)}_{(n,j)}(t) = \int \int \int \sum_{k<n} r((s,I),(j,k),u) g^k(w-u) f(v-u)$$

$$\int \int \int \left[ \frac{G^*(S-k-n)(t-w) - G^*(S-k-n+1)(t-w)}{1 - G(v-w)} \right] dv \, dw \, du$$
For $1 \leq n \leq j$

\[
p(s,I)(n,j) = \int_0^t \int_0^t \int_0^t \sum_{u,v,w} r(s,I),(j,k),u)g^k(w-u)f(v-u)g^*(S-k-n)(t-w)G^*(S-k-n+1)(t-w) \left[ \frac{1 - G(v-w)}{G^*(S-k-n)(t-w) - G^*(S-k-n+1)(t-w)} \right] dv \, dw \, du
\]

\[
+ \int_0^t \int_0^t \int_0^\infty \sum_{u,v,w} r((s,I),(j,k),u)g^{(j-n)}(w-u)g^k(v-u)G^*(j-n)(v-w)G^*(j-n+1)(v-w) \left[ \frac{1 - G(v-w)}{G^*(j-n)(v-w) - G^*(j-n+1)(v-w)} \right] dv \, dw \, du
\]

Finally for $n = 0$,

\[
p(o,j)(t) = \int_0^t \int_0^t \int_0^\infty \sum_{u,v,w} r(s,I),(j,k),u)g^j(w-u)g^k(v-w)G^*(v-w)G^*(k-j+1)(v-w) \left[ \frac{1 - G(t-w)}{G^*(v-w) - G^*(k-j+1)(v-w)} \right] f(v-w)dv \, dw \, du
\]

5.3 CORRELATION BETWEEN THE NUMBER OF DEMANDS DURING A LEAD TIME AND THE INVENTORY DRY PERIOD

Let $J$ be the random variable representing an ordering
level and $Z$ be the length of the subsequent dry period. Then $J=j$, for $j=0,1,2,\ldots,c-1$ if the number of demands during the previous lead time was $j$ and $J=c$ if the number of demands during the previous lead time was larger than or equal to $c$.

Now for $0 \leq j \leq c-1$,

$$p_j = \Pr \{J=j\} = \int_0^\infty f(y) \{G^j(y) - G^{j+1}(y)\} \, dy$$

and

$$p_c = \Pr \{J=c\} = \int_0^\infty f(y) G^c(y) \, dy$$

For $0 \leq j \leq c$,

$$\Pr \{J = j, Z = 0\} = p_j \int_0^\infty f(y) G^j(y) \, dy$$

and

$$\Pr \{J = j, z < Z \leq z+dz\} = p_j \left( \int_{z}^{\infty} f(y) g^j(y-z) \, dy \right) dz$$

for $z > 0$ and $0 \leq j \leq c$.

Then

$$E(e^{-\alpha Z J}) = \sum_{j=0}^{c} p_j r^j \int_0^\infty f(y) G^j(y) \, dy$$

$$+ \sum_{j=0}^{c} p_j r^j \int_0^\infty e^{-\alpha z} \left( \int_{z}^{\infty} f(y) g^j(y-z) \, dy \right) dz$$

$$= \sum_{j=0}^{c} p_j r^j - \alpha \sum_{j=0}^{c} p_j r^j \int_0^\infty e^{-\alpha y} f(y)$$

$$\left( \int_{0}^{y} e^{\alpha x} G^j(x) \, dx \right) dy. \quad \text{(5.2.3)}$$
From this we have

\[
E(Z) = - \frac{\partial}{\partial \alpha} E \left( e^{-\alpha Z} r^J \right) \bigg|_{\alpha=0, \ r=1} \\
= \sum_{j=0}^{c} p_j \int_{0}^{\infty} f(y) \left( \int_{0}^{\infty} G^j(x) \, dx \right) dy
\]

\[
E(Z^2) = \frac{\partial^2}{\partial \alpha^2} E \left( e^{-\alpha Z} r^J \right) \bigg|_{\alpha=0, \ r=1} \\
= 2 \sum_{j=0}^{c} p_j \int_{0}^{\infty} f(y) \int_{0}^{\infty} G^j(u) \, du \, dy
\]

Similarly,

\[
E(J) = \frac{\partial}{\partial r} E \left( e^{-\alpha Z} r^J \right) \bigg|_{\alpha=0, \ r=1} \\
= \sum_{j=0}^{c} j p_j
\]

and \(E(J^2) = \sum_{j=0}^{c} j(j-1)p_j\).

Finally,

\[
E(ZJ) = \sum_{j=0}^{c} \int_{0}^{\infty} f(y) \left( \int_{0}^{\infty} G^j(x) \, dx \right) dy
\]

Substituting these values in \(\text{Cov}(Z,J) = E(ZJ) - E(Z) \ E(J)\) and using the fact that correlation coefficient between \(Z\) and \(J\) is \(\text{Cov}(Z,J) / \sqrt{\text{Var}(Z) \ \text{Var}(J)}\) we get the required expression.
AN ILLUSTRATION

Assume that \( G(x) = 1 - e^{-x} \)

\[ F(x) = 1 - e^{-2x}, \text{ and} \]

\[ c = 3. \]

Then one easily computes

\[ p_0 = \frac{2}{3}, \quad p_1 = \frac{2}{9}, \quad p_2 = \frac{2}{27}, \quad p_3 = \frac{1}{27} \]

\[ E(Z) = 0.489026063, \quad E(Z^2) = 0.852537722 \]

\[ E(J) = \frac{13}{27}, \quad E(J^2) = \frac{10}{27} \]

\[ \text{Cov} (Z, J) = -0.18813189 \]

\[ \text{Var} (J) = 0.138545953 \]

\[ \text{Var} (Z) = 0.613391231 \]

and \( \text{Correlation coefficient} = -0.64535208. \)

The negative correlation indicates that when the ordering level increases the length of the subsequent dry period decreases.