CHAPTER 1

PRELIMINARIES

The principal goal of this chapter is to highlight some fundamental definitions and results which are absolutely necessary to carry out our main studies and investigations. These concepts and results are collected from different literature, monographs and published papers mainly considered for some preliminary definitions, some basic results of dynamical systems.

A. Fundamental Concepts and results:

1.1 $C^r$ Homeomorphism

$f: I \rightarrow J$ is said to be $C^r$ homeomorphism, if $f$ is one-one, onto, $r^{th}$ derivative $f^r$ exists and continuous, $f^{-1}$ is continuous. Further if both $f$ and $f^{-1}$ are $C^r$ homeomorphism, then $f$ is said to be $C^r$ diffeomorphism.

1.2 Unimodal map:

Let $f: I \rightarrow J$ be a continuous map. If $f$ has a maximum at a point $x_c$ in $I$, then $f$ is said to be unimodal map.

1.3 Dynamical systems:

Let $(X, d)$ be a metric space. Let $A \subset X$ and let $T \subset R$. For any fixed $a \in A$, $t_0 \in T$, a mapping $p(., a, t_0): T_{a,t_0} \rightarrow X$ is called a motion if $p(t_0, a, t_0) = a_0$, where $T_{a,t_0} = [t_0,t_1) \cap T$, $t_1 > t_0$. 
A subset $S$ of the set $\bigcup_{(a,t_0)\in A \times \mathcal{T}} \{T_{a,t_0} \to X\}$ is called a family of motions if for every $p(., a, t_0) \in S$, we have $p(t_0, a, t_0) = a_0$.

The four-tuple $\{T, X, A, S\}$ is called a dynamical system[57]. In other words a dynamical system consists of an abstract space $X$ under the dynamical rule $p$ by which the future state of the state variable, i.e. $p(t, a, t_0)$ for all $t$ contained in $T_{a,t_0}$, can be obtained, when the present state of the state variable i.e. $p(t_0, a, t_0)$ is known. Therefore mathematically a dynamical system may be considered as an initial value problem.

Dynamical system is deterministic if $p(t, a, t_0)$ can be obtained uniquely out of the present state $p(t_0, a, t_0)$ while it is said to be random or stochastic if the future state is obtained with the help of some probability distributions.

If $T \subset \mathbb{R}^+$, then it is called a continuous dynamical system. If $T \subset \mathbb{R}^+ \cap \mathbb{N}$, then it is a discrete dynamical system. It may be considered as $X$ as the state space, $T$ as the time set, $t_0$ as the initial time, $a$ as the initial condition of the motion $p(t, a, t_0)$, $A$ as the set of initial conditions.

1.4 Autonomous and nonautonomous system:

An $n^{th}$ order autonomous dynamical system is defined as $\dot{x} = f(x)$, where $\dot{x} = \frac{dx}{dt}$, $x(t) \in \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^n$.

Non autonomous dynamical system: An $n^{th}$ order non autonomous dynamical system is defined by $\dot{x} = f(x, t)$, where $\dot{x} = \frac{dx}{dt}$, $x(t_0) = x_0$. Unlike the autonomous case, the initial time cannot be set as 0.

If there exists a $T > 0$ such that $f(x, t) = f(x, t+T)$ for all $x$ and for all $t$, then the system is said to be time periodic with period $T$. 
An $n^{th}$ order time periodic non-autonomous system can always be converted to an $(n+1)^{th}$ order autonomous system by taking an extra state $\theta = \frac{2\pi t}{T}$, then the autonomous system is given by

$$\dot{x} = f\left(x, \frac{\theta t}{2\pi}\right), x(0) = x_0, \dot{\theta} = \frac{2\pi}{T}, \theta(0) = \frac{2\pi t_0}{T}$$

1.5 Bounded motion:

A motion $p(.,a,t_0)$ is said to be bounded, if there exists an $x_0 \in X$ and $\epsilon > 0$ such that $d(p(t,a,t_0),x_0) < \epsilon$ for all $t \in T_{a,t_0}$ i.e. a trajectory is bounded if the future state stay within some specified region. However in dynamical system we are much more interested in the bounded trajectories which exhibit some interesting phenomenon than the unbounded trajectories.

1.6 Invariant set:

Let $\{T, X, A, S\}$ be a dynamical system, a set $M \subseteq A$ is said to be invariant with respect to $S$, if $a \in M$ implies $p(t,a,t_0) \in M$ for all $t_0 \in T$, all $t \in T_{a,t_0}$ and all $p(.,a,t_0) \in S$.

In other words the trajectory $p$ if it starts from "$a$" which is in $M$, it stays within $M$ for all values of $t$ or we may say trajectory starting from $M$ stays within $M$ for ever if $M$ is an invariant set.

1.7 Fixed point of a system:

Let $\{T, X, A, S\}$ be a dynamical system, a point $x_0 \in A$ is called a fixed point if the set $\{x_0\}$ is invariant with respect to $S$ or we can say if the invariant set $M$ contains
only one point, let's say $x_0$ then the trajectory stays in that point for ever and therefore the future points of the trajectory will be just the repetition of the point $x_0$.

### 1.8 Periodic points of a system:

A motion $p(.,a,t_0) \in S$ in a dynamical system $\{T,X,A,S\}$ is said to be periodic, if there exists a constant $c$, such that $t+c \in T$ for every $t \in T$, $t \geq 0$, such that $p(t+c,a,t_0)=p(t,a,t_0)$.

### 1.9 Stable and unstable set:

An invariant set $M$ with respect to a dynamical system $\{T,X,A,S\}$ is said to be stable, if for every $\epsilon > 0$, and every $t_0 \in T$, there exists a $\delta$ (dependent on $\epsilon$ and $t_0$) > 0, such that $d(p(t,a,t_0),M) < \epsilon$ for all $t \in T_{a,t_0}$ and for all $p \in S$, whenever $d(a,M) < \delta$.

Further if $\lim_{t \to \infty} d(p(t,a,t_0),M) = 0$, for all $p(.,a,t_0) \in S$, then $M$ is said to be attractive with respect to $S$. $M$ is said to be asymptotically stable if it is both stable and attractive. $M$ is said to be unstable, if it is not stable with respect to $S$.

Let $f$ be a map on $\mathbb{R}^n$ and let $p$ be an attracting fixed point or periodic point of $f$, then the basin of attraction of $p$ is the set of points $x$, such that $|f^k(x) - f^k(p)| \to 0$ as $k \to \infty$.

### 1.10 Saddle point:

Let $f$ be a map on $\mathbb{R}^m$, $m \geq 1$. Let $f(p)=p$, then the fixed point $p$ is called saddle point if at least one of the eigen values of $Df(p)$, where $Df(p)$ is the Jacobian matrix of the map $f$ at the fixed point $p$ has magnitude greater than 1 and at least one eigen value has magnitude less than 1.
If \( p \) is a periodic point of period \( n \) then \( f^n(p) = p \) and \( p \) is said to be saddle if at least one eigen value of \( Df^n(p) \), where \( Df^n(p) \) is the Jacobian matrix of the composition function \( f \) to itself \( n \) times, has magnitude greater than 1 and at least one eigen value has magnitude less than 1.

1.11 Stable and unstable manifolds:

For a saddle fixed point some initial values nearby the fixed point may converge while other may not converge. The set of initial values that converge to the saddle will be called the stable manifold[3] of the saddle.

Let \( f \) be a \( C^1 \) diffeomorphism on \( \mathbb{R}^n \) and let \( p \) be a saddle fixed point or periodic point of \( f \), then the stable manifold of \( p \) denoted by \( S(p) \), is the set of points \( x \in \mathbb{R}^n \) such that \( |f^n(x) - f^n(p)| \to 0 \) as \( n \to \infty \).

The unstable manifold of \( p \) is the set of points \( y \in \mathbb{R}^n \), such that \( |f^{-n}(x) - f^{-n}(p)| \to 0 \) as \( n \to \infty \).

1.11.1 Stable manifold theorem:

Let \( f \) be a diffeomorphism of \( \mathbb{R}^2 \) and \( x \) be the saddle point of \( f \) such that \( Df(x) \) has one eigen value \( s \) with \( |s| < 1 \) and one eigen value \( \mu \) with \( |\mu| > 1 \). Let \( V^s \), \( V^u \) be the eigen vectors corresponding to \( s \) and \( u \) respectively.

Then both the stable and unstable manifolds \( S \) and \( U \) of \( p \) are one dimensional manifolds that contain \( p \). Furthermore, the vector \( V^s \) is tangent to \( S \) at \( p \), and \( V^u \) is tangent to \( U \) at \( p \).
1.12 Homoclinic and Heteroclinic points:

Let $f$ be an invertible map of $\mathbb{R}^n$ and $p$ be a fixed point which is saddle. A point which is in both the stable and unstable manifold of $p$ and is distinct from $p$ is called a homoclinic point. If $x$ is a homoclinic point, then $f^n(x) \to p$ and $f^{-n}(x) \to p$ as $n \to \infty$. The orbit of homoclinic point is called a homoclinic orbit.

A point in the stable manifold of a fixed point $p$ and in the unstable manifold of a different fixed point $q$ is called a heteroclinic point. The orbit of a heteroclinic point is called a heteroclinic orbit.

1.13 Poincare maps:

There are many situations where discrete time dynamical system naturally appears in the study of continuous time dynamical system defined by differential equations. These maps help us to use their results in the concerning differential equations. These maps arising from ordinary differential equations are called Poincare maps.

Let us consider a continuous time dynamical system defined by:

$$\dot{x} = f(x), x \in \mathbb{R}^n,$$

where $f$ is smooth.

Let us consider that the above system has a periodic orbit $K_0$. Let us consider a point $x_0 \in K_0$. We introduce a cross section in the cycle say $\Sigma$ at this point $x_0$. $\Sigma$ is a smooth hyper surface of dimension $n-1$ intersecting $K_0$ at a non zero angle. We suppose that $\Sigma$ is defined near the point $x_0$ by the zero level set of a smooth scalar function $g : \mathbb{R}^n \to \mathbb{R}^1$, $g(x_0) = 0$. Then

$$\Sigma = \{x \in \mathbb{R}^n : g(x) = 0\}$$
A non zero intersecting angle means that the gradient $\Delta g(x)$, defined as $\Delta g(x) = \left( \frac{\partial g(x)}{\partial x_1}, \frac{\partial g(x)}{\partial x_2}, \ldots, \frac{\partial g(x)}{\partial x_n} \right)^T$ is not orthogonal to $K_0$ at $x_0$. The simplest choice of $\Sigma$ is a hyper plane orthogonal to $K_0$ at $x_0$. Let us now consider the orbits of $\dot{x} = f(x)$ near the cycle $K_0$. An orbit starting at a point $x \in \Sigma$ sufficiently close to $x_0$ also returns to $\Sigma$ at some point $\bar{x} \in \Sigma$ near $x_0$. Moreover nearby orbits will also intersect the hyper plane transversally. Thus a map $P: \Sigma \rightarrow \Sigma$ is constructed.

![Diagram of Poincare section in three dimensional state space.](image)

**Fig: 1.13.1 Poincare section in three dimensional state space.**

**1.13.2 Poincare-Benedixson theorem:**

Suppose that

1) $R$ is closed, bounded subset of the plane.

2) $\dot{x} = f(x)$ is continuously differentiable vector field on an open set containing $R$

3) $R$ does not contain any fixed point and

4) There exists a trajectory $C$ that is confined in $R$ in the sense that it starts in $R$ and stays in $R$ for all future time.

Then either $C$ is a closed orbit, or it spirals towards a closed orbit $t \rightarrow \infty$. 

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1.14 Bifurcations:

The qualitative structure of the flow can change as control parameters are varied. Particularly fixed points may be created or destroyed or their stability may be changed. These qualitative changes in the dynamics are called bifurcations\[90\], and the parameter values at which they occur are called bifurcation points.

Some types of bifurcations are:

(i) **Saddle node bifurcation:**

The saddle node bifurcation is the basic mechanism by which fixed points are created and destroyed. As a parameter is varied, two fixed points move toward each other, collide and mutually annihilate.

For example: We consider the following one dimensional dynamical system depending on one parameter:

Let us assume that \( \dot{x} = \alpha + x^2 = f(x, \alpha) \). For \( \alpha < 0 \), the system has two fixed points one of which is stable and the other is unstable. For \( \alpha = 0 \) the two fixed points become one and then disappear when \( \alpha > 0 \).

![Graph of \( f(x) \) for \( \alpha < 0 \)](image)

Fig: 1.14.1 \( \alpha < 0 \)(abscissa represents \( x \) while ordinate represents \( f \))
In fig: 1.14.1, fixed points are those points of the x-axis which are cut by the curve. It can be observed that left part of the curve is decreasing, so the fixed point lying in the left part has derivative less than 0, so it is a stable fixed point while the fixed point lying in the right part has derivative greater than 0, hence it is an unstable fixed point.

\[ f \]

\[ x \]

Fig: 1.14.2 \( a = 0 \) (abscissa represents x and ordinate represents f)

In fig 1.14.2 the first derivative of the fixed point is 0 and hence bifurcation occurs at the parameter value \( a = 0 \).

\[ f \]

\[ x \]

Fig: 1.14.3 \( \alpha > 0 \) (abscissa represents x and ordinate represents f)
In fig 1.14.3 we can see that fixed points disappear for $\alpha > 0$.

(ii) **Transcritical bifurcation:**

There are certain scientific situations where a fixed point exists for all values of a parameter and can never be destroyed. However such a fixed point may change its stability as the parameter is varied. The transcritical bifurcation is the standard mechanism for such changes in stability.

For example: let $\dot{x} = rx - x^2$, $f(x, r) = rx - x^2$ where $r$ is a parameter. Clearly $x=0$ is a fixed point. When $r < 0$, $x = 0$ is a stable fixed point and $x=r$ is an unstable fixed point. For $r = 0$, the two fixed points become the same and for $r > 0$, $x=r$ becomes stable and $x = 0$ becomes unstable. We may say that in the whole process the stability has been exchanged between the two fixed points. However like the saddle node bifurcation, the fixed points are not vanished.

![Graph](image)

**Fig: 1.14.4** $r < 0$ (abscissa represents $x$ and ordinate represents $f$)
(iii) **Pitchfork bifurcation:**

This bifurcation generally appears in the physical problems which have symmetry. There are two types of pitch fork bifurcations, one is supercritical and the other subcritical.

For example, let us consider the differential equation $\dot{x} = rx - x^3$, $f(x, r) = rx - x^3$, we can see that the equation is invariant, if we replace $x$ by $-x$. For $r < 0$, the fixed point $x = 0$ is the only fixed point which is stable. On increasing $r$
when it becomes equal to 0, \( x = 0 \) is still a stable fixed point but is very much weak. That means convergence to the fixed point 0 is very much slow, which may be termed as "critical slowing down". For \( r > 0 \), the fixed point \( x = 0 \) becomes unstable and two new fixed points appear on either side of the origin in a symmetry.

**Fig: 1.14.7**  \( r < 0 \) (abscissa represents \( x \) and ordinate represents \( f \))

**Fig: 1.14.8**  \( r = 0 \) (abscissa represents \( x \) and ordinate represents \( f \)).
In fig. 1.14.9, $x = 0$ becomes unstable and two new fixed points are produced at equal distance of $x = 0$. Near both the fixed points on two sides of $x=0$, $f$ is decreasing means both of them are stable. Near $x=0$, $f$ is increasing. Hence 0 is unstable.

Another example, let $\dot{x} = rx + x^3$, $f(x, r) = rx + x^3$ gives subcritical pitch fork bifurcation. When $r < 0$, the two fixed points $x = \pm \sqrt{-r}$ are unstable but
0 is still stable and when \( r > 0 \), \( x = 0 \) becomes unstable as \( f \) near 0 is increasing.

\[ f \]

\[ \begin{array}{c}
\text{Fig: 1.14.10 } r < 0 \text{ (abscissa represents } x \text{ and ordinate represents } f) \\
\text{Fig: 1.14.11 } r = 0 \text{ (abscissa represents } x \text{ and ordinate represents } f) 
\end{array} \]
(iv) **Hopf bifurcation:**

If two complex conjugate eigenvalues simultaneously cross the imaginary axis into the right half plane, Hopf bifurcation occurs.

Two types of Hopf bifurcations are:

a) **Supercritical hopf bifurcation**

![Supercritical Hopf Bifurcation Diagram](image)

**Fig: 1.14.13** Supercritical Hopf bifurcation at the parameter value $\beta = 0$

b) **Subcritical hopf bifurcation**

![Subcritical Hopf Bifurcation Diagram](image)

**Fig: 1.14.14** Subcritical Hopf bifurcation at the parameter value $\beta = 0$
1.15 Topologically transitive:

Let \( \{T, X, A, S\} \) be a dynamical system. A mapping \( p(.,a,t_0) \) is said to be topologically transitive if for any pair of open sets \( U \subset A, V \subset X \), there exists some \( t \) such that \( U \subset \bigcap_{a \in U} \{p(t, a, t_0)\} \cap V \neq \phi \).

1.16 Sensitive dependence of initial condition:

Let \( \{T, X, A, S\} \) be a dynamical system. A mapping \( p(.,a,t_0) \) is said to have sensitive dependence on initial conditions\([19]\), if there exists a \( \delta > 0 \) such that for any \( a \in A \) and any neighbourhood \( N \subset A \) of “\( a \)”, there exists \( b \in N \) and \( t > 0 \) such that \( d(p(t,a,t_0), p(t,b,t_0)) > \delta \). Generally the sensitiveness of initial condition is detected by the exponential divergence of the two initial state variables. Again the exponential divergence means that if two initial points have negligible difference then soon after some time elapsed (for continuous time dynamical system) or after some iteration (for discrete time dynamical system) the difference between two trajectories can be observed and hence the trajectories of two points whose initial difference is negligible cannot be predicted.

1.17 Chaotic map:

Let \( \{T, X, A, S\} \) be a dynamical system. A mapping \( p(.,a,t_0) \) is said to be chaotic\([19]\) if

(i) \( p \) has sensitive dependence to the initial condition.

(ii) \( P \) is topologically transitive.

(iii) The periodic points of \( p \) are dense in \( X \).
1.18 *Hyperbolic periodic point*:

Let \( p \) be a periodic point of period \( n \) of the function \( f \). If the eigen values of \( J_n(p) \) does not lie on the unit circle, then \( p \) is said to be hyperbolic. Here \( J_n(p) \) means the Jacobian of the vector valued function \( f \) at \( p \).

1.19 *Schwarzian derivative*:

The Schwarzian derivative[15] of a function \( f \) at \( x \) is

\[
S_f(x) = \frac{f''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2
\]

1.19.1 *Proposition*:

Let \( f(x) \) be a polynomial. If all the roots of \( f'(x) \) are real and distinct then \( S_f(x) < 0 \).

1.19.2 *Proposition*:

Let \( S_f < 0 \) and \( S_g < 0 \) then \( S(f \cdot g) < 0 \).

1.20 *Normal form[48]*:

Fortunately bifurcation diagrams for different dynamical systems are not entirely chaotic. Different aspects of bifurcation diagrams of different systems interact each other with certain rules. This makes the bifurcation diagrams arising out in different applications look similar.
Let us consider two different dynamical systems (say continuous)

\[ \dot{x} = f(x, \alpha), x \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{R}^m \]  \hspace{1cm} (1.20.1)

And \[ \dot{y} = g(y, \beta), y \in \mathbb{R}^n \text{ and } \beta \in \mathbb{R}^m \]  \hspace{1cm} (1.20.2)

with same number of variables and parameters. Then the dynamical system (1.20.1) is called topologically equivalent to dynamical system (1.20.2) if

i. There exists a homeomorphism of the parameter space \( p: \mathbb{R}^m \rightarrow \mathbb{R}^m, \beta = p(\alpha) \).

ii. There exists a parameter dependent homeomorphism of the phase space \( h_\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n, y = h_\alpha(x) \), mapping orbits of the system (1.20.1) at the parameter \( \alpha \) on to the orbit of the system (1.20.2) at parameter values \( \beta = p(\alpha) \).

Let us consider a system \[ \dot{\xi} = g(\xi, \beta; \sigma), \xi \in \mathbb{R}^k, \sigma \in \mathbb{R}^l \]  \hspace{1cm} (1.20.3)

which has at \( \beta = 0 \) a fixed point \( \xi = 0 \) satisfying \( k \) bifurcation conditions determining a co-dimensional \( k \) bifurcation of this equilibrium. Here \( \sigma \) is a vector of the coefficients \( \sigma_i, i = 1, 2, \ldots, l \).

Let us consider another system \[ \dot{x} = f(x, \alpha), x \in \mathbb{R}^n, \alpha \in \mathbb{R}^k \]  \hspace{1cm} (1.20.4)

having an equilibrium at \( \alpha = 0 \) is \( x = 0 \).

The system (1.20.3) is called the normal form for bifurcation if any generic system (1.20.4) with the equilibrium \( x = 0 \) satisfying the same bifurcation conditions at \( \alpha = 0 \) is locally topologically equivalent near the origin to (1.20.3) for some values of the coefficients \( \sigma_i \). Here generic means the systems satisfy finite number of genericity conditions which have the form of nonequalities.
1.21 Feigenbaum’s Universality:

Let us consider a one dimensional map of the interval \([-1,1]\). \(x \rightarrow f(x, \alpha)\), where \(x \in \mathbb{R}\) and \(\alpha\) is a parameter. We are interested in the maps with quadratic maxima and smooth and as \(\alpha\) increases a stable fixed point loses its stability giving rise to a periodic point of period 2, again as \(\alpha\) is increased further periodic point of period 2 loses its stability and periodic points of period 4 appears and so on. It has been observed that the map with the above physical properties follows the following conditions [48]:

i. \(f:[-1,1] \rightarrow [-1,1]\) is an even smooth function

ii. \(f'(0) = 0, x = 0\) is the only maximum and \(f(0) = 1\)

iii. \(f(1) = -\alpha < 0\)

iv. \(b = f(\alpha) > \alpha\)

v. \(f(b) = f^2(\alpha) < \alpha\), where \(a\) and \(b\) are positive.

Feigenbaum showed that the bifurcation parameters \(\alpha_1, \alpha_2, ..., \alpha_n\) of the map \(f(x, \alpha)\) satisfying the above conditions form asymptotically a geometric progression:

\[
\frac{\alpha_i - \alpha_{i-1}}{\alpha_{i+1} - \alpha_i} = \delta
\]

Fig: 1.21.1 Feigenbaum tree

for large value of \(i\) and the value of \(\delta = 4.66920\ldots\), which is independent of the system.
Moreover at $\alpha_\infty$, the system show chaotic nature i.e. there is no periodic or regular behavior and the structure of the attractor $F$ in particular its Hausdorff dimension at the parameter $\alpha_\infty$, is independent of the system. The above properties are based on the fact that any map satisfying the Feigenbaum condition can be written of the form $(Tf)(x) = -\frac{1}{a}f(-ax)$, where $a = f(1)$ and $a$ depends on $f$. This map $T$ is called the doubling operator. This map $T$ has a fixed point $\psi$ (say), i.e. $T\psi = \psi$.

Further it has been observed that the size scaling of the period doubling sequence is universal and is defined as:

$$\alpha = \lim_{n \to \infty} \frac{d_n}{d_{n+1}}$$

where $d_n$ is the size of the bifurcation pattern of period $2^n$ just before it gives birth to period $2^{n+1}$. The relation between $\alpha$ and $\delta$ is given by the equation

$$-\alpha(-\alpha + 1) = \delta$$

**1.22 Sharkovskii’s theorem:**

Sharkovskii’s theorem gives a scheme for ordering the natural numbers such that for each natural number $n$, the existence of a period-$n$ point implies the existence of all the higher order periodic points where ordering is in the Sharkovskii’s manner. The ordering is as follows:

$$3 < 5 < 7 < 9 < \ldots < 2^{2}3 < 2^{2}5 < \ldots < 2^{3}3 < 2^{3}5 < \ldots < 2^{4}3 < 2^{4}5 < \ldots < 2^{2} < 2^{1}$$

The theorem states that if $f$ is a continuous map on an interval and has a periodic $p$ orbit. If $p < q$, then $f$ has a period $q$ orbit.