Chapter 5

Generation of Exact S-Wave Solution of Schrödinger Equation from Non Power-Law Potential

5.1 Introduction

The number of exactly solvable potentials (ESPs) is limited in physics. Morse potential [17, 18, 124], Hulthén potential [19, 20], Rosen-Morse potential [21, 22, 122, 123, 125], Eckart potential [21], Scarf Potential [21, 23], Pöschl-Teller potential [24, 25, 126], pseudoharmonic potential [127] etc. are some well known central potentials for which the exact s-wave solutions for Schrödinger equation are available. So far various methods [18, 27, 28, 31, 66, 81, 95, 100, 128, 129] have been developed to get such ESPs.

In this chapter, we take generalized Hulthén potential, i.e., $V(r) = \frac{\alpha^2}{1-e^{-\alpha r}} - (\lambda^2 + \alpha^2)\frac{e^{-\alpha r}}{1-e^{-\alpha r}} - \frac{1}{2}\alpha^2 \frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}$ as our input reference potential. The Hulthén potential has already been solved for the bound states with Nikiforov-Uvarov method [19], SUSY method [20], shifted 1/N expansion method [50] and asymptotic iteration method [100]. It has been
5.2. The Extended Transformation Method

widely used to describe the bound state or continuum states of the interaction systems. In this work, we treat the Hulthen potential within the framework of the ET method [81, 83, 87, 88, 96, 97, 120] and try to generate new ESPs which give rise to bound state solutions of $D$-dimensional radial Schrödinger equation. When ET method is applied to the non power-law type ESPs, the newly generated ESPs are always Sturmian (i.e., principal quantum number $n$-dependent potential). As there is no specific technique to convert a Sturmian ESP to normal ESP, we employ different QS specific regrouping techniques to produce a normal QS.

5.2 The Extended Transformation Method

For a QS, say A-QS, the radial part of the Schrödinger equation [14, 130] for the potential $V_A(r)$ in $D_A$ dimensional Euclidean space (in natural units $\hbar = 1 = 2m$) is

$$\psi''_A(r) + \frac{(D_A - 1)}{r}\psi'_A(r) + \left( E_A - V_A(r) - \frac{\ell_A(\ell_A + D_A - 2)}{r^2}\right)\psi_A(r) = 0 \quad (5.1)$$

where $r$ is a dimensionless spatial coordinate.

The ET method includes a coordinate transformation, which is followed by a functional transformation and a set of plausible ansatze to restore the transformed equation to normal standard form of the Schrödinger equation.
We now apply the coordinate transformation

\[ r \rightarrow g_B(r) \]  

(5.2)

which is followed by the functional transformation

\[ \psi_B(r) = f_B^{-1}(r)\psi_A(g_B(r)) \]  

(5.3)

where the transformation function \( g_B(r) \) and the modulated amplitude \( f_B(r) \) have to be specified within the framework of ET method. The transformed Schrödinger equation then takes the form:

\[
\psi''_B(r) + \left( \frac{d}{dr} \ln \frac{f_B^{D_A-1}}{g_B} \right) \psi'_B(r) + \left[ \left( \frac{d}{dr} \ln f_B \right) \left( \frac{d}{dr} \ln \frac{f_B^{D_A-1}}{g_B} \right) + g_B^2 \left( E_A - V_A(g_B(r)) - \frac{\ell_A \ell_A + D_A - 2}{g_B^2} \right) \right] \psi_B(r) = 0
\]

(5.4)

where the prime denotes differentiation with respect to the variable \( r \).

The dimension of the Euclidean space of the transformed QS, henceforth called the B-QS, can be chosen arbitrarily. Let it be denoted by \( D_B \). Then,

\[
\frac{d}{dr} \ln \frac{f_B^{D_A-1}}{g_B} = \frac{D_B - 1}{r}
\]

(5.5)
5.2. The Extended Transformation Method

Expression (5.5) fixes \( f_B(r) \) as a function of \( g_B(r) \) and its derivative. We get

\[
f_B(r) = N_B g_B^{(D_A-1)/2} g_B^{-1/2} \frac{D_{A-1} - D_B - 3}{2} \frac{1}{r^2} \psi_A(g_B(r))
\]  
(5.6)

where \( N_B \) is the normalization constant.

From equations (5.3) and (5.6), we find

\[
\psi_B(r) = N_B g_B^{(D_A-1)/2} g_B^{-1/2} r^{-(D_A-1)/2} \psi_A(g_B(r))
\]  
(5.7)

where the transformation function \( g_B(r) \) is at least three times differentiable.

The corresponding \( D_B \) dimensional Schrödinger equation for B-QS can be rewritten as

\[
\psi''_B(r) + \frac{D_B - 1}{r} \psi'_B(r) + \left[ \frac{1}{2} \{g_B, r\} - \frac{D_A - 1}{2} \left( g_B' \right)^2 + \frac{D_B - 1}{2} \right] \psi_B(r) = 0
\]  
(5.8)

where

\[
\{g_B, r\} = \frac{g_B^{(D_B)}(r)}{g_B'(r)} - \frac{3}{2} \frac{g_B''(r)}{g_B'(r)}
\]  
(5.9)

In case of multi-term A-QS, we have to select a term of \( V_A(g_B(r)) \) as a working potential (WP) to implement ET method and is designated as \( V^w_A(g_B(r)) \).
In order to mould equation (5.8) to the standard form of the Schrödinger equation, the following plausible ansätze have to be made, which are integral part of the transformation method.

\[ g_B^2 V_A(g_B(r)) = -E_B \]  
\[ V_B^{(1)}(r) = -g_B^2 E_A \]  
\[ V_B^{(2)}(r) = g_B^2 \left( V_A(g_B(r)) - V_A^{W}(g_B(r)) \right) \]  
\[ V_B^{(3)}(r) = -\frac{1}{2} \{g_B, r\} \]  
\[ \frac{g_B^2 (\ell_A + \frac{\ell_B}{2} - 1)^2}{g_B^2} = \frac{(\ell_A + \frac{\ell_B}{2} - 1)^2}{r^2} \]

We obtain the new potential \( V_B(r) \) as

\[ V_B(r) = V_B^{(1)}(r) + V_B^{(2)}(r) + V_B^{(3)}(r) \]
5.3 Generation of ESPs from generalized Hulthén potential

Finally, the radial Schrödinger equation for B-QS for the potential $V_B(r)$ can be read as

$$\psi''_B(r) + \frac{D_B - 1}{r}\psi'_B(r) + \left( E_B - V_B(r) - \frac{\ell_B(\ell_B + D_B - 2)}{r^2} \right)\psi_B(r) = 0 \quad (5.16)$$

An important property of the transformation method is that the wave functions of the generated QSs are almost always normalizable. The normalizability condition for the eigenfunctions of $D_B$-dimensional B-QS is

$$\int_0^\infty \psi^2_B(r)r^{D_B-1}dr = \frac{1}{|N_B|^2} \text{ finite} \quad (5.17)$$

Using equation (5.7) in equation (5.17), the normalization constant becomes

$$N_B = \left[ \frac{-E_B}{\langle V_A^{(\ell)}(g_B) \rangle} \right]^{\frac{1}{2}} \quad (5.18)$$

The expectation value of the ESP $V_A(g_B)$ is always finite; therefore, the expectation value of a part of it is also finite. Hence, all the $\psi_B(r)$ are normalizable for which $E_B \neq 0$.

5.3 Generation of ESPs from generalized Hulthén potential

We consider a non power-law type multiterm quantum ESP (the generalized Hulthén potential) as our A-QS which is exactly solvable for $\ell = 0$ case only.
The potential is given by [96]

\[ V_A(r) = \frac{\alpha^2 \mu^2}{1 - e^{-2\alpha r}} - (\lambda^2 + \alpha^2) \frac{e^{-2\alpha r}}{1 - e^{-2\alpha r}} - \frac{3}{4} \alpha^2 \frac{e^{-4\alpha r}}{(1 - e^{-2\alpha r})^2} \]  

(5.19)

where \( \alpha, \mu \) and \( \lambda \) are parameters of the A-QS potential.

The energy eigenvalues are

\[ E_n^4 = \frac{1}{4} \frac{\lambda^2}{(n + \mu)(n + \mu + 1)} \]  

(5.20)

The constraint equation for the parameters is

\[ (n + \mu)(n + \mu + 1) - \frac{\lambda^2}{\alpha^2} = 0 \]  

(5.21)

which yields

\[ \mu = \frac{1}{2} \left( 1 + \frac{4\lambda^2}{\alpha^2} \right) - \left( n + \frac{1}{2} \right) > 0 \]  

(5.22)

The number of bound states is restricted by the integral part of \( n + \mu \).

The exact s-wave eigenfunction is given as

\[ \psi_A(r) = N_A r^{-1} (1 - e^{-2\alpha r})^{\frac{1}{2}} P_{n\mu}^\nu \left(1 - e^{-2\alpha r}\right)^{\frac{1}{2}} \]  

(5.23)

where \( N_A \) is the normalization constant.
5.3. Generation of ESPs from generalized Hulthen potential

The A-QS potential (5.19) is a three term potential. In case of multiterm potential, the transformation procedure may be applied repeatedly by selecting different WPs to generate a variety of solved QSs.

5.3.1 Generation of ESP using the first WP

Choosing the WP \( V_A^{W}(g_B(r)) \) and putting it in equation (5.10), we get

\[
\frac{\alpha^2 \mu^2}{1 - e^{-2\alpha g_B}} = -E_n^g
\]

(5.24)

Integrating (5.24), we find the transformation function as

\[
g_B(r) = \frac{1}{\alpha} \ln \cosh \alpha k_B r
\]

(5.25)

where

\[
k_B = \sqrt{-E_n^g \alpha^2 \mu^2}
\]

(5.26)

We set the integration constant to zero for \( g_B(r) \) to satisfy the required local property \( g_B(0) = 0 \).

Equations (5.11) and (5.25) yield

\[
V_B^{(1)}(r) = C_B^2 \tanh^2 \alpha k_B r
\]

(5.27)
We put

\[ k_B^2(-E_n^B) = C_B^2 \quad (5.28) \]

where \( C_B^2 \) is the characteristic constant of the generated B-QS obtained from the transformation of A-QS. Equations (5.12) and (5.25) lead to

\[ V_{B}^{(2)}(r) = -(\alpha^2 + \alpha^2)k_B^2\text{sech}^2 \alpha k_B r - \frac{3}{4} \alpha^2 k_B^2 \text{sech}^4 \alpha k_B r \quad (5.29) \]

Equations (5.13) and (5.25) give

\[ V_{B}^{(3)}(r) = \alpha^2 k_B^2 \text{sech}^2 \alpha k_B r + \frac{3}{4} \alpha^2 k_B^2 \text{sech}^4 \alpha k_B r \quad (5.30) \]

Then from equation (5.15), the B-QS potential is obtained as

\[ V_B(r) = -C_B^2 \text{sech}^2 \alpha k_B r \quad (5.31) \]

The presence of \( E_n^B \) in \( V_B(r) \) makes it a Sturmian potential. The equation (5.26) shows that \( E_n^B \) or equivalently \( k_B \) is \( n \)-dependent (energy dependent). Therefore the combination \( \alpha K_B \) is also \( n \)-dependent. In our endeavor to make \( V_B(r) \) normal/physical potential, we have to transform \( \mu \rightarrow \mu_n \), an \( n \)-dependent parameter in equation (5.21). This will make the ratio \( \frac{\sqrt{-E_n^B}}{\mu_n} \) \( n \)-independent, restoring \( C_B^2 \) as a characteristic constant of B-QS. We have introduced the scale parameter \( \beta \) as \( \frac{\sqrt{-E_n^B}}{\mu_n} = \beta \), i.e., \( \alpha k_B = \beta \). The normal B-QS potential
5.3. Generation of ESPs from generalized Hulthén potential

becomes

\[ V_B(r) = -C_B^2 \text{sech}^2 \beta r \] \hspace{1cm} (5.32)

This potential belongs to the family of Pöschl-Teller potential [19].

The energy eigenvalues of \( V_B(r) \) given by equations (5.21) and (5.28) are

\[ E_n^B = -\beta^2 \left[ \frac{1}{2} \left( 1 + \frac{4C_B^2}{\alpha^2} \right) - \left( n + \frac{1}{2} \right)^2 \right] \] \hspace{1cm} (5.33)

The exact energy eigenfunction of the B-QS is obtained from equations (5.7) and (5.25) as

\[ \psi_B(r) = N_Br^{-\frac{(D_B-1)}{2}} P_{n+m}^m(\tanh \beta r) \] \hspace{1cm} (5.34)

In Figure (5.1), we have shown the behavior of the potential and its corresponding wave functions given by expressions (5.32) and (5.34) as function of \( r \). The familiar Schrödinger equation in \( D_B \) dimensional Euclidean space can be written as

\[ \psi_B''(r) + \frac{D_B-1}{r} \psi_B'(r) + \left( E_n^B + C_B^2 \text{sech}^2 \beta r \right) \psi_B(r) = 0 \] \hspace{1cm} (5.35)
5.3. Generation of ESPs from generalized Hulthen potential

![Figure 5.1](image)

Figure 5.1: The exactly solvable potential $V(r)$ with ground-state wave function $\psi_0(r)$ ($n = 0$) and excited-state wave functions $\psi_1(r)$ and $\psi_2(r)$ ($n = 1, 2$) given by expressions (5.32) and (5.34) plotted as function of $r$ for the set of parameters $C_B = 10, \beta = 1, \mu = 1$. The graphs are drawn in arbitrary units.

5.3.2 Generation of ESP using the second WP

Putting the second choice of WP $V_A^W(g_B(r)) = -(\alpha^2 + \alpha^2)^{-\frac{1}{2}}$ in equation (5.10), we get

$$-g_B^2 \times (\alpha^2 + \alpha^2) \frac{e^{-2\alpha g_B}}{1 - e^{-2\alpha g_B}} = -E_n^B$$  \hspace{1cm} (5.36)

Integrating (5.36), we get $g_B(r)$ as

$$g_B(r) = \frac{1}{\alpha} \ln \sec \alpha K_B r$$  \hspace{1cm} (5.37)

where

$$K_B = \sqrt{\frac{E_n^B}{\lambda^2 + \alpha^2}}$$  \hspace{1cm} (5.38)
5.3. Generation of ESPs from generalized Hulthén potential

Equations (5.11) and (5.37) yield

$$V_B^{(1)}(r) = K_B^2(-E_n^A) \tan^2 \alpha K_B r$$  \hspace{1cm} (5.39)

We put

$$K_B^2(-E_n^A) = C_B^2$$  \hspace{1cm} (5.40)

where $C_B^2$ is the characteristic constant of the new QS.

Equations (5.12) and (5.37) lead to

$$V_B^{(2)}(r) = \alpha^2 K_B^2 \mu^2 \sec^2 \alpha K_B r - \frac{3}{4} \alpha^2 K_B^2 \cot^2 \alpha K_B r$$  \hspace{1cm} (5.41)

Equations (5.13) and (5.37) give

$$V_B^{(3)}(r) = -\alpha^2 K_B^2 \sec^2 \alpha K_B r + \frac{3}{4} \alpha^2 K_B^2 \cot^2 \alpha K_B r + \frac{3}{4} \alpha^2 K_B^2 \tan^2 \alpha K_B r + \frac{3}{2} \alpha^2 K_B^2$$  \hspace{1cm} (5.42)

Then from equation (5.15), the B-QS potential is found as

$$V_B(r) = \alpha^2 K_B^2 \mu^2 \sec^2 \alpha K_B r - \frac{1}{2} \alpha^2 K_B^2 \tan^2 \alpha K_B r$$  \hspace{1cm} (5.43)

The quantized energy eigenvalues of B-QS are subsequently obtained as

$$E_n^B = \frac{\alpha^2 K_B^2}{2} [2(n + \mu)(n + \mu + 1) + 1]$$  \hspace{1cm} (5.44)
5.3. Generation of ESPs from generalized Hulthén potential

\( V_B(r) \) is a Sturmian potential as \( K_B \) in equation (5.38) is \( n \)-dependent. To transform it to a normal potential, we shall make the combination \( aK_B \) \( n \)-independent. This can be achieved if the A-QS parameters \( \alpha \rightarrow \alpha_n \) and \( \lambda \rightarrow \lambda_n \) are made \( n \)-dependent. The re-defined parameters \( \alpha_nK_B = \gamma \) and \( \gamma\mu = \sigma \) are the \( n \)-independent scale parameters which restore the characteristic constant \( C_B^2 \) of B-QS.

Consequently the potential \( V_B(r) \) given by

\[
V_B(r) = \sigma^2 \sec^2 \gamma r - \frac{1}{2} \gamma^2 \tan^2 \gamma r
\]  

(5.45)

becomes a normal potential and is similar to Pöschl-Teller I potential [21]. The energy eigenvalue expression of the normal B-QS is

\[
E_n^B = 2C_B^2 \left[ 2(n + \mu)(n + \mu + 1) + 1 \right]
\]  

(5.46)

Utilizing equations (5.7) and (5.37), we get the eigenfunction expression as

\[
\psi_B(r) = N_B r^{\frac{-(D_{B-1})}{2}} \cos^\frac{1}{2}(\gamma r)P_{n+\mu}^{\mu}(\sin \gamma r)
\]  

(5.47)

Figure (5.2) shows the behavior of the potential and its corresponding wave functions given by expressions (5.45) and (5.47) as function of \( r \).
5.3 Generation of ESPs from generalized Hulthén potential

Figure 5.2: The exactly solvable potential $V(r)$ with ground-state wave function $\psi_0(r)$ ($n = 0$) and excited-state wave functions $\psi_1(r)$ and $\psi_2(r)$ ($n = 1, 2$) given by expressions (5.45) and (5.47) plotted as function of $r$ for the set of parameters $\sigma = 10$, $\gamma = 1$, $\mu = 1$. The graphs are drawn in arbitrary units.

5.3.3 Generation of ESP using the third WP

Proceeding in a similar way, using the WP $V_A^W(g_B(r)) = -\frac{3}{4} \alpha^2 \frac{\xi}{(1 - e^{2\alpha\xi r})^2}$ in equation (5.10) gives the transformation function as

$$\psi_B(r) = -\frac{1}{2\alpha} \ln (1 - e^{2\alpha\xi r})$$  \hspace{1cm} (5.48)

where

$$\xi = \sqrt{\frac{4 E_B}{3 \alpha^2}}$$  \hspace{1cm} (5.49)

Equations (5.11), (5.12), (5.13), (5.15) and (5.48) yield the new potential of B-QS as

$$V_B(r) = -(\lambda^2 + \alpha^2) \xi^2 \frac{e^{2\alpha\xi r}}{1 - e^{2\alpha\xi r}} + \alpha^2 \mu^2 \xi^2 \frac{e^{2\alpha\xi r}}{1 - e^{2\alpha\xi r}} - \alpha^2 \xi^2 \frac{e^{4\alpha\xi r}}{(1 - e^{2\alpha\xi r})^2}$$  \hspace{1cm} (5.50)

The Sturmian B-QS can be converted to normal QS by a case specific regrouping technique, where we have to redefine the A-QS parameters. To make $\xi$ $n$-independent, we
choose $\alpha \rightarrow \alpha_n$ and $\lambda \rightarrow \lambda_n$. Fortunately, the ratio $\frac{e^{2r}}{\alpha^2}$ becomes $n$-independent. This will make $V_B(r)$ normal/physical potential and can be written as

$$V_B(r) = -(\tau^2 + \eta^2) \frac{e^{2\eta r}}{1 - e^{2\eta r}} + \eta^2 \mu^2 \frac{e^{2\eta r}}{1 - e^{2\eta r}} - \eta^2 \frac{e^{4\eta r}}{(1 - e^{2\eta r})^2}$$  \hspace{1cm} (5.51)$$

where we have introduced scale parameters $\eta = \alpha_n \xi$ and $\tau = \lambda_n \xi$.

For the potential $V_B(r)$, we found constant energy eigenvalues as

$$E_B = \frac{3}{4} \eta^2$$  \hspace{1cm} (5.52)$$

This leads to a set of iso-spectral QSs having only the ground states with $n$ playing the role of system index.

The corresponding bound state wave function is

$$\psi_B(r) = N_B r^{-(\theta r + 1)} e^{\frac{r}{2}} \left( \frac{e^{2\eta r}}{1 - e^{2\eta r}} \right)^{-\frac{1}{2}} P_{n+\mu}^\mu (e^{2\eta r})$$  \hspace{1cm} (5.53)$$

To summarize, we have listed the potentials and their associated energy eigenvalues and wave functions generated from generalized Hulthén potential in Table 5.1.

5.4 Discussion and Conclusion

We have generated a few exactly solved QSs in non-relativistic quantum mechanics using the ET method in arbitrary $D$-dimensional Euclidean spaces. The selection of WP
### Table 5.1: Summary of the quantum systems obtained from generalized Hulthén potential

<table>
<thead>
<tr>
<th>$WP$</th>
<th>$V_B(r)$</th>
<th>$E_n$</th>
<th>$\Psi_B(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\alpha^2}{1-e^{-2\alpha r}}$</td>
<td>$-C_0^2 \text{sech}^2 \beta r$</td>
<td>$-\beta^2 \left[ \frac{1}{2} \left( 1 + \frac{4\alpha^2}{\omega^2} \right) - \left( n + \frac{1}{2} \right) \right]^2$</td>
<td>$r^{\frac{\omega^2}{2\alpha^2}} P_{n\mu}(\text{tanh} \beta r)$</td>
</tr>
<tr>
<td>$-(\lambda^2 + \alpha^2) \frac{e^{2\alpha r}}{1-e^{-2\alpha r}}$</td>
<td>$\alpha^2 \sec^2 \gamma r - \frac{\gamma^2}{2} \tan^2 \gamma r$</td>
<td>$2C_0^2 \left[ 2(n + \mu)(n + \mu + 1) + 1 \right]$</td>
<td>$r^{\frac{\omega^2}{2\alpha^2}} \cos \frac{1}{2} (\gamma r) P_{n\mu}(\sin \gamma r)$</td>
</tr>
<tr>
<td>$\frac{-3}{4} \frac{\alpha^2 - e^{2\alpha r}}{(1-e^{-2\alpha r})}$</td>
<td>$-(\nu^2 + \eta^2) \frac{e^{2\nu r}}{1-e^{-2\nu r}}$</td>
<td>$\frac{1}{4} \eta^2$</td>
<td>$r^{\frac{\omega^2}{2\alpha^2}} e^{\frac{1}{2} \gamma r} \left( \frac{e^{\nu r}}{1-e^{-2\nu r}} \right)^{-\frac{1}{2}} \times P_{n\mu}(e^{\nu r})$</td>
</tr>
</tbody>
</table>

can be made in principle in $(2^3 - 1) = 7$ different ways for three term potential. We however restrict ourselves to taking one term WP. Two or multiterm WP are usually avoided as they offer mathematical difficulties. The generated QSs, in general, are Sturmian potentials. There is no general procedure that can be applied to convert them into normal QSs. However, in some instances, the case-specific procedures can be applied to convert a set of Sturmian QSs to normal/physical QSs by redefining certain quantities.