CHAPTER-8

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In this short chapter, we wish to concentrate on some motivations of the results described in this monograph. These motivations give rise to many interesting research problems in the field of our study. As we know, though the growth of the general theory on bifurcations and chaos seem to be old, the real breakthrough for the development of these areas came with the profound work of the Feigenbaum on universality in nonlinear systems. Since then, this field is not only confined in a small area in mathematical sciences with an independent life, but also serves as a unifying thread interlacing many other branches of mathematics and science such as Functional analysis, Measure theory, Fluid mechanics, Quantum mechanics, Physics, Chemistry, Biosciences, Engineering, Medical sciences etc., and thus, it suggests a very wide scope of doing research. Some problems are posed as follows:

OPEN PROBLEMS:

1. In chapter-2, we have discussed analytically the existence of fixed points and periodic points of the generalised logistic map, then in another part we discussed about the numerical procedure to get the bifurcation in a particular case of that class. Now the questions are:
   i. Can we get a rigorous analytical method which tells about the uniqueness of the stable fixed points or periodic points.
   ii. Can we develop rigorous analytical methods to establish the existence of different bifurcation values and the corresponding periodic points.
iii. Under what condition a polynomial model of degree n: 
\[ f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n \] 
can be transformed to generalised logistic form which shows Feigenbaum tree of bifurcation points leading to chaos?

iv. If some smooth function is added to generalised logistic map such that unimodality property is not preserved then how the Feigenbaum criteria is related to the map?

2. In chapter 3, we have discussed the stability for the map \( f \) analytically. Can we give a similar analytical approach for \( f^2, f^4 \) etc. Further we are able to calculate the various bifurcation points and ultimately the accumulation point for a particular case of that class. Now can we find a rigorous analytical method for the same purpose? We can propose the same problem as in one dimensional i.e. if the unimodality criteria is removed then how Feigenbaum universal constant \( \delta \) will be related provided chaos exists through period doubling route?

3. Fractal objects often come in some way or another while studying non linear models. So, nowadays it has become essential to study the fractal objects and quantify them in some way or another to give some concrete conclusions. In chapter 4 we proved numerically that we have encountered a fractal object by quantifying it in different manner. For that we calculated Lyapunov exponent numerically. Also, we calculated various kinds of fractal dimensions near the accumulation point numerically. Is it possible to get rigorous analytical method by which it is possible to get Lyapunov function of the generalised logistic map?

Can we get an analytical method by which box-counting dimension, correlation dimension, information dimension, Hausdorff dimension calculation is possible?
4. In chapter 4 also we calculated Lyapunov exponents and various dimensions to quantify the chaotic property of the strange attractor near the accumulation point. Grassberger calculated the box counting dimension of logistic map at the accumulation point by the renormalization method which is universal in one dimensional map with quadratic maximum. Is it extendable to two dimensional unimodal map also? Further if unimodal property is removed then how box counting dimension value will behave?

5. Can we develop a suitable theory and thereby formula in order to determine Hopf Bifurcation analytically for chaotic maps and differential equations? Another idea may be posed: how can one control chaos in two or higher dimensional chaotic population models?

6. In the one dimensional and two dimensional models discussed in chapter -2 and chapter - 3 respectively, can we determine periodic orbits of periods in the Sarkovskii orders: $3 \rightarrow 5 \rightarrow 7 \rightarrow \ldots \ 2 \times 3 \rightarrow 2 \times 5 \rightarrow \ldots \ 2^{2} \times 3 \rightarrow 2^{2} \times 5 \rightarrow \ldots \ldots \ ?$

7. One important method of characterizing an attractor makes use of a probability distribution function. This notion becomes particularly important as the number of state space dimensions increases. For a larger number of state space dimensions, we have more and more geometric possibilities for attractors. For higher dimensional state spaces, we need more abstract and less geometric methods of characterizing the attractor / strange attractor. Various kinds of probability
distributions are useful in this case. Now we can ask: what is the probability that a
given trajectory point of the dynamical system falls within some particular region
of a state space? This question is an important one in various ways and one of
them is that it will help in calculation of information dimension analytically.
Another aspect is that it will say about the geometric aspect of the chaotic
attractor i.e whether it is fractal or multifractal e.t.c.

8. We consider the following predator-prey model:

\[
\begin{align*}
\frac{dx}{dt} &= r x \left(1 - \frac{x}{k}\right) - yp(x) \\
\frac{dy}{dt} &= ys(1 - \frac{hx}{x})
\end{align*}
\]

\(x(0) > 0, y(0) > 0, r, s, k, h > 0\), where \(x\) is the population of the prey and \(y\) is the
population of the predator. It is assumed that the prey grows logistically with carrying
capacity \(K\) with intrinsic growth rate \(r\) in the absence of predation. The predator
consumes the prey according to the functional response \(p(x)\) and grows logistically with
intrinsic growth rate \(s\) and carrying capacity proportional to the population size of prey.
The parameter \(h\) is the number of pre required to support one predator at equilibrium
when \(y\) equals \(x/h\). In [42], the functional response \(p(x)\) is classified into three types.
When the functional response \(p(x)\) is of type 1, i.e., \(p(x) = mx\), then we have the Leslie-
Gower model [49], when the functional response \(p(x)\) is of type 2, in particular, \(p(x) =
\frac{mx}{A+x}\), then we have the Holling-Tanner models [44,5360,61 73]

Now the questions are:

i. Under what conditions of the parameter period doubling bifurcations, Hopf
bifurcation occur?

ii. What will be the Poincare map of the above system?
iii. How the dynamical behavior of the above system can be predicted from the stability of the fixed points of its Poincare map?

iv. If some properties of the function \( p(x) \) for example unimodality or continuity of the first derivative is added or \( p(x) \) becomes some popular one dimensional ecological map (say logistic map which has the Feigenbaum universal behavior) e.t.c. then how the system behaves dynamically.

9. Dimension function is a concept which extends the Renei’s dimension. It is based on statistical property of the distribution function \( p(\delta, n) \) of nearest neighbor distance \( \delta \) between points on the attractor. Now the question is: The two dimensional ecological model discussed in chapter 3 and the generalized logistic map discussed in chapter 2, how dimension function can be implemented to express the geometry of the fractal sets encountered in chapter 2 and 3 respectively, also how capacity dimension, correlation dimension, information dimension e.t.c can be yield with the help of dimension function at the parameter values where bifurcations occur.

10. As we have discussed the two models in chapter 2 and chapter 3 where we have shown numerically the bifurcation points for some particular parameters. The question is how symbolic dynamics is useful in predicting the bifurcation points as well as how much it helps in predicting Feigenbaum universal behavior.

11. In dynamical system the concept of entropy may be applied in predicting the dynamical behavior of a system. Actually the changes in entropy are much more important than the value of the entropy itself. For example if a set of initial conditions are chosen in two cells and if the motion is regular, then the entropy
value would not be 0, but it would remain constant as the system evolves. The change in entropy is characterized by the Kolmogorov-Sinai entropy rate, which describes the rate of change of entropy as the system involves. The K-S entropy $K_n$ after $n$ units of time is defined to be

$$K_n = \frac{1}{\tau} (S_{n+1} - S_n)$$

i.e.

The average K-S entropy over the entire attractor is defined as follows:

$$= \lim_{N \to \infty} \frac{1}{N \tau} \sum_{n=0}^{N-1} (S_{n+1} - S_n)$$

$$= \lim_{N \to \infty} \frac{1}{N \tau} \sum_{n=0}^{N-1} (S_N - S_0)$$

Further putting $\tau \to 0, L \to 0$ we have,

$$K = \lim_{N \to \infty} \lim_{\tau \to 0} \lim_{L \to 0} \frac{1}{N \tau} \sum_{n=0}^{N-1} (S_N - S_0).$$

Now the question is how K-S entropy is related to various kinds of fractal dimensions and Lyapunov exponent.

12. In chapter 7, we have discussed about the controlling of chaos which is applied to the one dimensional generalized logistic map as well as two dimensional ecological map. Two methods have been discussed. One is periodic proportional pulse method and the other is proportional pulse method. Now the question is can we make a rigorous description of the proportional pulse method by which chaos control behavior of the fixed point and periodic points can be explained?