Appendix C

Basic properties of Laguerre and Jacobi EOPs as well as Standard ESPs referred in Chapter 3

C.1 Basic properties of Laguerre EOPs

The $X_1$ Laguerre EOP, denoted by $L_n^a(z); \ n = 1, 2, 3, ..., \ a > 0$, has the following properties [85]:

\[ L_n^a(z) = -z - \alpha - 1, \quad L_n^a(z) = z^2 - \alpha(\alpha + 2), \ldots \]  
(C.1)

\[ \frac{d^2}{dz^2} - \frac{z - \alpha}{z + \alpha} \left( (z + \alpha + 1) \frac{d}{dz} - 1 \right) L_n^a(z) = -(n - 1) L_n^a(z) \]  
(C.2)

\[ \int_0^{\infty} L_n^a(z) L_n^a(z) \frac{z^\alpha e^{-z}}{(z + \alpha)^2} \, dz = \delta_{\alpha,n} \frac{\Gamma(n + \alpha + 1)}{(n + \alpha - 1)(n - 1)!} \]  
(C.3)
C.2. Basic properties of Jacobi EOPs

The $X_1$ Jacobi EOP, denoted by $\hat{R}_n^{(\alpha,\beta)}(z)$; $n = 1, 2, 3, ..., \alpha, \beta > -1$, $\alpha \neq \beta$, has the following properties [85]:

$$
\hat{R}_1^{(\alpha,\beta)}(z) = -\frac{1}{2}z^2 - \frac{2 + \alpha + \beta}{2(\alpha - \beta)}
$$

$$
\hat{R}_2^{(\alpha,\beta)}(z) = -\frac{\alpha + \beta + 2}{4}z^2 - \frac{\alpha^2 + \beta^2 + 2(\alpha + \beta)}{2(\alpha - \beta)}z - \frac{\alpha + \beta + 2}{4},...
$$

(C.5)

$$
\left[(z - 1)\frac{d^2}{dz^2} + 2a\left(\frac{1 - bz}{b - z}\right)\left(z - c\right)\frac{d}{dz} - 1\right] \hat{R}_n^{(\alpha,\beta)}(z) = (n - 1)(\alpha + \beta + n) \hat{R}_n^{(\alpha,\beta)}(z)
$$

(C.6)

where the real parameters $a$, $b$ and $c$ are given by

$$
a = \frac{\beta - \alpha}{2}; \quad b = \frac{\beta + \alpha}{\beta - \alpha}; \quad c = b + \frac{1}{a}
$$

$$
\int_{-1}^{1} \frac{(1 - z)^{\alpha}(1 + z)^{\beta}}{(z - b)^2} \left(\hat{R}_n^{(\alpha,\beta)}(z)\right)^2 dz = \frac{(\alpha + n)(\beta + n)}{4(\alpha + n - 1)(\beta + n - 1)} C_n^{-1}
$$

(C.7)

where

$$
C_n = \frac{2^{2\alpha+\beta+1}}{(\alpha + \beta + 2n + 1)\Gamma(n + 1)\Gamma(\alpha + \beta + n + 1)} \frac{\Gamma(\alpha + n + 1)\Gamma(\beta + n + 1)}{(\alpha + \beta + n + 1)\Gamma(n + 1)\Gamma(\alpha + \beta + n + 1)}
$$
\[ p_{n}^{(\alpha,\beta)}(z) = -f_{n}p_{n}^{(\alpha,\beta)}(z) + 2b_{n}g_{n-1}p_{n-1}^{(\alpha,\beta)}(z) - h_{n}r_{n-2}^{(\alpha,\beta)}(z) \]  
(C.8)

where

\[ f_{n} = \frac{n(\alpha + \beta + n)}{(\alpha + \beta + 2n - 1)(\alpha + \beta + 2n)} \]

\[ g_{n} = \frac{(\alpha + n)(\beta + n)}{(\alpha + \beta + 2n - 2)(\alpha + \beta + 2n)} \]

\[ h_{n} = \frac{(\alpha + n)(\beta + n)}{(\alpha + \beta + 2n - 2)(\alpha + \beta + 2n - 1)} \]

C.3 Standard ESPs referred

The radial oscillator potential \((V_{RO})\) in \(D\)-dimensional Euclidean space [83] is

\[ V_{RO}(r) = \frac{1}{4} \omega^2 r^2 + \frac{\ell(\ell + D - 2)}{r^2} \]  
(C.9)

The corresponding exact energy eigenvalues and the wave function are given by

\[ E_{n\ell}^{RO} = \omega(2n_{r} + \ell + \frac{D}{2}) \]  
(C.10)

\[ \psi_{RO}(r) = Nr^{\ell} \exp\left(-\frac{\omega r^2}{4}\right) I_{n_{\omega}}^{(\ell)} \left(\frac{\omega r^2}{2}\right) \]  
(C.11)
The Morse potential \((V_M)\) is given by [21, 22]

\[
V_M(r) = -B(2A + a)e^{-ar} + B^2e^{-2ar}
\]  
(C.12)

The corresponding exact energy eigenvalues and the wave function are given by

\[
E_n^M = -(A - na)^2
\]  
(C.13)

\[
\psi_M(r) = Ng^{2n}e^{-\frac{1}{2}I_n^{(2i-2n)}(g(r))}
\]  
(C.14)

where

\[
g(r) = \frac{2B}{a}e^{-ar}, \quad s = \frac{A}{a}
\]

The Scarf I potential \((V_S)\) is given by [23]

\[
V_S(r) = (a^2 + b^2 - a\alpha)\sec^2\alpha r - b(2a + \alpha)\tan\alpha r \sec\alpha r
\]  
(C.15)

The corresponding exact energy eigenvalues and the wave function are given by

\[
E_n^S = (n\alpha + a)^2
\]  
(C.16)

\[
\psi_S(r) = N(1 - \sin\alpha r)^{\frac{1}{2}}(1 + \sin\alpha r)^{\frac{1}{2}}P_n^{r-\frac{1}{2}}(\sin\alpha r)
\]  
(C.17)
where

\[ \gamma = \frac{a-b}{\alpha}; \quad \delta = \frac{a+b}{\alpha} \]

The Rosen-Morse potential \((V_{RM})\) is given by [21]

\[ V_{RM}(r) = -A(A + a) \text{sech}^2 ar + 2B \tanh ar \]  

(C.18)

The corresponding exact energy eigenvalues and the wave function are given by

\[ E_n^{RM} = -(A - na)^2 - \frac{B^2}{(A - na)^2} \]  

(C.19)

\[ \psi_{RM}(r) = N(1 - \tanh ar)^{-\frac{A}{a}}(1 + \tanh ar)^{-\frac{B}{a}} F_{\frac{s-n}{2},\frac{s-n}{2}}(r\tanh ar) \]  

(C.20)

where

\[ s = \frac{A}{a} \]