Spinor Solution in Bianchi Type III Cosmology

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ABSTRACT

In this paper an exact solution of the Dirac equation in Bianchi type III metric is presented.

Key Words : Bianchi type III metric, Dirac equation, cosmology.

INTRODUCTION

In quantum mechanics Klein-Gordon, Dirac and Weyl equations represent the equations of motion in flat space-times. But with the formulation of General Theory of Relativity (GRG), the study of gravitational interactions on quantum mechanical systems in curved space-times have become more important due its wide applications in astrophysics and cosmology. The study of the quantum effects in gravitational field in creation of particles and antiparticles is a must due to presence of strong gravitational field. This is why much interest has been evinced by a large number of authors to carry out quantum theoretical investigations and to examine the creation of particles and anti particles in curved space-times (Castagnino et. al., 1984; Helliwell and Konkowski, 1986; Parkar, 1987; Kröri et. al., 1990). Recently Weinberg (1972), Chimento and Mollerach (1987) have studied the particle creation in Robertson-Walker metrics and extended the study to Bianchi type I metrics. In this context it would be of relevant interest to derive exact solution of generalized Dirac equation in different cosmological models.

The same equation in curved space-time represented by Bianchi type III metric is examined in this study. Since various metric represent various evolutionary phases of the universe, the aim of the present study is to extend to the subsequent phase and to find out whether the exact solution of the Dirac equation is derivable here as well. In this communication, the exact solution of the Dirac equation in Bianchi type III metric is presented.
GENERALIZED DIRAC EQUATION AND SOLUTION

The generalized Dirac equation in curved space-time is

\[ \gamma^\mu \nabla_\mu \psi (x,t) = 0 \]  

(1)

where

\[ \nabla_\mu = \delta_\mu - \sigma_\mu \]  

(2)

and

\[ \sigma_\mu = (1/4) \gamma^{(0)} \gamma^{(0)} V_{(\alpha)}^\nu \cdot V_{(\beta)}^\nu, \mu \]  

(3)

Here \( \sigma_\mu \) are spinorial affine connections. \( \gamma^{(0)}, \gamma^{(0)} \) are flat space-time Dirac matrices. \( \gamma^\mu \) are curved space-time Dirac matrices. \( V_{(\alpha)}^\nu \) are four vector fields called vierbeins and are related to the matrices by the equation

\[ V_{(\alpha)}^a \cdot V_{(\beta)}^b \eta^{(\alpha)ab} = g^{ab} \]  

(4)

\( \eta^{(\alpha)ab} \) are Minkowski metric with signature \{ -1, -1, -1, +1 \}.

The Dirac matrices in the flat and curved space-times are connected by the relation

\[ \gamma^\mu = V_{(\alpha)}^\mu \gamma^{(a)} \]  

(5)

Now let us start with the Bianchi type III metric in the form

\[ ds^2 = dt^2 - A^2 dx^2 - B^2 dy^2 - C^2 dz^2 \]  

(6)

where \( A = A(t), B = B(t), C = C(t) \).

Obviously \( g^{\alpha\nu} \) can be written as

\[ g^{11} = -1/A^2, \quad g^{22} = -1/B^2 e^{2x}, \]
\[ g^{33} = -1/C^2, \quad g^{00} = 1. \]

Applying equation (4), we obtain the vierbeins for the Bianchi type III metric

\[ V_{(1)}^1 = 1/A, \quad V_{(2)}^2 = 1/B e^x, \]
\[ V_{(3)}^3 = 1/C, \quad V_{(0)}^0 = 1. \]  

(7)
Now with the help of (5) and (7) the Dirac matrices in curved space-time may be obtained as

\[ \gamma^1 = \gamma^{(1)}A, \quad \gamma^2 = \gamma^{(2)}B e^x, \]
\[ \gamma^3 = \gamma^{(3)}C, \quad \gamma^0 = \gamma^{(0)}. \]  

(8)

The term \( V_{(g)\mu,\nu} \) in Eq. (3) is given by

\[ V_{(g)\mu,\nu} = \delta_{\mu} V_{(g)\nu} - \Gamma_{\nu\mu}^{\lambda} V_{(g)\lambda}. \]

(9)

Here, \( \Gamma_{\nu\mu}^{\lambda} \) are Christoffel's symbols whose non-vanishing components are

\[ \Gamma_{11}^0 = A A^0, \quad \Gamma_{10}^1 = \Gamma_{01}^1 = A^0 A, \]
\[ \Gamma_{12}^2 = \Gamma_{21}^2 = 1, \quad \Gamma_{12}^1 = - B^2 e^{2x}/A^2, \]
\[ \Gamma_{22}^0 = B B^0 e^{2x}, \quad \Gamma_{20}^2 = \Gamma_{02}^2 = B^0 B, \]
\[ \Gamma_{33}^0 = C C^0, \quad \Gamma_{30}^3 = \Gamma_{03}^3 = C^0 C. \]

(10)

The spinorial affine connections \( \sigma_\mu \) can now be written as

\[ \sigma_1 = \{A/2\} \gamma^{(0)} \gamma^{(1)} \]
\[ \sigma_2 = \{B e^x/2A\} \gamma^{(1)} \gamma^{(2)} + \{B e^x/2\} \gamma^{(0)} \gamma^{(2)} \]
\[ \sigma_3 = \{C/2\} \gamma^{(0)} \gamma^{(3)} \]
\[ \sigma_0 = 0 \]

(11)

The Dirac equation (1) can now be solved by applying the method of separation of variables. We write the solution in the form

\[ \psi(x, t) = (2\pi)^{-3/2} (A B C e^x)^{-1/2} \left\{ \exp\{ i t \eta_{p} dt \} \right\} \exp\{ i t A_{p} dt \} \exp\{ i t B_{p} dt \} \exp\{ i t C_{p} dt \} \cdot e^{ikx}. \]

(12)

where \( k_x = k_1 x + k_2 y + k_3 z \)
Substituting equation (12) in equation (1), we obtain the following matrix equation.

\[
\begin{bmatrix}
\eta_p - m & 0 & E_3 & E_2 \\
0 & \xi_p - m & E_2 & -E_3 \\
E_3 & E_1 & A_p + m & 0 \\
E_2 & -E_3 & 0 & B_p + m
\end{bmatrix}
\begin{bmatrix}
\exp\left\{ \int_0^t \eta_p \, dt \right\} \\
\exp\left\{ \int_0^t \xi_p \, dt \right\} \\
\exp\left\{ \int_0^t A_p \, dt \right\} \\
\exp\left\{ \int_0^t B_p \, dt \right\}
\end{bmatrix} = 0
\]  

(13)

where

\[E_j = k_j V_{(j)} \mathcal{J}, \quad j = 1, 2, 3 \quad \text{only and}
\]

\[E_1 = k_1 / A + i k_2 / B e^x = \tilde{E}_1 + i \tilde{E}_2
\]

\[E_2 = k_1 / A - k_2 / B e^x = \tilde{E}_1 - i \tilde{E}_2
\]

\[E_3 = k_3 / C
\]

(14)

It may be seen that the solution of equation (13) will be nontrivial if

\[E^2 - (B_p + m)(E_1 + m)(E_2 - m)(E_3 + m) + \tilde{E}_3^2(E_1 - m)(E_2 - m) = 0
\]

(15)

here

\[E^2 = \tilde{E}_1^2 + \tilde{E}_2^2 + \tilde{E}_3^2 = k_1^2 / A^2 + k_2^2 / B^2 e^{2x} + k_3^2 / C^2
\]

Let us now assume that the particles, whose equation of motion is under consideration, are travelling in the z-direction. Hence, \(k_1 = 0 = k_2\) and \(k_3 = k\) (say).

Eq. (13) reduces to

\[
\begin{bmatrix}
\eta_p - m & 0 & k/C & E_2 \\
0 & \xi_p - m & E_2 & \frac{-k}{C} \\
\frac{k}{C} & E_1 & A_p + m & 0 \\
E_2 & \frac{-k}{C} & 0 & B_p + m
\end{bmatrix}
\begin{bmatrix}
\exp\left\{ \int_0^t \eta_p \, dt \right\} \\
\exp\left\{ \int_0^t \xi_p \, dt \right\} \\
\exp\left\{ \int_0^t A_p \, dt \right\} \\
\exp\left\{ \int_0^t B_p \, dt \right\}
\end{bmatrix} = 0
\]  

(16)
To facilitate the solution of Eq. (16) let us introduce a condition

\[(\eta_p - m)(A_p + m) = k^2/C^2 = (B_p + m)(\xi_p - m)\]  

(17)

The column matrix may now be expressed as

\[
\begin{bmatrix}
\exp\left\{ \int \eta_p \, dt \right\} \\
\exp\left\{ \int \xi_p \, dt \right\} \\
\exp\left\{ \int A_p \, dt \right\} \\
\exp\left\{ \int B_p \, dt \right\}
\end{bmatrix}
\begin{bmatrix}
(A_p + m) - k/C \\
(B_p + m) - k/C \\
(\eta_p - m) - k/C \\
(k/C - (\xi_p - m))
\end{bmatrix}
\]

(18)

Operating Eq. (16) from the left with the operator \[\gamma (0) \partial_0 + i \tilde{E}_1 \gamma (0) - m\] and substituting (18) we get the following four differential equations \(\eta_p, \xi_p, A_p\) and \(B_p\):

\[
\eta_p \eta_p' - \left[ (k/C) + m \right] \eta_p + (C' / C) \eta_p^2 - \left[ (2mC'/C) + (kC'C^2) \right] \eta_p
+ \left[ (m^2C'/C) + (mkC'C^2) \right] = 0
\]

(19)

\[
\xi_p \xi_p' - \left[ (k/C) + m \right] \xi_p + (C' / C) \xi_p^2 - \left[ (2mC'/C) + (kC'C^2) \right] \xi_p
+ \left[ (m^2C'/C) + (mkC'C^2) \right] = 0
\]

(20)

\[
A_p A_p' - \left[ (k/C) - m \right] A_p + (C' / C) A_p^2 + \left[ (2mC'/C) - (kC'C^2) \right] A_p
+ \left[ (m^2C'/C) - (mkC'C^2) \right] = 0
\]

(21)

\[
B_p B_p' - \left[ (k/C) - m \right] B_p + (C' / C) B_p^2 + \left[ (2mC'/C) - (kC'C^2) \right] B_p
+ \left[ (m^2C'/C) - (mkC'C^2) \right] = 0
\]

(22)
All the above four equations are similar to the Abel equation of the second kind and the exact values of $\eta_p, \xi_p, A_p, B_p$ may be obtained. With appropriate choice of the integration constants the solutions are

$$\eta_p = \frac{(2k/C)}{m}, \quad \xi_p = \frac{(2k/C)}{m},$$
$$A_p = \frac{(2k/C)}{m}, \quad B_p = \frac{(2k/C)}{m} \quad (23)$$

We thus obtain the exact solution of the Dirac equation in Bianchi type III metric.

ACKNOWLEDGEMENT

We express our sincere thanks to Prof. K.D.Krori for his encouragement in carrying out this investigation.

REFERENCES

EXACT SCALAR AND SPINOR SOLUTIONS IN ROBERTSON-WALKER UNIVERSE

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Received in September 2002; Revised and accepted in March 2003

In this paper exact solutions of the Klein-Gordon and Weyl equations in cylindrically symmetric Robertson-Walker universe are presented.

INTRODUCTION

Gravitation, astrophysics and cosmology are exciting and rapidly advancing fields of research on theoretical, observational and experimental fronts. Researchers are now convinced about the crucial role played by quantum mechanical effects in the space-time singularity, as in the last stages of a collapsing star or in the early stages of the universe. Hence the study of the gravitational interactions on quantum mechanical systems in curved space-times has been an active field of investigation. Many workers are engaged in this field to study quantum mechanics in curved space-times and to examine the creation of particles and ant-particles in curved space times\(^1\)\(^-\)\(^3\). Klein-Gordon and Weyl equations we studied in Godel universe by Pimentel and Mačias\(^4\). The study of Weyl equation was further extended to other Godel type universes later on by Pimentel, Camacho and Mačias\(^6\). Both Klein-Gordon and Weyl equations were also studied by Krori et al\(^6\)\(^,\)\(^7\) by considering Som-Raychaudhuri, Hoenselaers-Vishveshwara and Rebouças universes and the field of stationary cosmic field metric. Some more work on the spin-0 fields in a static cosmic string metric has been done by Helliwell and Konkowski\(^2\) and Parker\(^3\).

The purpose of this paper is to present exact solutions of Klein-Gordon and Weyl equations in cylindrically symmetric Robertson Walker(RW) Universe. We could solve massive scalar equation and the massless Weyl equation in the field of RW
metric during the present investigation. The solutions could also satisfy usual boundary conditions.

SCALAR FIELD EQUATION

The governing equation for a massive scalar field with arbitrary coupling to the gravitational field can be taken in the form

$$(- \nabla^2 \psi + \xi R_c + m_0^2) \psi = 0$$  \hspace{1cm} (1)

where $\xi$ is the real dimensionless coupling constant and $R_c = R_{\mu\nu}g^{\mu\nu}$ is the Reimann scalar. The $\nabla^2 \psi$ of Eq.(1) may be expressed as

$$\nabla^2 \psi = \psi_{,\alpha} = [ \partial \psi / \partial x^\alpha ] + \{ \nu \sigma, \sigma \} \psi^\alpha$$  \hspace{1cm} (2)

where $\{ \nu \sigma, \sigma \}$ is a Christoffel's symbol of the second kind.

The cylindrical symmetrical non-singular RW metric is

$$ds^2 = dt^2 - (1 + \beta t^2)^2 [ dr^2/(1 + m^2 r^2) + r^2 d\theta^2 + (1 + m^2 r^2) dz^2 ]$$  \hspace{1cm} (3)

here $\beta$ and $m$ are constants.

The Ricci scalar for (3) is

$$R_c = -(24 \beta^4 - 6m^2)/(1 + \beta^2 t^2)^2 - 12\beta^2/(1 + \beta^2 t^2)$$  \hspace{1cm} (4)

The term $\nabla^2 \psi$ of Eq.(1) is evaluated with the help of equation (3) as

$$\nabla^2 \psi = [- \{(1 + m^2 r^2)/(1 + \beta^2 t^2)\partial_{tt} - [(3m^2 t^2 + 1)/(1 + \beta^2 t^2) r^2(1 + 1/\beta^2 t^2) r^2] \partial_{\theta\theta}^2

- [1/(1 + \beta^2 t^2)(1 + m^2 r^2)] \partial_{zz}^2 + \partial_{tt}^2 + (6\beta^2 t/(1 + \beta^2 t^2)) \partial_t] \psi$$  \hspace{1cm} (5)

Putting equation (5) in Equation (1), we get

$$\left[ \{(1 + m^2 r^2)/(1 + \beta^2 t^2)\partial_{tt} + [(3m^2 t^2 + 1)/(1 + \beta^2 t^2) r^2(1 + 1/\beta^2 t^2) r^2] \partial_{\theta\theta}^2

+ [1/(1 + \beta^2 t^2)(1 + m^2 r^2)] \partial_{zz}^2 - \partial_{tt}^2 - (6\beta^2 t/(1 + \beta^2 t^2)) \partial_t + \xi R_c + m_0^2 \right] \psi = 0$$  \hspace{1cm} (6)
Now to solve the above differential equation we choose a most general solution, keeping in view the expanding universe, in the form

$$\psi = R(r)T(t)e^{ia\theta}e^{ibz}$$

(7)

where $a$ and $b$ are constants. Substituting (7) in (6) and separating the variables we obtain the following two equations

$$(1+m^2r^2)R''(r) + \left\{\frac{am}{r} + 1\right\} R'(r) - \left\{\frac{a^2r^2}{2} + \frac{b^2}{1+m^2r^2} + K\right\} R(r) = 0$$

(8)

$$(1+\beta^2t^2)^2 T''(t) + 6\beta^2t(1+\beta^2t^2)T'(t) - \left[(\xi R_c + m_0^2)(1+\beta^2t^2) + K\right] T(t) = 0$$

(9)

where $K$ is a separation constant, $K$ has arisen when we equated both space variable part and the time dependent part of equation (6) [after we apply solution in equation (7)] separately to it to arrive at equations (8) and (9).

Now we proceed to solve the equation (8) first. To solve this equation we shall have to substitute the variables at quite a few stages by trial. Reason being that it is really difficult to solve the same in present form. Each time we substitute, the resultant equation comes out simpler. We continue this exercise till a standard form of equation emerges and a plausible solution is obtained.

Thus we substitute $1+ m^2r^2 = x$ and $R(r) = X(x)$ and obtain

$$x^2(1-x)^2X'' + x(1-x)(1-2x)X' + \left[\frac{b^2}{4m^2} + \{-(a^2/4) - (b^2/4m^2) + (K/4m^2)\}x - (K/4m^2)x^2\right] X = 0$$

(10)

Then we substitute $X(x) = U(z)$ and $2x = 1-z$ in equation (10) and obtain

$$(1-z^2)^2 U'' - 2z(1-z^2)U' + \left[\frac{1}{2}.\left\{\frac{b^2}{m^2} + (K/2m^2) - a^2\right\}\right.$$

$$+ \left.\{a^2 + (b^2/m^2)\}z - (K/4m^2)z^2\right] U = 0$$

(11)
Substituting again $1+z = 2v$ and $U(z) = (1+z)^r \cdot (1-z)^s \cdot y(v)$ in equation (11) we finally obtain

$$v (1-v) y'' + \left[ c_1 - (1+a_1+b_1) v \right] y' - a_1 b_1 y = 0$$

(12)

with

$$c_1 = (a^2/4) - (b^2/4m^2) - (K/8m^2) + \sqrt{(4m^2a^2+K)/4m} + \sqrt{(K-4b^2)/4m}$$

$$+ \sqrt{(4m^2a^2+K)/8m^2}$$

$$a_1 = \frac{1}{2} + \frac{\sqrt{(K-4b^2)/8m} + \sqrt{(4m^2a^2+K)/4m} + \sqrt{16m^2+12b^2+13K^2}}{8m}$$

$$b_1 = \frac{1}{2} + \frac{\sqrt{(K-4b^2)/8m} + \sqrt{(4m^2a^2+K)/4m} - \sqrt{16m^2+12b^2+13K^2}}{8m}$$

Equation (12) is the Gauss' hyper-geometric equation. The solution is

$$y = C_1 y_1 + C_2 y_2$$

(13)

with

$$y_1 = {}_2F_1 (a_1, b_1, c_1; v)$$

$$y_2 = v^{2-c} \cdot {}_2F_1 (1+a_1-c_1, 1+b_1-c_1, 2-c_1; v)$$

where

$$2F_1 (a_1, b_1, c_1; v) = 1 + [a_1b_1/c_1]v + [(a_1(a_1+1).b_1(b_1+1))/(2!c_1(c_1+1))]v^2 + \ldots$$

Now we proceed to examine (9). Putting equation (4) in equation (9) we get

$$(1+\beta^2t^2)^2 T''(t) + 6\beta^2t(1+\beta^2t^2) T'(t) + [(12\beta^2 \xi - m_0^2 - K - 6\xi m^2) + (36\beta^2\xi - 2m_0^2) \beta^2t]^2$$

$$- m_0^2 \beta^4 t^4 \] T(t) = 0$$

(14)
There is no loss of generality if we arbitrarily take $\beta = 1/3$ and substitute $T(t) = y(x)$, $t = iax$ in the above equation, which reduces to

$$(1-x^2)^2 y'' - 2x(1-x^2) y' + [ A_0 + A_2 x^2 + A_4 x^4 ] y = 0$$

(15)

where $A_0 = 12\beta^2\xi - m_0^2 - K - 6\xi m^2$

$A_2 = -36\beta^2\xi + 2m_0^2$

$A_4 = -m_0^2$

Equation (15) is a spheroidal wave equation. It can be rearranged as

$$d/dx[(1-x^2)dy/dx] + \left[A-h^2x^2 - \eta^2/(1-x^2)\right]y = 0$$

(16)

where $A = -(A_2 + A_4)$

$$h^2 = A_4$$

$$\eta^2 = (A_0 + A_2 + A_4)$$

The solution $y(x)$ of equation (16) could be expressed in terms of a series of associated Legendre function

$$y(x) = \sum_n n^2 P^n \eta^n(x) = (1-x^2)^{n/2} \sum_n d_n T^n(x)$$

(17)

where the recursion formula relating successive coefficients is

\[
\begin{align*}
\{n(n-1)h^2\} / \{(2n+2\eta-1)(2n+2\eta-3)\} \cdot d_{n-2} \\
+ h^2 \{n+2\eta+1)(n+2\eta+2)\} / \{(2n+2\eta+3)(2n+2\eta+5)\} \cdot d_{n+2} \\
+ h^2 (2(n+\eta)(n+\eta+1) - \eta^2 - 1) / (2n+2\eta+3)(2n+2\eta-1) + (n+\eta)(n+\eta+1) - A \cdot d_n = 0
\end{align*}
\]

(18)

And $P^n \eta^n(x)$ is Legendre function of first kind $T^n(x)$ is Gegenbauer function.
The general form of the Weyl equation for a massless spin-$\frac{1}{2}$ field is

$$\gamma^0 \nabla_\sigma \psi = 0$$

i.e.,

$$[\gamma^1 \nabla^1 + \gamma^2 \nabla_\tau + \gamma^0 \nabla_\theta + \gamma^2 \nabla_z] \psi = 0 \quad (19)$$

and

$$[1 + \gamma^5] \psi = 0 \quad (20)$$

where

$$\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

and

$$\gamma^a = h^{(a)} \gamma^{(a)} \quad (21)$$

Here $\gamma^a$ are generalized Dirac matrices and are given in terms of the flat space-time Dirac matrices $\gamma^{(a)}$ and $h^{(a)}$, vierbeins defined by the relations

$$h^{(a)} h^{(b)} \eta^{(a)(b)} = g^{ab} \quad (22)$$

Now we consider the Weyl equation in the field of a Robertson-Walker metric given by the metric (3). For this metric the set of vierbeins that we shall use are

$$h^{(1)} = \sqrt{(1+m^2r^2)/(1+\beta^2t^2)}$$,

$$h^{(2)} = 1/(1+\beta^2t^2)$$,

$$h^{(3)} = 1/(1+m^2r^2)(1+\beta^2t^2)$$,

$$h^{(0)} = 1 \quad (23)$$

From equations (21) and (23) the Dirac matrices in curved space-time are

$$\gamma^1 = \gamma^{(1)} \sqrt{(1+m^2r^2)/(1+\beta^2t^2)}$$,

$$\gamma^2 = \gamma^{(2)} / r (1+\beta^2t^2)$$,

$$\gamma^3 = \gamma^{(3)} / \sqrt{(1+m^2r^2)(1+\beta^2t^2)}$$,

$$\gamma^0 = \gamma^{(0)} \quad (24)$$
Now we have

\[ \nabla_a \psi = [ \partial_a - \Gamma_a ] \psi \]  

(25)

where \[ \Gamma_a = -\langle \gamma_a \rangle \gamma^{(b)} h_{(a) b} h_{(b) a} \]  

(26)

and \[ h_{(b) c a} = \partial_a h_{(b) c} - T_{b c} \gamma^{(b)} \]  

(27)

Here \( T_{b c} \) are Christoffel's symbols and \( \Gamma_a \) are spinorial affine connections.

Using Eqs. (25), (26) and (27), we get

\[ \Gamma_1 = -\gamma^{(0)} \gamma^{(1)} \beta^2 t / \sqrt{1+m^2 r^2} \]  

(28)

\[ \Gamma_2 = -\gamma^{(0)} \gamma^{(2)} \sqrt{1+m^2 r^2} - \gamma^{(0)} \beta^2 t \]  

(29)

\[ \Gamma_3 = -\gamma^{(0)} \gamma^{(3)} m^2 r - \gamma^{(0)} \beta^2 t \sqrt{1+m^2 r^2} \]  

(30)

\[ \Gamma_0 = 0 \]  

For separation of variables we take

\[ \psi = T(t) e^{i k \theta} e^{i m z} \begin{pmatrix} \eta_1(r) \\ \eta_2(r) \end{pmatrix} \]  

(31)

where \( k, m \) are constants and \( \eta_1 \) and \( \eta_2 \) are two-component spinors which can be expressed as follows by using the standard representation of gamma matrices:

\[ \eta_1(r) = \eta_2(r) = \begin{pmatrix} R_1(r) \\ R_2(r) \end{pmatrix} \]  

(32)

On using Eqs. (25) – (30) in Eq. (19) we get two sets of equations which on separation of variables give the following equations –

\[ (1+\beta^2 r^2) T' - (3\beta^2 t + \chi) T = 0 \]  

(33)

\[ \sqrt{1+m^2 r^2} R_2' + \frac{k}{r} - \frac{1+2m^2 r^2}{2r} \sqrt{1+m^2 r^2} R_2 + \frac{\chi + im}{\sqrt{1+m^2 r^2}} R_1 = 0 \]  

(34)

\[ \sqrt{1+m^2 r^2} R_1' - \frac{k}{r} + \frac{1+2m^2 r^2}{2r} \sqrt{1+m^2 r^2} R_1 + \frac{\chi - im}{\sqrt{1+m^2 r^2}} R_2 = 0 \]  

(35)
A solution can be found by integrating equation (31) and it is

\[ T = \sqrt{1 + \beta^2 t^2} \cdot e^{(\beta t)^3} \tan^{-1}(\beta t) \]  

(34)

Now putting the values of \( R_2 \) and \( R_2' \) from (33), equation (32) becomes

\[ f_{11}(r).R_1''(r) + f_{12}(r).R_1'(r) + f_{13}(r).R_1(r) = 0 \]  

(35)

Similarly, putting the values of \( R_1 \) and \( R_1' \) from (32), equation (33) becomes

\[ f_{21}(r).R_2''(r) + f_{22}(r).R_2'(r) + f_{23}(r).R_2(r) = 0 \]  

(36)

[NOTE : the exhaustive mathematical expressions for co-efficient \( f_{11}(r) \), \( f_{12}(r) \), \( f_{13}(r) \), \( f_{21}(r) \), \( f_{22}(r) \), and \( f_{23}(r) \) appearing in the equations (35) and (36) are shown in the ANNEXURE.]

The equations (35) and (36) can not ordinarily be solved and hence we take two cases to solve the same

**CASE 1**: For small \( r \) [i.e., \( r \sim 0 \)] we get,

\[ a_1 R_1''(r) + (b_1/r).R_1'(r) + (c_1/r^2).R_1(r) = 0 \]  

\[ a_2 R_2''(r) + (b_2/r).R_2'(r) + (c_2/r^2).R_2(r) = 0 \]  

(37)

where

\[ a_1 = -1/(\chi - im), \quad a_2 = -1/(\chi + im), \]

\[ b_1 = 1/(\chi - im), \quad b_2 = 1/(\chi + im), \]

\[ c_1 = (4k^2 - 4k - 3) / 4(\chi - im), \]

\[ c_2 = (4k^2 + 4k - 3) / 4(\chi - im), \]

**CASE II**: For large \( r \) [i.e, \( r \gg 0 \)] we get,

\[ a_3 r^2 R_1''(r) + b_3 r R_1'(r) + c_3 R_1(r) = 0 \]  

\[ a_4 r^2 R_2''(r) + b_4 r R_2'(r) + c_4 R_2(r) = 0 \]  

(38)
where

\[ a_3 = -\frac{m^2}{\chi}, \quad a_4 = -\frac{m^2}{\chi} \]

\[ b_3 = -\frac{m^2}{\chi}, \quad b_4 = -\frac{m^2}{\chi} \]

Now all the equations in (37) and (38) can be solved in the following way –

Equation (37) can be written as

\[ a_1 r^2 R_1''(r) + b_1 r. R_1'(r) + c_1. R_1(r) = 0 \]

\[ a_2 r^2 R_2''(r) + b_2 r. R_2'(r) + c_2. R_2(r) = 0 \]

These are Euler's equations and the solutions are

\[ R_1 = r^{p_1} \left[ C_1 \cos(q_1 \log r) + C_2 \sin(q_1 \log r) \right] \] (40)

\[ R_2 = r^{p_2} \left[ C_3 \cos(q_2 \log r) + C_4 \sin(q_2 \log r) \right] \] (41)

where \( x_1 = p_1 + i q_1 \) and \( x_2 = p_1 - i q_1 \) are the two roots of the characteristics equation

\[ x^2 = (b_1 - 1) + c_1 = 0 \]

and \( x_3 = p_2 + i q_2 \)

and \( x_4 = p_2 - i q_2 \)

are the roots of the characteristic equation

\[ x^2 = (b_2 - 1) + c_2 = 0 \]

Equation (38) can now be written as

\[ a_3 r^2 R_1''(r) + b_3 r. R_1'(r) + c_3. R_1(r) = 0 \]

\[ a_4 r^2 R_2''(r) + b_4 r. R_2'(r) + c_4. R_2(r) = 0 \] (42)
These are also Euler's equations and the solutions here are

\[ R_1 = C_5 r^{x_1} + C_6 r^{x_2} \]  \hspace{1cm} (43)

\[ R_2 = C_7 r^{x_3} + C_8 r^{x_4} \]  \hspace{1cm} (44)

where \( x_1, x_2 \) are the real roots of the characteristic equation

\[ x^2 + (b_3 - 1) + c_3 = 0 \]

and \( x_3, x_4 \) are the two real roots of the characteristic equation

\[ x^2 + (b_4 - 1) + c_4 = 0 \]

CONCLUSION

We have solved exactly the massive scalar equation in the background of a cylindrically symmetric non-singular RW metric. The radial solution is of the Gauss hyper-geometric type. On the other hand, the temporal solution turns out to be of the spheroidal wave type.

We have also exactly solved the massless spin-\( \frac{1}{2} \) Weyl equation in the field of RW metric. The solution to the resulting temporal equation had an exponential component and hence conform to the requirements of the physically observed phenomenon of spreading wave-function since the Big Bang. The radial equations can be reduced to the form of Euler's equations and we have been able to solve the same for both small and large \( 'r' \). The expressions for both \( R_1 \) and \( R_2 \) in the equations (40), (41), (43) and (44) show that the solutions are also physically acceptable when \( r \to 0 \) and \( r \to \infty \).
With reference to the equations (35) and (36), the values of the coefficients $f_{11}(r)$, $f_{12}(r)$, $f_{13}(r)$, $f_{21}(r)$, $f_{22}(r)$ and $f_{23}(r)$ may be calculated as –

$$f_{11}(r) = \sqrt{(1+m^2 r^2)^3},$$

$$f_{21}(r) = -\sqrt{(1+m^2 r^2)^3},$$

$$f_{12}(r) = \frac{\sqrt{(1+m^2 r^2)^3(1+2m^2 r^2)} - 2m^2 r \sqrt{(1+m^2 r^2)}}{r \sqrt{(1+m^2 r^2)} + im} + \frac{m^2 r (1+m^2 r^2)}{\left[\sqrt{(1 + m^2 r^2)} - im\right]^2},$$

$$f_{22}(r) = \frac{\sqrt{(1+m^2 r^2)^3(1+2m^2 r^2)} - 2m^2 r \sqrt{(1+m^2 r^2)}}{r \sqrt{(1+m^2 r^2)} + im} + \frac{m^2 r (1+m^2 r^2)}{\left[\sqrt{(1 + m^2 r^2)} + im\right]^2},$$

$$f_{13}(r) = 2m^2 \sqrt{(1+m^2 r^2)} + km + \chi \sqrt{(1+m^2 r^2)} + im$$

$$\chi \sqrt{(1 + m^2 r^2)} - \sqrt{(1+m^2 r^2)}$$

$$- [2k(1+m^2 r^2) + \sqrt{(1+m^2 r^2)}(1+2m^2 r^2)] \left[\chi \sqrt{(1+m^2 r^2)} + m^2 r \sqrt{(1+m^2 r^2)} - im\right]$$

$$2 r^2 \left[\chi \sqrt{(1+m^2 r^2)} - im\right]^2$$

$$+ \left[2k \sqrt{(1+m^2 r^2)} + (1+2m^2 r^2) \right] \left[2k \sqrt{(1+m^2 r^2)} - (1+2m^2 r^2)\right]$$

$$4r^2 \sqrt{(1+m^2 r^2)} \left[\chi \sqrt{(1+m^2 r^2)} - im\right]$$
\[ f_{2a}(r) = - \frac{2m^2 \sqrt{1+m^2 r^2} - km^2}{\chi \sqrt{1 + m^2 r^2} + \text{im}} + \frac{x \sqrt{1+m^2 r^2} - \text{im}}{\sqrt{1+m^2 r^2}} \]

\[ + \frac{2k(1+m^2 r^2) - \sqrt{1+m^2 r^2}(1+2m^2 r^2)}{2r^2 \chi \sqrt{1+m^2 r^2} + \text{im}} \]

\[ + \frac{2k \sqrt{1+m^2 r^2} - (1+2m^2 r^2)}{2k \sqrt{1+m^2 r^2} + (1+2m^2 r^2)} \times 4r^2 \sqrt{1+m^2 r^2}. \chi \sqrt{1+m^2 r^2} + \text{im} \]

ACKNOWLEDGEMENT

We express our sincere thanks to Prof. K. D. Krori for encouragement in presenting this investigation.

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EXACT SPINOR AND SCALAR SOLUTIONS IN KANTOWSKI-SACHS COSMOLOGY

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Received in February 2004 : Revised and accepted in July 2004

In this paper we present solutions of the Dirac and Klein-Gordon equations in inflationary Kantowski-Sachs cosmology.

Key Words : Curved space-time, cosmology

INTRODUCTION

The present day universe appears, on astronomical considerations, to be of Friedmann-Robertson-Walker (FRW) type. But there is no evidence of the fact the early universe was also of the same type. Perhaps in the early era, some other type of cosmological model evolved and, then at some stage it changed over to FRW Model. Hence it is reasonably relevant to think of different types of cosmological models in the context of early phases of the universe. We have, therefore, chosen Kantowski-Sachs (KS) cosmological background for the work presented in this paper.

The study of gravitational interaction on quantum mechanical systems is a field of investigation recently explored by number of authors. The Klein-Gordon equation and the Dirac equation in covariant form have been used in curved space-times for such studies. The electromagnetic equation in Gödel's universe have been studied by Cohen et al. The Klein Gordon and the Weyl equations have also been studied in the same universe by Pimentel and Macias. Krori et al have considered the same two equations in three other well-known rotating universes, viz, Som-Raychaudhuri, Hoenselaers-Vishveshwara and Rebouças universe and also in the field of a stationary cosmic string. The quantum theory was studied by several workers who also examined in creation of particles and anti-particles in curved space-time. Recently Chimento and Mollerach have studied the particle creation in Robertson-Walker metrics and
extended the study to Bianchi type I metrics. In view of what has been explained above in this paper the Dirac and Klein-Gordon equations in curved space time represented by an inflationary Kantowski-Sachs metric. Exact solutions of the equations are derived here.

**DIRAC EQUATION AND ITS SOLUTION**

The generalized Dirac equation in a curved space-time is

\[ [\gamma^\mu \nabla_\mu - m] \psi (x,t) = 0 \]  \hspace{1cm} (1)

where \( \nabla_\mu = \partial_\mu - \sigma_\mu \) \hspace{1cm} (2)

and \( \sigma_\mu = (\frac{i}{2}) \gamma (a) \gamma (b) V_{(a)} V_{(b)} \gamma_\mu \) \hspace{1cm} (3)

Here \( \sigma_\mu \) are spinorial affine connections. \( \gamma (a), \gamma (b) \) are Dirac matrices in Minkowski space-time and \( \gamma^\mu \) are Dirac matrices in space-time and \( V_{(a)}V_{(b)} \) are four-vector fields called vierbeins and are related to the metrics by the equation

\[ V_{(a)}^a V_{(b)}^b \eta (\alpha) (\beta) = g^{ab} \] \hspace{1cm} (4)

where \( \eta (\alpha) (\beta) \) are Minkowski metric with signature

\[ \{ -1, -1, -1, +1 \} \].

Also the Dirac matrices in the two space-times are connected by the relation

\[ \gamma^\mu = V_{(\alpha)}^\mu \gamma (\alpha) \] \hspace{1cm} (5)

Now the anisotropic Kantowski-Sachs metric is

\[ ds^2 = dt^2 - A^2 dr^2 - \Gamma^2 (d\theta^2 + \sin^2 \theta d\phi^2) \] \hspace{1cm} (6)

where \( r, \theta, \phi \), are co-moving co-ordinates and \( A \) and \( \Gamma \) are functions of . The Einstein's equations for the metric are
\[2 \Lambda \Gamma / \Gamma + (1 + \Gamma^{-2})/\Gamma^2 = 8\pi \rho \] \hspace{1cm} (7)

\[2 \Gamma''/\Gamma + (1 + \Gamma^{-2})/\Gamma^2 = -8\pi \rho \] \hspace{1cm} (8)

\[\Lambda^{\prime\prime}/\Lambda + \Gamma''/\Gamma + \Lambda^{\prime}/\Lambda = -8\pi \rho \] \hspace{1cm} (9)

From (8) and (9) we get,

\[\Lambda \Gamma'' - \Gamma \Lambda^{\prime\prime} - \Lambda^{\prime} \Gamma + (\Lambda + \Lambda^{\prime\prime})/\Gamma = 0 \] \hspace{1cm} (10)

Now from Bianchi identity we get,

\[(R^{ij} - \frac{1}{2} g^{ij} R)_{;j} = 0 \] \hspace{1cm} (11)

which leads to the equation

\[dp/dt + (p + p)[2\Lambda^{\prime}/\Lambda + 4\Gamma''/\Gamma] = 0 \] \hspace{1cm} (12)

A possible solution of (6) is

\[\Lambda = B_0 t^2, \quad \Gamma = t / \sqrt{3} \] \hspace{1cm} (13)

where \(B_0\) is a constant.

With the help of Eq.(4) vierbeins worked out that for metric (6) are

\[V_{(1)} = 1 / B_0 t^2, \quad V_{(2)} = \sqrt{3} / t, \quad V_{(3)} = \sqrt{3} / t \sin \theta, \quad V_{(0)} = 1 \] \hspace{1cm} (14)

Now with the help of (5) and (14) the Dirac matrices in curved space–times may be obtained as

\[\gamma^1 = \gamma^{(1)} / B_0 t^2, \quad \gamma^2 = \gamma^{(2)} \sqrt{3} / t, \quad \gamma^3 = \gamma^{(3)} \sqrt{3} / t \sin \theta, \quad \gamma^0 = \gamma^{(0)} \] \hspace{1cm} (15)

In Eq.(3) \(V_{(\beta)\nu,\mu}\) are expressed as

\[V_{(\beta)\nu,\mu} = \partial_\nu V_{(\beta)\nu} - \Gamma^{\lambda}_{\nu\mu} V_{(\beta)\lambda} \] \hspace{1cm} (16)
where $\Gamma^\lambda_{\mu \nu}$ are Christoffel's symbols non-vanishing components are
\[
\begin{align*}
\Gamma_{11}^0 &= 2 B_0 t^3, & \Gamma_{22}^0 &= t/3, \\
\Gamma_{10}^1 &= \Gamma_{01}^1 = 2/t, & \Gamma_{33}^0 &= -t/3 \sin^2 \theta, \\
\Gamma_{23}^3 &= \Gamma_{32}^3 = \cot \theta, & \Gamma_{33}^2 &= -\sin \theta \cdot \cos \theta, \\
\Gamma_{22}^2 &= \Gamma_{02}^2 = \Gamma_{30}^3 = \Gamma_{03}^3 = 1/t
\end{align*}
\] (17)

The connections $\sigma_\mu$ can now be written as
\[
\begin{align*}
\sigma_1 &= -B_0 t \gamma^{(0)} \gamma^{(1)}, \\
\sigma_2 &= (1/2\sqrt{3}) \gamma^{(0)} \gamma^{(2)}, \\
\sigma_3 &= -\sin \theta / 2\sqrt{3} \gamma^{(0)} \gamma^{(3)} - (\cos \theta / 2) \gamma^{(2)} \gamma^{(3)}, \\
\sigma_0 &= 0
\end{align*}
\] (18)

With the help of equations (15) and (18), Dirac equation can now be written in Kantowski-Sachs space-time. The resulting equation can now be solved by the method of separation of variables. We try a solution in the form
\[
\psi(r, \theta, \phi, t) = \left[(\delta/3) \right. \pi^3 B_0 t^4 \sin \theta \left. \right]^{-\frac{1}{2}} \exp \left\{ \int_0^t A_\rho dt \right\} \exp \left\{ \int_0^t B_\rho dt \right\} \exp \left\{ \int_0^t \eta_\rho dt \right\} \exp \left\{ \int_0^t \xi_\rho dt \right\} \exp \left\{ \int_0^t \sigma_0 dt \right\} e^{ikt} \] (19)

where $\eta_\rho$, $\xi_\rho$, $A_\rho$ and $B_\rho$ are spinor functions of the particles which are invariant under co-ordinate transformations. The time dependence of $\psi$ should be taken care of by the exponential functions of the spinors in the above column matrix.
Inserting (19) in (1) we get the following matrix equation

\[
\begin{pmatrix}
\eta_p+(4i/t)-m & 0 & \xi_p+(4i/t)-m & 0 \\
0 & E_1 & 0 & 0 \\
\xi_p+(4i/t)-m & E_2 & -E_3 & 0 \\
E_3 & 0 & A_p+(4i/t)+m & 0 \\
\end{pmatrix}
\begin{pmatrix}
\exp\left\{\int_0^t \eta_p \, dt\right\} \\
\exp\left\{\int_0^t \xi_p \, dt\right\} \\
\exp\left\{\int_0^t A_p \, dt\right\} \\
\exp\left\{\int_0^t B_p \, dt\right\}
\end{pmatrix} = 0
\tag{20}
\]

where

\[
E_1 = k_1/B_0 t^2 + (\sqrt{3}/2t) \cot \theta - i\sqrt{3}k_2/t - E_1 + \xi_2 \\
E_2 = k_1/B_0 t^2 + (\sqrt{3}/2t) \cot \theta + \sqrt{3}k_2/t - E_1 - \xi_2 \\
\xi_3 = \sqrt{3}k_3/t \sin \theta \quad \text{and} \\
k_x = k_1 r + k_2 \theta + k_3 \varphi
\tag{21}
\]

Equation (20) will have nontrivial solutions if

\[
[ E^2 - (B_p + (4i/t) + m)(\eta_p + (4i/t) - m) ] [ E^2 - (A_p + (4i/t) + m)(\xi_p + (4i/t) - m) ] \\
+ \xi_3^2 (\eta_p - \xi_p)(B_p - A_p) = 0
\]

with \(E^2 = \xi_1^2 + \xi_2^2 + \xi_3^2\)

\tag{22}

For simplification, we now consider that the particle travels in the radial direction and \(\theta = 90^\circ\) so that \(k = k_1/B_0\) and \(k_2 = 0 = k_3\). Then Eq.(20) reduces to
Solving Eq. (23) and introducing the condition that
\[
(\eta \times (4i/t) - m) \times (A \times (4i/t) + m) = (B \times (4i/t) + m) \times (\xi \times (4i/t) - m) = k^2/t^4
\] (24)

The column matrix may now be expressed as
\[
\begin{pmatrix}
\exp\left\{ \omega \int \eta dt \right\} & \exp\left\{ \omega \int \xi dt \right\} & \exp\left\{ \omega \int A dt \right\} & \exp\left\{ \omega \int B dt \right\}
\end{pmatrix}
\begin{pmatrix}
\eta \\
\xi \\
A_p \\
B_p
\end{pmatrix}
\begin{pmatrix}
(\eta \times (4i/t) - m) \\
(\xi \times (4i/t) - m) \\
(A \times (4i/t) + m) \\
(B \times (4i/t) + m)
\end{pmatrix}
= 0
\] (23)

Operating Eq. (23) from the left with the operator \[\gamma^{(0)} \partial + iE_1 \gamma^{(1)} - m \] and substituting (25) we get the following four differential equations in \(\eta_p\), \(\xi_p\), \(A_p\) and \(B_p\):
\[
\eta_p \times [\eta_p - kt^{-2} + 4it^{-1} - m] + \eta_p [2t^{-1} \eta_p - 2kt^{-3} + 12it^{-2} - 4mt^{-1}]
+ [ - 4ikt^4 + 2(mk - 8)t^3 - 12imt^2 + 2m^2t^{-1}] = 0
\] (26)
\[
\xi_p \times [\xi_p - kt^{-2} + 4it^{-1} - m] + \xi_p [2t^{-1} \xi_p - 2kt^{-3} + 12it^{-2} - 4mt^{-1}]
+ [ - 4ikt^4 + 2(mk - 8)t^3 - 12imt^2 + 2m^2t^{-1}] = 0
\] (27)
\[ \begin{align*}
A_p^* \left[ A_p - k t^2 + 4 i t^{-1} + m \right] + A_p \left[ 2 t^1 A_p - 2 k t^3 + 12 i t^2 + 4 m t^1 \right] \\
+ \left[ - 4 i t k t^4 - 2 (mk + 8) t^3 + 12 i m t^2 + 2 m^2 t^1 \right] &= 0 
\end{align*} \]

\[ \begin{align*}
B_p^* \left[ B_p - k t^2 + 4 i t^{-1} + m \right] + B_p \left[ 2 t^1 B_p - 2 k t^3 + 12 i t^2 + 4 m t^1 \right] \\
+ \left[ - 4 i t k t^4 - 2 (mk + 8) t^3 + 12 i m t^2 + 2 m^2 t^1 \right] &= 0
\end{align*} \] (28)

All the four above differential equations are similar to the Abel equation of the second kind and may be solved accordingly. With suitable values of integration constants, the solutions are

\[ \begin{align*}
\eta_p &= - k/t^2 - (4i/t) + m \\
\xi_p &= - k/t^2 - (4i/t) + m \\
A_p &= - k/t^2 - (4i/t) - m \\
B_p &= - k/t^2 - (4i/t) - m
\end{align*} \] (30)

We thus obtain the exact solution of the Dirac equation in the Kantowski-Sachs cosmology.

**KLEIN-GORDON EQUATION AND ITS SOLUTION**

The governing equation for massive scalar field with arbitrary coupling to the gravitational field can be taken in the form

\[ \begin{align*}
&\left[ - \nabla_\alpha \nabla^\alpha + \xi R_{\alpha} + m_0^2 \right] \psi = 0 
\end{align*} \] (31)

where \( \xi \) is a real dimensionless coupling constant and \( R_{\alpha} = R_{\mu \nu} g^{\mu \nu} \) is the Riemann scalar. The term \( \nabla_\alpha \nabla^\alpha \) of Eq.(31) may be expressed as
\[ \nabla_\alpha \nabla^\alpha \psi = \psi^\alpha = \left[ \partial \psi^\alpha / \partial x^\alpha \right] + \{ \nu_\sigma , \sigma \} \psi^\nu \quad (32) \]

where \( \{ \nu_\sigma , \sigma \} \) is a Christoffel symbol of the second kind.

From equation (6) we get Ricci scalar as

\[ R_c = -2 \left[ \Lambda^{**}/\Lambda + 2\Gamma^{**}/\Gamma + 2\Lambda^* \Gamma^*/\Lambda \Gamma + (1 + \Gamma^{*2})/\Gamma^2 \right] \quad (33) \]

Taking the value of \( \Lambda \) and \( \Gamma \) as \( \Lambda = B_0 t^2 \) and \( \Gamma = t^2 \) for an inflationary phase and putting (6) and (32) in the equation (31) we get

\[ \frac{1}{B_0^2 t^4} \frac{d}{dr^2} + \frac{3}{t^2} \frac{d}{d\theta}^2 + \left( \frac{3}{t^2} \sin^2 \theta \right) \frac{d}{d\psi^2} - \frac{1}{t^2} - \frac{4}{t} \frac{d}{dt} + \left( \frac{3 \cot \theta}{t^2} \right) \frac{d}{d\psi} \]

\[ - 20 \frac{d^2}{d\psi^2} + m_0^2 \] \( \psi = 0 \quad (34) \]

Let us now choose \( \psi \) in the form

\[ \psi = T(t) S(\theta) e^{ilp} e^{ir} \quad (35) \]

where \( l \) and \( \varepsilon \) are constants.

Substituting (35) in (34) and separating the variables we obtain the following two equations

\[ S(\theta)'' + \{ \cot \theta \} S(\theta)' + \{ \alpha / 3 - 1^2 \csc^2 \theta \} S(\theta) = 0 \quad (36) \]

\[ t^4 T(t)'' + 4t^3 T(t)' + \left[ -m_0^2 \right] t^4 (20 \xi + 3) + t^2 + \varepsilon^2 / B_0^2 \] \( T(t) = 0 \quad (37) \]

where \( \alpha \) is a separation constant

Equation in \( \theta \), i.e., (36) may be handled first by substituting \( S(\theta) = u(z) \) and \( z = \cos \theta \), (36) thus reduces to

\[ (1-z^2)^2 u'' - 2z (1-z^2) u' - \left[ 1^2 - s(s+1)(1-z^2) \right] u = 0 \quad (38) \]

where \( s = \frac{1}{2} \left[ -1 \pm \sqrt{1+4\alpha/3} \right] \)
It is an associated Legendre differential equation and here \( l \), \( s \) are not integers. The general solution is usually given in terms of a Legendre function of first kind, \( P^l_s(z) \), of order \( l \), and of degree \( s \); and a Legendre function of a second kind, \( Q^l_s(z) \). The results refer to real \( z = \cos \theta \), \(-1 < z < 1\). They can be generalized as

\[
\begin{align*}
  u_1 &= P^l_s(z) = \left[ \frac{1}{\Gamma(1-l)} \right] \left( \frac{z+1}{(z-1)} \right)^{1/2} \, _2F_1\left( l, 1+s, 1-l; z \right) \\
  u_2 &= Q^l_s(z) = \left( \frac{\pi}{2} \right) \csc \pi \left[ \frac{P^l_s(z) \cos \pi - \{ \Gamma(1+l+s) / \Gamma(1-l+s) \} P^l_s(z) }{} \right] \tag{39}
\end{align*}
\]

where \( _2F_1(a,b,c;z) = 1 + (ab/c)z + \{a(a+1)b(b+1)/2c(c+1)\}z^2 + \ldots \).

The equation (37) in \( t \) is solved piecewise for some temporal phases.

**Phase I**: If \( t \) is very small so that

\[
- \frac{m^2 t^4}{2} + (20\xi + \alpha) t^2 \ll \frac{\varepsilon^2}{B_0^2}
\]

then equation (37) reduces to

\[
t^4 T^{-4}(t) + 4t^3 T^{-3}(t) \left( \frac{\varepsilon^2}{B_0^2} \right) T(t) = 0 \tag{40}
\]

With the substitution \( T(t) = t^{-3/2}. y(x) \), \( x = -\varepsilon/B_0 t \), the above mentioned equation again reduces to

\[
4x^2 y^{(-4)}(x) + 4xy^{(-3)}(x) - \left[ (1 + 2n)^2 - 4x^2 \right] y(x) = 0 \tag{41}
\]

with \( n = 1 \).

This is Bessel's equation and its general solution is

\[
y = C_1 J_{3/2}(x) + C_2 J^{-3/2}(x) \tag{42}
\]

where \( J_{3/2}(x) = \sqrt{(2/\pi)}x^{3/2}(\sin x - x \cos x) \) and \( J^{-3/2}(x) = \sqrt{(2/\pi)}x^{-3/2}(\sin x - x \cos x) \)
Phase II: If $t$ is moderately small so that

$$(20\zeta + \omega)^2 \sim \varepsilon^2/B_0^2 \gg \left| m_0^2 t^4 \right|;$$

then equation (37) reduces to

$$t^2 T''(t) + 4t T'(t) + \left[ (20\zeta + \omega) + \left( \frac{\varepsilon^2}{B_0^2} \right) t^{-2} \right] T(t) = 0 \quad (43)$$

Now substituting $T(t) = t^{3/2} \cdot y(x)$, $x = -\varepsilon/B_0 t$, we get Bessel's equation in another form

$$x^2 y''(x) + x y'(x) - \left[ q^2 - x^2 \right] y(x) = 0 \quad (44)$$

where $q^2 = \left[ \frac{1}{4} - 20\zeta - \omega \right]$.

Taking the value of $20\zeta + \omega = 5/4, -7/4, -27/4, -55/4 \ldots \ldots$. $q$ becomes positive or negative integers: $1, 2, 3, 4, \ldots \ldots$

Hence the general solution of (44) is

$$y = C_3 J_q(x) + C_4 Y_q(x) \quad (45)$$

where $J_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+n+q)} \left( x/2 \right)^{n+2n}$

and $Y_q(x) = \csc(q\pi) \left[ J_q \cos(q\pi) - J_{-q} \right]$.

Again for $\omega = -20\zeta$, $q$ becomes a half integer and hence the general solution is similar to Phase I as

$$y = C_5 J_{3/2}(x) + C_6 J_{-3/2}(x) \quad (46)$$

This is, however, the exact solution for a mass-less scalar particle.
**Phase III:** If \( t \) is large so that
\[
- m_0^2 t^4 \sim (20\xi + \alpha) t^2 \gg \epsilon^2 B_0^2 ;
\]
then equation (37) reduces to
\[
t^2 T''(t) + 4t T'(t) + [-m_0^2 t^2 + (20\xi + \alpha)] T(t) = 0 \quad (47)
\]
Now putting \( T(t) = t^{-3/2} y(x), x = im_0 t \), in equation (47), we get Bessel equation of the form similar to Phase II.
\[
x^2 y''(x) + x y'(x) - [q_1^2 - x^2] y(x) = 0 \quad (48)
\]
where
\[
q_1^2 = [9 - 4(20\xi - \alpha)]
\]
Taking the value of \( 20\xi + \alpha = 5/4, -7/4, -27/4, -55/4 \ldots \ldots \)
\( q \) becomes positive or negative integers: 1, 2, 3, 4, .......

Hence the general solution of (48) is
\[
y = C_7 J_{q_1}(x) + C_8 Y_{q_1}(x) \quad (49)
\]
Again for \( 20\xi + \alpha = 27/16 \), \( q_1 \) becomes a half integer and hence the general solution is similar to Phase I as
\[
y = C_9 J_{3/2}(x) + C_{10} J_{3/2}(x) \quad (50)
\]

**Phase IV:** If \( t \) is very large so that
\[
| - m_0^2 t^4 | \gg (20\xi + \alpha) t^2 ;
\]
then equation (37) reduces to
t. T''(t) + 4T'(t) - m_0^2 t. T(t) = 0 \quad (51)

Now putting T(t) = t^{3/2} y(x), x = im_0 t, equation (51) gives Bessel equation of the following form

\[ 4 x^2 y''(x) + 4 x y'(x) - \left[ (1 + 2n)^2 - 4x^2 \right] y(x) = 0 \quad (52) \]

with \( n = 1 \).

The general solution is similar to Phase I and is

\[ y = C_{11} J_{3/2}(x) + C_{12} J_{3/2}(x) \quad (53) \]

**SUMMARY**

We have thus solved exactly the Dirac equation in K-S metric. We have also solved exactly the massless scalar equation in the background of an inflationary Kantowski-Sachs cosmology for \( \theta \). The exact solution to the temporal equation for the massive case for \( 0 \leq t \leq \infty \) has remained, however, a separate problem to be dealt with in another paper. Here we have solved the equation piece-wise for four temporal phases. The solution thus derived piece-wise are all physically acceptable different forms of Bessel solutions. This shows that the complete solution will be composed of all these Bessel solutions appropriately pieced together.

**ACKNOWLEDGEMENTS**

The authors are grateful to Prof. K. D. Krori for encouragement in presenting this investigation and are indebted to him for bearing all the pains to guide us to reach this level of confidence.
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This is an investigation of the problem of quantum fluctuations near space-time singularity in Chodos metric. It is seen that quantum fluctuations do not vanish near space-time singularity. So we have come to the conclusion that such singularity in Chodos metric is unlikely to occur in nature.

PACS numbers: 0420D, 0460.

Key Words: Quantum fluctuation, space-time singularity

INTRODUCTION

Space-time singularity is an inevitable phenomenon in classic theory of gravitation (General Theory of Relativity) in both cosmology and astrophysics (black holes). Although a correct quantum theory of gravity is not yet available, some techniques have been applied to the study of quantum effects near the space-time singularities. One of the methods is path-integral technique used by Narlikar et al. and another is the operator formalism used by Joshi et al. and later by Krori et al. The purpose of this paper is to examine the quantum fluctuations near the space-time singularity represented by Chodos Metric using operator formalism. Alan Chodos et al. had investigated and obtained a simple solution to the vacuum field equation of general relativity in $4+1$ space-time dimensions which would lead to cosmology having $3+1$ observable dimensions at present epoch in which Einstein-Maxwell equations are obeyed.

We also do not have correct singularity theorem till date despite engagement of several workers in this field. The fact was realized by several scientists in this field in the past. Several singularity theorems were put forward by several authors to study the same and put a plausible explanation and circumstances leading to such phenomenon. The earliest such theorem was by Penrose in 1965 and the most popular was by Hawking
and Penrose in 1970. In fact, the definition of the singularity is undergoing modifications till today and most acceptable explanation is yet to emerge.

CALCULATIONS

With the basic idea of operator formalism, we proceed to quantize the conformal degree of freedom for the metric tensor, which leaves the fundamental causal structure of the underlying space-time invariant. We define the conformal fluctuations of the space-time

\[ g^{ik} \rightarrow \Omega^2 g^{ik} \equiv (1 + \phi) g^{ik} \]  

(1)

where \( \Omega \) is a conformal function of time with the property that \( 0 < \Omega < \infty \). We also define a quantum operator \( J \) and a quantum uncertainty \( \chi \), as follows

\[ J = \phi^2 \quad \text{with} \quad \chi = \langle J \rangle \]  

(2)

Now, if we take that the time-evolutions of \( \chi \) is governed by the Hamiltonian \( H \) of the system then using Heisenberg representation and equation we arrive at

\[ \frac{i}{\hbar} \frac{d\langle J \rangle}{dt} = \langle [J, H] \rangle + i \frac{\partial \langle J \rangle}{\partial t} \]  

(3)

throughout the calculations, we choose \( \hbar = c = G = 1 \)

From classical point of view the state \( \phi = 0 \) occurs with certainty and satisfies Einstein's equations. But from the non-classical point of view, the state with \( \phi \neq 0 \) can occur with a finite probability and can be non-singular.
The quantum mechanical state of the object under consideration is represented by a wave function $\psi = (\psi, t)$. There will always be a quantum mechanical spread around the classical state $\phi = 0$ and one investigates the time-evolution of this spread as classical singularity is approached, assuming that average of the quantum ensemble i.e., $<\phi> = 0$, represents the classical state.

To examine the quantum fluctuations near the space–time singularity represented by Chodos Metric, we take the following form:

$$ds^2 = -dt^2 + \frac{t}{t_0} dx_1^2 + \frac{t}{t_0} dx_2^2 + \frac{t}{t} dx_3^2 + \frac{t}{t_0} dx_5^2$$  \hspace{1cm} (4)

Now using exterior calculus we find the Ricci scalar from

$$R_{\mu\nu} - (\frac{1}{2}) g_{\mu\nu} R = 8\pi T_{\mu\nu}$$  \hspace{1cm} (5)

where $R_{\mu\nu}, T_{\mu\nu}, R$ are the Ricci tensor, energy-momentum tensor and the Ricci scalar respectively.

as

$$R = 0$$  \hspace{1cm} (6)

which means that the Chodos metric represents empty exterior space-time. Now under conformal fluctuations, the Ricci scalar, $R$, transforms as

$$R \rightarrow R^* = R (1 + \phi)^{-2} + 6 (1 + \phi)^{-3} \Box \phi$$  \hspace{1cm} (7)

where $\Box \phi = g^{\alpha\beta} \partial_\alpha \phi \partial_\beta$.

The classical Hilbert action, therefore, transforms to

$$S_H = \frac{1}{16\pi} \int_V R^* \sqrt{-g^*} . d^4x$$

$$= \frac{1}{16\pi} \int_V [(1 + \phi)^2 R - 6 \phi \phi^1] \sqrt{-g} . d^4x$$  \hspace{1cm} (8)
where $\phi$, denotes the gradient $\frac{\partial \phi}{\partial x^i}$, $x^i$ are the four co-ordinates. $V$ is the space-time four volume over which the action is defined and the indices are raised and lowered by the classic metric "g". On rigorous evaluation of the equation (8) it will be seen that there arises a second order derivative in $\phi$ due to term $\Box \phi$ in equation (7). Here we have to use Green's theorem to get equation (8) from the second order derivative of equation (7) together with a surface term defined over $dv$. This surface term is again removed by cancellation with a similar term which comes from the conformal transform of the surface term introduced by Gibbons and Hawking$^{10}$. This process effectively removes the second derivatives of the metric tensor from the Hilbert's action.

Now putting the value of Ricci's scalar from equation (6) and $\sqrt{-g}$ from the Chodos metric, we get

$$S = V \int \phi^*^2 \cdot t \cdot dt$$

(9)

where dot denotes differentiation with respect to $t$ and $V$ is the co-ordinate volume of the region under consideration.

Now we write the action $S_\phi = \int L \ dt$

(10)

where $L$ is the Lagrangian of the system given by

$$L = V t \phi^*^2$$

(11)

according to equation (9)

If $P$ is the conjugate momentum associated with $\phi$, then from (11) we have

$$P = \frac{\delta L}{\delta \phi} = 2V t \phi^*^2$$

(12)

The Hamiltonian of the system is written as

$$H = P \phi^* - L = V t \phi^*^2 = (\frac{1}{2}) P \phi^*$$

(13)
The commutation of the operators $\phi$ and $P$ with $H$ are given by,

\[
[\phi, H] = i \frac{\delta H}{\delta \phi} = \frac{iP}{2} \tag{14}
\]

\[
[P, H] = -i \frac{\delta H}{\delta \phi} = 0 \tag{15}
\]

In order to find the time evolution of $\chi$ we make use of the equation (3) with $< \delta J > = 0$, then using equation (14) in equation (3), we obtain

\[
\frac{\delta \chi}{\delta t} = \frac{1}{2V} < \phi P + P\phi > \tag{16}
\]

Now, the time evolution of quantum spread, $\chi$, will be incomplete unless we get a second order differential equation in $\chi$. We need to find out the value of $d < \phi P + P\phi > d t$ by using equation (3) again. We thus have

\[
\frac{d}{dt} < \phi P + P\phi > = < P^2 > \tag{17}
\]

Equation (16) now gives,

\[
\chi'' = < P^2 > \tag{18}
\]

Here we make use of the definition

$\chi = < \phi^2 >$ and $\omega = < P^2 >$

For a general wave function of the universe representing the system as

$\psi = \psi(\phi, t)$, the quantum uncertainties have to satisfy the inequality

$\chi \omega \geq \frac{1}{4}$.
Considering the Gaussian wave-packet form i.e., \( \chi = 1/4 \), equation (18) reduces to
\[
\chi \dot{\chi} = \frac{1}{4} \quad (19)
\]
where the dot denotes differentiation with respect to \( t \). The first integral of equation (19) is
\[
\chi^2 = (\frac{1}{4}) \log \chi + C \quad (20)
\]
While attempting a solution, Equation (20) will result in another equation, viz.,
\[
\left[ d \chi / \sqrt{\log \chi} \right] = t = \text{a finite quantity} \quad (21)
\]
On mathematical evaluation of the left-hand side of eq. (21), it would reveal that the value of it would tend to infinity as \( \chi \to 0 \). Since right-hand side of the above equation is finite, the value of \( \chi \) must have to be finite (may be of very small value) to make equation (21) true. This means that there will always be non-zero quantum fluctuations near the space-time singularity in Chodos metric.

Thus we may conclude that such space-time singularity in Chodos metric is unlikely to occur in nature.

ACKNOWLEDGEMENTS

The authors express their gratitude to Prof. K. D. Kröri for all the encouragements to proceed for the investigations presented in this paper. One of the authors (D. Das Kar) expresses her deep appreciation for the award of grants for Minor Research Project by University Grants Commission, Guwahati Chapter to carry out the investigations presented here.
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Quantum Effects in Robertson-Walker and Bianchi Type III Cosmologies

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Received in January, 2008 Revised and accepted in May, 2008

1. INTRODUCTION

The classical singularity of space-time which inevitably results from Einstein's General Theory of Relativity when applied to cosmology or to a gravitationally collapsing object has become a subject of intense theoretical interest for about three decades. It is believed that during the final stages of a collapsing body or at the time of creation of universe when matter density and space-time curvature were supposed to be extremely large, quantum effects come into operation and play a very crucial role in the process of evolution. These effects may give rise to large quantum fluctuations and may counteract the forces that lead to a classical singularity. Although we do not have any established theory of quantum gravity at the moment to study such situations, attempts are made by different methods to study conditions near space-time singularities. One method of study is the path integral technique developed by Narlikar et.al and another is the operator formalism used by Joshi et. al and later by Krori et. al.

In both these methods, quantum effects near the space-time singularities have been studied in terms of conformal fluctuations ($\phi$) considered as a function of time only for the sake of simplicity. In this paper, however, we study quantum effects near cosmological singularities in Robertson-Walker and Bianchi type III space-times in
terms of conformal fluctuations considered as a function of both space and time. Further, although the Robertson-Walker space-time has been a subject of quantum-theoretical investigations by several authors the Bianchi type III space-time is subjected to such investigations for the first time in this paper. We apply the operator formalism to the problems considered here.

In the operator formalism, one quantizes the conformal degree of freedom, \( \Omega \), for the metric tensor, \( g^{ik} \), and the fundamental causal structure of the relevant space-time is left invariant. The conformal fluctuation of the space-time is given by

\[ g^{ik} \rightarrow \Omega^2 g^{ik} \equiv (1 + \phi) g^{ik} \]

where \( \Omega \) is a conformal function of time with the property that \( 0 < \Omega < \infty \). Here we define a quantum operator \( J \) as

\[ J = \phi^2 \text{ with } \chi = \langle J \rangle \]

and consider that the time-evolutions of \( \chi \) is governed by the Hamiltonian \( H \) of the system. We also take \( h = c = G = 1 \) here.

Classically, the state \( \phi = 0 \) can occur with certainty and satisfies Einstein's equations. But quantum-mechanically, the states with \( \phi \neq 0 \) can occur with a finite probability and can be non-singular as well. The quantum mechanical state of an object under consideration is represented by a wave function \( \psi = \psi(\phi, t) \). There will always be a quantum mechanical spread around the classical state \( \phi = 0 \) and we may investigate the time-evolution of this spread as the classical singularity is approached, assuming that average of the quantum ensemble i.e., \( \langle \phi \rangle = 0 \), represents the classical state.

In the operator formalism we consider \( \phi \) to be function of both space \( (x) \) and time \( (t) \) since it is more appropriate to assume that \( \phi \) depends on \( x \) also. Under the conformal fluctuation the classical action of the gravitational field transforms to
where  is the Ricci Scalar,  is the space-time volume for integration and . is here a function of  and  For our calculation, we shall make use of a Taylor expansion of  in the neighborhood of a value  and restrict ourselves to the first order approximation

Thus

and

where a dash and dot indicate differentiation with respect to  and  respectively.

2. ROBERTSON-WALKER SPACE-TIME

The Robertson-Walker line element is

The parameter  takes values 0, 1 or -1 while  is an arbitrary real function of . The Ricci Scalar for Robertson-Walker space-time works out to

Therefore, equation (4) gives with (5) - (8) [ with  replaced by ]

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We now write the classical action in the form

$$S_1 = \int L \, dt$$

where $L$ is the Lagrangian of the system. Comparing (10) and (11), we obtain

$$L = \frac{\sqrt{V}}{16 \pi} \frac{Q^3}{\sqrt{(1 - k)}} \left[ R_1 (\phi^2 + \phi'^2 + 2 \phi \phi' - 1) + \frac{6(1 - k) \phi'^2}{Q^2} - 6 (\dot{\phi}^2 + 2 \phi \phi'') \right] \, dt$$

(12)

Denoting conjugate momenta associated with $\phi$ and $\phi'$ as $p_1$ and $p_2$ respectively, we have from (12)

$$p_1 = \frac{\delta L}{\delta \phi} = \alpha Q^3 [\phi + \phi''']$$

(13)

and

$$p_2 = \frac{\delta L}{\delta \phi''} = \alpha Q^3 \phi'$$

(14)

where

$$\alpha = -\frac{3\sqrt{V}}{4\pi \sqrt{(1 - k)}}$$

The Hamiltonian of the system is written as

$$H = \phi \cdot p_1 + \phi' \cdot p_2 - L$$

$$H = \frac{1}{2 \alpha Q^3} \left[ 2 p_1 p_2 - p_2^2 \right] + \frac{\alpha Q^3}{12} \left[ R_1 (\phi^2 + \phi'^2 + 2 \phi \phi' - 1) + \frac{6(1 - k) \phi'^2}{Q^2} \right]$$

(15)
The commutation relations for $\phi$, $\phi'$, $p_1$ and $p_2$ with $H$ are given by

\[
[\phi, H] = i \frac{\delta H}{\delta p_1} = -\frac{i p_2}{\alpha Q^3}
\]

\[
[p_1, H] = -i \frac{\delta H}{\delta \phi} = -\frac{i \alpha Q^3}{6} R_1 (\phi + \phi')
\]

\[
[\phi', H] = i \frac{\delta H}{\delta p_2} = -\frac{i}{\alpha Q^3} [p_1 - p_2]
\]  \hspace{1cm} (16)

\[
[p_2, H] = -i \frac{\delta H}{\delta \phi'} = -\frac{i \alpha Q^3}{6} [R_1 (\phi + \phi') + \frac{6(1-k)\phi'}{Q^2}]
\]

Now substituting (16) in (3) and considering $\frac{\delta J}{\delta t} = 0$, we derive two equations for Robertson-Walker metric -- one corresponding to $\phi$ and the other corresponding to $\phi'$. We shall use following set of definitions in our calculation.

\[
<\phi^2> = \xi_1
\]

\[
<\phi'^2> = \xi_2
\]

\[
<\phi \phi'> = \frac{1}{2} \left[ \xi - \xi_1 - \xi_2 \right]
\]  \hspace{1cm} (17)

\[
<p_1^2> = \omega_1
\]

\[
<p_2^2> = \omega_2
\]

\[
<(\phi + \phi')^2> = \xi
\]

\[
<(p_1 + p_2)^2> = \omega
\]

\[
<p_1 p_2> = \frac{1}{2} \left[ \omega - \omega_1 - \omega_2 \right]
\]
(i) Equation for $\phi$

Substituting (16) in (3) for $\phi$, we obtain (considering $\langle \delta \psi \rangle = 0$)

$$i \frac{d <\phi^2>}{dt} = <[\phi^2, H]>$$

In terms of $\xi_1$, this equation is

$$\frac{d \xi_1}{dt} = -\frac{1}{\alpha Q^3} <\phi p_2 + p_2 \phi>$$

Differentiating (19) with respect to $t$ and simplifying (using (17))

$$\xi_1^{**} + \frac{3Q^2}{Q} \xi_1^* + \frac{[R_1 - (1/k)]}{6} \xi_1 + \frac{[(1/k) + R_1]}{Q^2} (\xi_1 - \xi_2)$$

$$= \frac{2\omega_2}{\alpha^2 Q^6}$$

Using $<\phi> = 0$ and the relation $\psi$ as the wave function of the system

$$0 = \frac{d <\phi>}{dt} = i \int \psi^* [H \phi - \phi H] \psi \, d\sigma$$

It is not difficult to see that $<p_1> = 0$. Then, the quantum uncertainties involving $\phi$ and $p_1$ turn out to be

$$<\phi^2> - <\phi>^2 = <\phi^2> = \xi_1$$

$$<p_1^2> - <p_1>^2 = <p_1^2> = \omega_1$$
Hence uncertainty principle in quantum mechanics gives

\[ \xi_1, \omega_1 \geq \frac{\gamma}{4} . \]  

(24)

At the first instance, we shall proceed with our calculation with

\[ \xi_1, \omega_1 = \frac{\gamma}{4} . \]  

(25)

Proceeding in the same manner, in the case of \( \phi' \) (assuming \( <\phi'> = 0 \)), we obtain the corresponding uncertainty relation

\[ \xi_2, \omega_2 \geq \frac{\gamma}{4} \]  

(26)

and for this also, we shall first work with

\[ \xi_2, \omega_2 = \frac{\gamma}{4} . \]  

(27)

Using (9) and (27) in (20), we obtain the equation

\[ \xi_2 \left[ Q^6 \xi_1^{**} + 3 Q^5 Q^* \xi_1^* + Q^4 (QQ^{**} + Q^{*2} + 2k - 1) (\xi_1 - \xi_2) - Q^4 (QQ^{**} + Q^{*2} + 1) \right] = \frac{1}{2} \frac{1}{\alpha^2} \]  

(28)

(ii) Equation for \( \phi' \)

Here, (3) gives

\[ i \frac{d<\phi'^2>}{dt} = <\left[ \phi'^2, H \right]> \]  

(29)

Then proceeding as above and using (25) we obtain the equation

\[ \xi_1 \left[ Q^6 \xi_2^{**} + 3 Q^5 Q^* \xi_2^* - 2 (1 - k) Q^4 \xi_2 + \frac{2}{\alpha} (\omega - 2 \omega_2) \right] = \frac{1}{\alpha^2} \]  

(30)
From the above equation it may be seen that the quantum fluctuations represented by $\xi_1$ do not vanish near singularity. To study this phenomenon completely near singularity, let us consider

$$\phi \sim C Q^\varepsilon f(1 - k r^2) \tag{31}$$

where $C$ is any arbitrary constant and $\varepsilon$ is positive, so that

$$\xi_1 = \langle \phi^2 \rangle \sim C^2 Q^{2\varepsilon}$$

$$\xi_2 = \langle \phi'^2 \rangle \sim C^2 Q^{2\varepsilon}$$

$$\xi = \langle (\phi + \phi')^2 \rangle \sim C^2 Q^{2\varepsilon} \tag{32}$$

$$\omega_2 = \langle p_2^2 \rangle \sim C^2 Q^2 \varepsilon^2 Q^{2\varepsilon - 2}$$

$$\omega = \langle (p_1 + p_2)^2 \rangle \sim C^2 Q^2 \varepsilon^2 Q^{2\varepsilon - 2}$$

The study of the equations (28), (30), (31), (32) reveals that for acceptable solutions of equations (28) and (30) we must have

$$\varepsilon \geq \frac{1}{2} \tag{33}$$

Our purpose will be served if we take only equation (30). At singularity, as $t \to 0$, for $\varepsilon \geq \frac{1}{2}$ and hence $Q(t) \to 0$; $\xi_1 \to 0$ and as such left hand-side of (30) tends to zero. But it is incompatible with the finite non-zero right-hand side of the equation. On the other hand for $\varepsilon < 0$ as $t \to 0$ and hence $Q(t) \to 0$; $\xi_1 \to$ infinity and the left-hand side of equation (30) also tends to infinity. This is again incompatible with the finite right-hand side of this equation.
For values $0 < s < \frac{1}{2}$, the equation is still incompatible with the finite right-hand side since $\xi_1 \to 0$. We may conclude that there will be non-zero quantum fluctuations near a space-time singularity in the Robertson-Walker metric.

We shall arrive at the same result if, instead of taking $\chi_1 \omega_1$ (or $\chi_2 \omega_2$) = $\frac{1}{4}$, we take $\chi_1 \omega_1$ (or $\chi_2 \omega_2$) equal to some value $\delta > \frac{1}{4}$.

3. BIANCHI TYPE III SPACE-TIME

The Bianchi type III metric is

$$ds^2 = dt^2 - A^2 dx^2 - B^2 e^{2x} dy^2 - C^2 dz^2$$

(34)

where $A(t)$, $B(t)$, $C(t)$ are the cosmic scale functions. The Ricci Scalar for Bianchi III space-time works out to

$$R = -2 \left[ \frac{A'' + B'' + C'' + A'B' + B'C' + C'A'}{A B C AB BC CA} - \frac{1}{A^2} \right]$$

(35)

Therefore, equation (4) gives with (5) - (7) and (34)

$$S_2 = \frac{\nu}{16\pi} \int \left[ R_2 \phi^2 + (R_2 + \frac{6}{A^2}) \phi'^2 + 2 R_2 \phi \phi' - 6\phi^2 ight.$$

$$\left. - 12 \phi \phi'' - R_2 \right] ABC \cdot dt$$

(36)

Comparing (11) and (36), we obtain

$$L = \frac{\nu ABC}{16\pi} \left[ R_2 \phi^2 + (R_2 + \frac{6}{A^2}) \phi'^2 + 2 R_2 \phi \phi' - 6\phi^2 ight.$$

$$\left. - 12 \phi \phi'' - R_2 \right]$$

(37)
Again denoting conjugate momenta associated with $\phi$ and $\phi'$ as $p_1$ and $p_2$ respectively, we have from (37):

\[ p_1 = \frac{\delta L}{\delta \phi} = -\frac{3\nu'ABC}{4\pi} (\phi^* + \phi') \] (38)

and

\[ p_2 = \frac{\delta L}{\delta \phi'} = -\frac{3\nu'ABC}{4\pi} \phi^* \] (39)

The Hamiltonian of the system is then

\[
H = \left[ \phi^* p_1 + \phi' p_2 + L \right]
= \frac{2\pi}{3\nu'ABC} \left[ p_2^2 - 2p_1 p_2 \right]
- \frac{\nu'ABC}{16\pi} \left[ R_2 \phi^2 + (R_2 + 6\pi) \phi'^2 + 2R_2 \phi \phi' - R_2 \right] \] (40)

Again, the commutation relations for $\phi$, $\phi'$, $p_1$ and $p_2$ with $H$ are given by

\[
[\phi, H] = i \frac{\delta H}{\delta p_1} = -i \frac{4\pi p_2}{3\nu'ABC}
[\phi', H] = i \frac{\delta H}{\delta p_2} = i \frac{\nu'ABC}{8\pi} R_2 [\phi + \phi']
[\phi', H] = i \frac{\delta H}{\delta \phi} = i \frac{\nu'ABC}{3\nu'ABC} [p_2 - p_1]
[\phi, H] = -i \frac{\delta H}{\delta \phi'} = i \frac{\nu'ABC}{8\pi} \left[ R_2 \phi + (R_2 + 6\pi) \phi' \right] \] (41)
As in earlier case, we derive two equations for Bianchi type III metric — one corresponding to $\phi$ and the other corresponding to $\phi'$. We shall use following definitions in our calculation in this section -

\[
\begin{align*}
<\phi^2> &= \chi_1 \\
<\phi'^2> &= \chi_2 \\
<\phi\phi'> &= (\frac{1}{2}). [\chi - \chi_1 - \chi_2] \\
<p_1^2> &= \omega_1 \\
<p_2^2> &= \omega_2 \\
<(\phi+\phi')^2> &= \chi \\
<(p_1+p_2)^2> &= \omega \\
<p_1p_2> &= (\frac{1}{2}). [\omega - \omega_1 - \omega_2]
\end{align*}
\] (42)

(i) Equation for $\phi$

With the same consideration as in the previous case, we obtain

\[
\begin{align*}
\frac{d}{dt} <\phi^2> &= <\phi^2, H> \\
\frac{d}{dt} \chi_1 &= -\frac{4\pi}{3\sqrt{\text{ABC}}} <\phi\cdot p_2 + p_2\phi> 
\end{align*}
\] (43)
Differentiating (43) with respect to $t$ and simplifying by using (21) – (23), (35) and (42)

$$\chi_{1}^{\ddagger}\!+\! \left[ \frac{A^{\ast}}{A} + \frac{B^{\ast}}{B} + \frac{C^{\ast}}{C} \right] \chi_{1}^{\ddagger} + (\frac{1}{6}) \left( \frac{R_{2} - \frac{6}{A^{2}}}{}\right) \chi_{1} + (\frac{1}{6}) \left( \frac{R_{2} + \frac{6}{A^{2}}}{}\right) (\chi^{\prime} \chi_{2})$$

$$= \frac{32 \pi^{2} \omega}{9} A^{2} B^{2} C^{2}$$

(44)

Now using (27) in (44), we obtain the equation

$$\chi_{2} \left[ \frac{A^{2} B^{2} C^{2}}{\chi_{1}^{\ddagger}} + A^{2} B^{2} C^{2} \left\{ \frac{A^{\ast}}{A} + \frac{B^{\ast}}{B} + \frac{C^{\ast}}{C} \right\} \chi_{1}^{\ddagger} + A^{2} B^{2} C^{2} \left( \frac{R_{1} - 1}{6} \right) \chi_{1} \right.$$

$$+ A^{2} B^{2} C^{2} \left( \frac{R_{2} + 1}{6} \right) (\chi^{\prime} \chi_{2}) \left. \right] = \frac{8 \pi^{2}}{9}$$

(45)

(ii) Equation for $\phi^{l}$

In this case, (3) gives

$$i \frac{d \langle \phi^{l2} \rangle}{dt} = \langle [\phi^{l2}, H] \rangle$$

(46)

Then, proceeding as above and using (25) we obtain the equation

$$\chi_{1} \left[ \frac{A^{2} B^{2} C^{2}}{\chi_{2}^{\ddagger}} + A^{2} B^{2} C^{2} \left\{ \frac{A^{\ast}}{A} + \frac{B^{\ast}}{B} + \frac{C^{\ast}}{C} \right\} \chi_{2}^{\ddagger} - 2 B^{2} C^{2} \chi_{2} \right.$$

$$+ \frac{32 \pi^{2} \omega}{9} - \frac{64 \pi^{2} \omega_{2}}{9} \left. \right] = \frac{16 \pi^{2}}{9}$$

(47)

To study the quantum fluctuations near the space-time singularity completely, at this stage we consider a solution to the Bianchi type III metric as given by Ram
A (at + b)^\gamma = B \quad (48)

and

C = c_1 (at + b)^\gamma + c_2 (at + b)^{-\gamma} \quad (49)

where \gamma = (a^2 - 1)^{1/2} / a, a, b are arbitrary constants with |a| > 1.

Now putting values of A, B, C from (48) and (49) in (45) and (47), we may easily verify that the coefficients of \chi_1^{**}, \chi_2^{**}, \chi_1^{\prime}, \chi_2^{\prime}, \chi, \omega, \omega_1 are all finite provided \gamma \geq 1.

We further observe that from (48) and (49) and the form of Kretchmann scalar that the singularity occurs at \( t \to t_0 = -\frac{b}{a} \).

We now show that the quantum fluctuations represented by \chi_1 do not vanish as the singularity is approached (as \( t \to t_0 = -\frac{b}{a} \)).

For this purpose, let us suppose that near singularity

\[ \phi(x, t) \sim (t - t_0)^n \cdot f(e^x) \quad (50) \]

where n is positive. The factor f(e^x) has been introduced to take into account a function of x in terms of e^x occurring in a metric coefficient. Thus we have

\[ \chi_1 = \langle \phi^2 \rangle \sim (t - t_0)^{2n} \]

\[ \chi_2 = \langle \phi'^2 \rangle \sim (t - t_0)^{2n} \]

\[ \chi = \langle (\phi + \phi')^2 \rangle \sim (t - t_0)^{2n} \quad (51) \]

\[ \omega_2 = \langle p_2^2 \rangle \sim (t - t_0)^{2n-2} \]

\[ \omega = \langle (p_1 + p_2) \rangle \sim (t - t_0)^{2n-2} \]
If we study equation (47) together with (48) and (49), it may be readily seen that for \( n \geq 1 \), as \( t \to t_0 = -\left(\frac{b}{a}\right), \chi_1 \to 0 \) and hence the left-hand side of (47) tends to zero. This is, however, incompatible with the non-zero right-hand side of the equation.

Again, for \( 0 < n < 1 \), as \( t \to t_0 = -\left(\frac{b}{a}\right), \chi_1 \to 0 \) and the left-hand side of (47) tends to infinity. It is again incompatible with the finite right-hand side. For \( 0 > n \), as \( t \to t_0 = -\left(\frac{b}{a}\right), \chi_1 \to \infty \) and also the left-hand side of (47) tends to \( \infty \). This is also incompatible with the finite right-hand side. We, therefore, conclude that there will be a finite quantum fluctuation as the singularity is approached.

A similar result will be obtained, if instead of \( \chi_1 \omega_1 \) (or \( \chi_2 \omega_2 \)) we take \( \chi_1 \omega_1 \) (or \( \chi_2 \omega_2 \)) equal to some value \( \delta > \frac{1}{4} \).

ACKNOWLEDGEMENTS

The authors express their gratitude to Prof. K. D. Krori for the valuable suggestions and encouragement to proceed for the investigations presented in this paper. One of the authors (D. Das Kar) expresses her deep appreciation for the award of grants for Minor Research Project by University Grants Commission, Guwahati Chapter to carry out the investigations presented here.
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QUANTUM FLUCTUATIONS NEAR BLACK-HOLE SINGULARITY IN KERR METRIC

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Sent for publication on January 4, 2011

Abstract: We present in this paper an investigation of the problem of quantum fluctuations near black-hole singularity in Kerr metric. We have seen through mathematical calculations that quantum fluctuations do not vanish near black-hole singularity and thus have come to a conclusion that such singularity in Kerr metric may not occur in nature.

PACS numbers: 0420D, 0460.

Key Words: Quantum fluctuation, black-hole singularity,

INTRODUCTION

Presently a popular area of research in general relativity and relativistic astrophysics includes the study of black-holes and their possible astrophysical applications. In General Theory of Relativity, the Schwarzschild radius derived from it describes the horizon radius of a non-rotating black-hole, whereas Kerr metric defines the horizon for rotating black-hole. The limit for the Kerr metric’s horizon for non-rotating black-hole, however, falls back to Schwarzschild radius. In fact, Kerr metric was derived from the Schwarzschild metric by a special type of complex coordinate transformation in which radial and time coordinates were allowed to take complex values. In a paper Chandrasekhar also noted that 'there is no extent derivation of Kerr's solution that is direct and simple'. It is widely believed that the only stationary black-hole solutions in general relativity are those discovered by Roy P. Kerr(1963) and the black-holes, once formed consequentially to that, would settle down asymptotically to a Kerr black-hole in course of time, through the operation of Penrose process.

The Kerr Metric for a stationary, axis-symmetric gravitational field, also known as Kerr vacuum formulated by Kerr, describes the geometry of space-time around uncharged rotating massive body. The relativistic exterior solution of a massive spinning
body was derived by Kerr which is known as Kerr solution. Kerr metric is often used to describe rotating black-holes, which exhibit even more exotic phenomenon. Such black-holes have different surfaces where the metric appears to have a singularity; the size and shape of these surfaces depend on the mass and angular momentum of the black-hole. The outer surface encloses the ergosphere and inner surface marks the event horizon.

While proceeding to study a singularity problem, one may observe that space-time singularity is an inevitable phenomenon in classic theory of gravitation (General Theory of Relativity) in both cosmology and astrophysics (black-holes). But a correct quantum theory of gravity is not yet available. Some techniques have, however, been applied to the study of quantum effects near the space-time singularities. One of the methods is path-integral technique used by Narlikar et al \textsuperscript{3,4} and another is the operator formalism used by Joshi et al \textsuperscript{5-7} and later by Krori et al \textsuperscript{8-10}.

We also do not have correct singularity theorem till date despite engagement of several workers in this field. The fact was realized by several scientists in this field in the past. Several singularity theorems were put forward by several authors to study the same and put a plausible explanation and circumstances leading to such phenomenon. The earliest such theorem was by Penrose \textsuperscript{11} in 1965 and the most popular was by Hawking and Penrose \textsuperscript{12} in 1970. The definition of the singularity is undergoing evolutions till today and most acceptable explanation is yet to emerge.

Nevertheless, Joshi and Joshi \textsuperscript{13} had studied the quantum effects near the black-hole singularity in Kruskal space-time using operator technique. The quantum effects near the black-hole singularity in Dirac metric using operator technique was also studied by Krori et al \textsuperscript{8}. The basic purpose of this paper is to examine the quantum fluctuations near the black-hole singularity represented by Kerr metric using operator formalism.

We proceed in the following way.
CALCULATIONS:

We begin with the Kerr Metric, the transformed form of which is

\[ ds^2 = -dT^2 + \frac{2mr}{\rho^2} dR^2 + \rho^2 d\theta^2 + \left(\frac{r^2}{\rho^2} + a^2 \cos^2 \theta\right) d\phi^2 - Q d\phi dT \]  \hspace{1cm} (1)

where

\[ \rho^2 = r^2 + a^2 \cos^2 \theta \]

\[ P = \left(\frac{r^2 + a^2}{\rho^2}\right) \sin^2 \theta + \frac{2mr \cdot a}{\rho^2} \cdot \sin^4 \theta \]  \hspace{1cm} (2)

\[ Q = \frac{4mr \cdot a \sin^2 \theta}{\rho^2} \]

To arrive at the metric above we have used the following transformations:

\[ dT = dt + A(r, \theta) \, dr + B(r, \theta) \, d\theta \]

\[ dR = dt + C(r, \theta) \, dr + D(r, \theta) \, d\theta \]

\[ d\theta = E(r, \theta) \, dr + Q(r, \theta) \, d\theta \]  \hspace{1cm} (3)

\[ d\phi = d\phi + F(r, \theta) \, dr + G(r, \theta) \, d\theta \]

Now using (1), (2) and (3) and opting \( \theta = \pi/2 \), we get

\[ R - T = \frac{Q}{(2m)^{3/2} \cdot a} \int \left[ r^{5/2} \left\{ 1 + \frac{r}{2m} + \frac{r^2}{4m^2} + \frac{r^4}{4m^2a^2} \right\} \right] dr \]

\[ = \frac{Q}{(2m)^{3/2} \cdot a} \cdot (2/7) \cdot r^{7/2} \text{ for } (r/m) \ll 1, (r^2/a^2) \ll 1. \]

\[ = (\mu r)^{7/2} \text{ where } \mu = \left[ \frac{(2/7)Q}{(2m)^{3/2} \cdot a} \right]^{2/7}. \]  \hspace{1cm} (4)
It is now obvious from equation (4) that at ring singularity i.e., \( r = 0, \theta = \pi/2 \), we shall get \( R = T \). We shall now proceed to study the quantum fluctuation at this ring singularity of Kerr Metric.

With the basic idea of operator formalism \(^6\), we proceed to quantize the conformal degree of freedom for the metric tensor, which leaves the fundamental causal structure of the underlying space-time invariant. We define the conformal fluctuations of the space-time

\[
g^{ik} \to \Omega^2 g^{ik} \equiv (1 + \phi)^2 g^{ik} \tag{5}
\]

where \( \Omega \) is a conformal function of time with the property that \( 0 < \Omega < \infty \).

The quantum operators \( J, \chi_1 \), and \( \chi_2 \) are defined as follows

\[
J = \phi^2, \quad \chi_1 = < \phi^2 >, \quad \chi_2 = < \phi^2 > \tag{6}
\]

Now, if we take that the time-evolutions of \( \chi_1 \) is governed by the Hamiltonian \( H \) of the system \(^{14}\), then using Heisenberg representation and equation we arrive at

\[
\frac{i d <J>}{d T} = < [J, H] > + i < \frac{\partial J}{\partial T} > \tag{7}
\]

Under conformal fluctuations the Ricci Scalar, \( R_c \), transforms as

\[
R_c \to R_c^* = R_c (1 + \phi)^2 + 6 \phi \phi \tag{8}
\]

and the classical action \( S \) for gravitational field transforms as

\[
S \to S = \frac{1}{16\pi} \int_v \left[ (1 + \phi)^2 R_c - 6 \phi \phi, \phi \right] \sqrt{-g} \cdot d^4 x \tag{9}
\]

where \( \phi_i \) denotes the gradient \( \frac{\delta}{\delta x^i} \), \( x^i \) are the four co-ordinates. \( v \) is the space-time four volume over which the action is defined and the indices are raised and lowered by the classic metric " \( g \) ''. Further, \( \phi \) is considered here as function of \( R \).
and T. For our calculations, we shall make use of Taylor expansion of $\phi (R,T)$ in the neighborhood of $R = R_0$ and restrict ourselves to the first order approximation in $(R - R_0)$.

$$\phi (R,T) = \phi (R_0,T) + (R - R_0) \phi' (R_0,T) + \ldots$$  \hspace{1cm} (10)

with

$$\phi'' (R,T) = \phi'' (R_0,T) + (R - R_0) \phi''' (R_0,T) + \ldots$$  \hspace{1cm} (11)

and

$$\phi' (R,T) = \phi' (R_0,T) + \ldots$$  \hspace{1cm} (12)

where a dash and a dot denote differentiation with respect to $R$ and $T$ respectively.

Now, since Kerr metric at equation (1) covers the entire region $0 \leq r \leq \infty$, the metric represents an empty exterior space-time and hence Ricci scalar for the metric

$$R_c = 0$$  \hspace{1cm} (13)

With the help of equations (1), (10), (11), (12) and (13), equation (9) gives

$$S = \int \left[ k_2 F_2 \phi^2 + 2 k_2 F_3 \phi' \phi'' - k_1 F_4 \phi^2 \right] dT$$  \hspace{1cm} (14)

where

$$F_1(T) = \int \frac{\rho^2 \sqrt{(4P + Q^2)}}{r} dR$$  \hspace{1cm} (15)

$$F_2(T) = \int \frac{P \sqrt{r}}{\sqrt{(4P + Q^2)}} dR$$  \hspace{1cm} (16)

$$F_3(T) = \int \frac{P(R - R_0) \sqrt{r}}{\sqrt{(4P + Q^2)}} dR$$  \hspace{1cm} (17)

and

$$k_1 = \frac{3\pi}{8\sqrt{2m}}, \quad k_2 = \frac{3\pi \sqrt{m}}{\sqrt{2}}$$

But in classical physics, Action

$$S = \int L \, dT$$  \hspace{1cm} (18)
Comparing (14) and (18), we get the Lagrangian of the system

$$L = k_2 F_2 \phi'^2 + 2 k_2 F_3 \phi' \phi'' - k_1 F_1 \phi'^2$$  \hspace{1cm} (19)

Denoting conjugate momenta $p_1$ and $p_2$ corresponding to $\phi$ and $\phi'$, we get from (14)

$$p_1 = \frac{\partial L}{\partial \dot{\phi}} = 2 k_2 (F_2 \phi' + F_3 \phi'')$$  \hspace{1cm} (20)

$$p_2 = \frac{\partial L}{\partial \dot{\phi}'} = 2 k_3 F_3 \phi'$$  \hspace{1cm} (21)

The Hamiltonian of the system is written as

$$H = \phi' p_1 + \phi'' p_2 - L$$

$$= \frac{p_1 p_2}{2 k_2 F_3} - \frac{p_2^2 F_3^2}{2 k_2 F_3} + k_1 F_1 \phi'^2$$  \hspace{1cm} (22)

The commutation relations for $\phi$ and $\phi'$ are

$$[\phi, H] = i \frac{\partial H}{\partial p_1} = i \frac{p_1}{2 k_2 F_3}$$

$$[\phi', H] = i \frac{\partial H}{\partial p_2} = i \frac{p_2}{2 k_2 F_3} - i \frac{F_3 p_2}{2 k_2 F_3}$$  \hspace{1cm} (23)

$$[p_1, H] = -i \frac{\partial H}{\partial \phi} = 0$$

$$[p_2, H] = -i \frac{\partial H}{\partial \phi'} = -2 i k_1 F_1 \phi'$$

Now substituting (23) in (7) and considering $\langle \frac{\partial J}{\partial T} \rangle = 0$, we obtain

$$\frac{id \langle \phi'^2 \rangle}{dT} = < [\phi'^2, H] >$$  \hspace{1cm} (24)

We shall now make use of the following definitions in our calculations
\[
\left< \phi^2 \right> = \chi_1 \\
\left< p_i^2 \right> = \omega_i \\
\left< \phi'^2 \right> = \chi_2 \\
\left< p_2^2 \right> = \omega_2 \\
\left< (\phi + \phi')^2 \right> = \chi \\
\left< \phi \phi' \right> = (\psi \left< \chi - \chi_1 - \chi_2 \right> \\
\left< (p_i + p_2)^2 \right> = \omega
\]

In terms of \( \chi_2 \), the equation (24) is
\[
\frac{d\chi_2}{dT} = \frac{1}{2k_2F_3} \left< (p_i \phi' + \phi' p_i) \right> - \frac{F_2}{2k_2F_3} \left< (p_2 \phi' + \phi' p_2) \right> \\
\] (25)

Differentiating (26) with respect to \( T \) and simplifying [using (25)] we get
\[
\chi_2^{\ldots} + \frac{F_3^2}{F_3} \chi_2^{\ldots} - \frac{2F_1F_2k_1}{F_3^2k_2} \chi_2 + \frac{F_2}{2F_3^3k_2^2} \omega - \frac{F_2}{2F_3^3k_2^2} \left( 1 + (F_2/F_3) \right) \omega_2 \\
+ \left[ \frac{F_2^2}{2F_3^2k_2} - \frac{F_2F_3^2}{2F_3^3k_2} \right] \left< (p_2 \phi' + \phi' p_2) \right> = \frac{(F_2 + F_3)}{2F_3^3k_2^2} \omega_1 \\
\] (27)

Using \( \left< \phi \right> = 0 \) and the relation [with \( \psi \) as the wave function of the system]
\[
0 = \frac{d}{dT} \left< \phi \right> = i \left[ \psi^* \left[ H \phi - \phi H \right] \psi \right] d\sigma \\
\] (28)
It is not difficult to see that \( \langle p_1 \rangle = 0 \). Then, the quantum uncertainties involving \( \phi \) and \( p_1 \) turn out to be.

\[
\langle \phi^2 \rangle - \langle \phi \rangle^2 = \langle \phi^2 \rangle \equiv \chi_1
\]  

(29)

\[
\langle p_1^2 \rangle - \langle p_1 \rangle^2 = \langle p_1^2 \rangle = \omega_1
\]  

(30)

Hence uncertainty principle in quantum mechanics gives

\[
\chi_1 \cdot \omega_1 \geq \gamma_4 .
\]  

(31)

At the first instance, we shall proceed with our calculation with

\[
\chi_1 \cdot \omega_1 = \gamma_4 .
\]  

(32)

Using (32) in equation (27) we obtain the following equation

\[
\frac{F_3^3}{(F_2 + F_3)} \chi_1 \left[ \chi_2 \ast - \frac{F_3^*}{F_3} \chi_2 \ast - \frac{2F_1 F_2 k_1}{F_3^2 k_2} \chi_2 + \frac{F_2}{2F_3^3 k_2^3} \omega - \frac{F_2}{2F_3^3 k_2^3} [1 + (F_2 / F_3)] \omega_2 
\right.

\[
+ \left[ \frac{F_2^*}{2F_3^2 k_2} - \frac{F_2 F_3^*}{2F_3^3 k_2^2} \right] \langle p_2 \phi' + p_2 \phi' \rangle \rangle = \frac{1}{8k_2^2}
\]  

(33)

With the help of equations (15), (16) and (17), we have since found that all the quantities \( F_1, F_2, F_3, F_2^* , F_3^* \) have finite values as the singularity is approached as \( T \rightarrow R_0 \).
We can now show that the quantum fluctuations represented by $\chi_1$ do not vanish as the singularity is approached (as $T \to R_0$). For the purpose, let us suppose that near the singularity i.e., $R \approx T$,

$$\phi \sim (R - T)^n$$

(34)

where $n$ may be positive or negative quantity, so that

$$\chi_1 \sim (R_0 - T)^{2n}$$

$$\chi_2 \sim (R_0 - T)^{2n-2}$$

$$\omega \sim (R_0 - T)^{2n-2}, (R_0 - T)^{2n-4}$$

$$\omega_2 \sim (R_0 - T)^{2n-2}$$

and

$$<(p_2\phi' + p_1\phi')> \sim (R_0 - T)^{2n-2}$$

Let us now consider equation (33). It may be readily seen therefrom that

(i) For $n \geq 1$, as $T \to R_0$, $\chi_1 \to 0$ and the left-hand side of (33) tends to zero. This is, however, incompatible with the non-zero right-hand side.

(ii) For $0 < n < 1$, as $T \to R_0$, $\chi_1 \to 0$, but now the left-hand side of (33) tends to infinity. It is again incompatible with the finite right-hand side of the said equation.

(iii) For $n < 0$, as $T \to R_0$, both $\chi_1$ and also the left-hand side of (33) tend to infinity. This is again incompatible with the finite right-hand side of the equation.

Thus, we may conclude that there will be non-zero quantum fluctuations near the black-hole singularity in Kerr Metric.
We shall arrive at the same conclusion even if, instead of taking $\alpha = \frac{1}{4}$, we take $\alpha_1 \omega_1$ equal to some value $\delta > \frac{1}{4}$.

CONCLUSION

Starting from Kerr metric, which describes a stationary, axis-symmetric gravitational field around uncharged rotating black-hole, we have seen, through the above mathematical technique/calculations that quantum fluctuations do not vanish near black-hole singularity. Thus we may come to a conclusion that such singularity in Kerr metric would not occur in nature.

ACKNOWLEDGEMENTS

The authors express their gratitude to Prof. K. D. Krori for all the encouragements to proceed for the investigations presented in this paper. The authors sincerely acknowledge the unstinted support extended by Prof. Kalyanee Baruah, Head, Department of Physics, Gauhati to carry out the investigations presented over here.

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