Chapter 3

RELATIVE INJECTIVITY

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3. RELATIVE INJECTIVITY

In this chapter we discuss relative injectivity and injectivity of N-groups. This chapter has four sections.

3.1 PRELIMINARIES:

This section deals with some basic definitions and results which are used in the later sections.

**Definition 3.1.1:** Let E be an N-group. Then the singular subset of E is defined as the set

\[ Z(E) = \{ x \in E / Ix = 0 \text{ for some essential } \text{N-subgroup } I \text{ of } N \}. \]

An N-group E is called singular N-group if \( Z(E) = E \).

An N-group E is called non-singular N-group if \( Z(E) = 0 \).

**Definition 3.1.2:** If E is an N-group, the set \( Z_W(E) = \{ x \in E / Ix = 0 \text{ for some essential ideal } I \text{ of } N \} \) is weak singular subset of E.

An N-group E is called weak singular if \( Z_w(E) = E \).

An N-group E is called weak non-singular if \( Z_w(E) = 0 \).

**Example 3.1.3:** \( N = \mathbb{Z}_8 \) is a near-ring with two operations ' + ' as addition modulo 8 and ' * ' defined by following table:
Here \( I = \{0, 4\} \) is an essential \( N \)-subgroup of \( N \). Here \( \forall x \in N, Ix = 0 \). So \( Z(N) = N \), so \( N \) is singular.

But \( I = \{0, 4\} \) is also an essential ideal of \( N \). Hence \( Z_w(N) = N \) and so \( N \) is also weak singular.

Example 2.1.13 is an example of non-singular as well as weak non-singular \( N \)-group.

**Definition 3.1.4:** An \( N \)-monomorphism \( f : A \to B \) is said to be an essential \( N \)-monomorphism if \( fA \leq B \).

**Proposition 3.1.5:** An \( N \)-group \( C \) is singular if there exists a short exact sequence

\[
0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0
\]

such that \( f \) is an essential \( N \)-monomorphism.
Proof: Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence such that $f$ is an essential $N$-monomorphism. For any $b \in B$, we have a map $k : N \rightarrow B$ defined by $k(n) = nb$. By proposition 1.3.5, $k^{-1}(fA) \leq_{e} N$.

⇒ the $N$-subgroup $I = \{ n \in N \mid nb \in fA \}$ is an essential $N$-subgroup of $N$.

Now $Ib \leq fA = K_{erg}$.

Hence $g(Ib) = 0 \Rightarrow I(gb) = 0$ and so $gb \in Z(C)$.

Since $g$ is an $N$-epimorphism, we get $Z(C) = C \Rightarrow C$ is singular.

Corollary 3.1.6: If $A$ is an essential ideal of $B$, then $B/A$ is singular.

Proof: We consider the short exact sequence $0 \rightarrow A \xrightarrow{i} B \xrightarrow{g} B/A \rightarrow 0$.

As $A \leq_{e} B$, from above proposition $B/A$ is singular.

Proposition 3.1.7: If $B$ is Non-singular and $B/A$ is singular then $A \leq_{we} B$.

Proof: If $B/A$ is singular and $x$ is non-zero element of $B$, then $Ix = \bar{0}$ for some essential $N$-subgroup $I$ of $N \Rightarrow Ix \leq A$. As $B$ is non-singular, we have $Ix \neq 0$ and thus $Nx \cap A \neq 0$.

Therefore $A \leq_{we} B$.

Proposition 3.1.8: If $N$ is a dgnr and $\{N_{e}\}_{e \in E}$ is an independent family of normal $N$-subgroups of $N$-group $E$ then $E$ is a homomorphic image of $\bigoplus_{e \in E} N_{e}$.

Proof: Let $f_{e} : N_{e} \rightarrow E$ be defined by $f_{e}(ne) = ne$.

Then $f_{e}$ is $N$-homomorphism.
Let $f_{e_i} : \mathbb{N}e_i \to E$ be defined by $f_{e_i}(n, e_i) = n e_i$ and $f_{e_j} : \mathbb{N}e_j \to E$ be defined by $f_{e_j}(n, e_j) = n e_j$

Let $f_{e_i} + f_{e_j} : \mathbb{N}e_i \oplus \mathbb{N}e_j \to E$ be defined by $(f_{e_i} + f_{e_j})(n, e_i + n, e_j) = (f_{e_i}(n, e_i) + f_{e_j}(n, e_j))$.

Obviously it is well-defined.

Let $(n/e_i + n/e_j), (n/e_i + n/e_j) \in \mathbb{N}e_i \oplus \mathbb{N}e_j$ and $(f_{e_i} + f_{e_j})(\sum_{i=1}^{n} s_i (n/e_i + n/e_j))$

$= (f_{e_i} + f_{e_j})(s_1(n/e_i + n/e_j) + s_2(n/e_i + n/e_j) + \ldots + s_n(n/e_i + n/e_j))$ [since $\mathbb{N}e_i$'s are normal $\mathbb{N}$-subgroups]

$= (f_{e_i} + f_{e_j})(s_1(n/e_i + n/e_j) + \sum_{i=1}^{n} s_i n_i e_i)$

$= (f_{e_i} + f_{e_j})(\sum_{i=1}^{n} s_i n_i e_i) + (n/e_i + n/e_j)$

Next for $n \in \mathbb{N}$, $(f_{e_i} + f_{e_j})(n(n/e_i + n/e_j)) = (f_{e_i} + f_{e_j})(\sum_{i=1}^{n} s_i (n/e_i + n/e_j))$ [since $\mathbb{N}$ $\mathbb{N}$-normal]

$= (f_{e_i} + f_{e_j})(s_1(n/e_i + n/e_j) + s_2(n/e_i + n/e_j) + \ldots + s_n(n/e_i + n/e_j))$

$= (f_{e_i} + f_{e_j})(s_1(n/e_i + s_2 n/e_i + \ldots + s_n n/e_i) e_i + (s_1 n_i/e_i + s_2 n_i/e_i + \ldots + s_n n_i)) e_j$

$= (f_{e_i} + f_{e_j})(\sum_{i=1}^{n} s_i n_i) e_i + (\sum_{i=1}^{n} s_i n_i) e_j$

$= (f_{e_i} + f_{e_j})(nn/e_i + nn/e_j)$

$= (nn/e_i + nn/e_j)$
Thus $(f_{e_1} + f_{e_j})$ is an $N$-homomorphism.

Similarly if we define \( f = \sum_{e \in E} f_e : \oplus_{e \in E} N e \to E \) by \( (\sum_{e \in E} f_e) (\sum_{e \in E} n_e) = (\sum_{e \in E} f_e(n_e)) \), \( n \in N \), it is an $N$-homomorphism.

Obviously it is an $N$-monomorphism.

Again for any \( e_k \in E \) we get \( e_k \in N e_k \in \oplus_{e \in E} N e \). So \( f \) is onto.

Hence \( E \) is a homomorphic image of \( \oplus_{e \in E} N e \).

**Theorem 3.1.9:** For a short exact sequence \( 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \) if \( A \) and \( C \) are finitely generated then \( B \) is also finitely generated.

**Proof:** As \( \beta : B \to C \) is an epimorphism, \( C \cong \frac{B}{\text{Ker} \beta} \Rightarrow C \cong \frac{B}{\text{Ker} \beta} \).

For identity map \( \alpha \), \( C \cong \frac{B}{A} \).

So if an $N$-group \( B \) has finitely generated $N$-subgroup \( A \) and factor $N$-group \( \frac{B}{A} \), then \( B \) is also finitely generated.
Definition 3.1.10: For an N-group E an element x is called a nilpotent element if \(x^k = 0\) for some \(k \in \mathbb{N}^+\).

3.2 E-injectivity and injectivity:

In this section we define relative injective N-groups, and some special relative injective N-groups and investigate various characteristics of these N-groups.

In the third section of the chapter we study direct sums of relative injective N-groups and N-subgroups, direct product of relative injective N-groups. Using the notion of dominance of an element of an N-group by another N-group direct sums of relative injective N-groups are established.

In the last section we are trying to relate direct sums of relative injective N-groups and chain conditions, relative injectivity of simple, semi-simple, strictly semi-simple, singular N-groups and chain conditions.

Throughout the remaining section of this chapter we consider all N-groups unitary N-groups unless otherwise specified.

Definition 3.2.1: Let E and U be N-groups. U is called E-injective or U is injective relative to E if for each N-monomorphism \(f: K \rightarrow E\), every N-homomorphism from K into U can be extended to an N-homomorphism from E into U. i.e. The diagram

\[
\begin{array}{ccc}
K & f & E \\
g & & h \\
& U & \\
\end{array}
\]
commutes, i.e. \( g = hf \).

An N-group \( A \) is injective if it is \( E \)-injective for every N-group \( E \) of \( N \). So if an N-group \( A \) is injective it is \( E \)-injective for any N-group \( E \).

**Proposition 3.2.2:** Let \( N \) be a dgnr, \( E \) be an N-group and \( F \) be a commutative N-group.

Then the set \( \text{Hom}_N(E, F) = \{ f / f: E \to F \text{ is an N-homomorphism} \} \) is an abelian group where addition is defined as: for \( f, g \in \text{Hom}_N(E, F) \), \( (f + g)(e) = f(e) + g(e) \).

**Proof:** As \( F \) is an abelian N-group, for \( f, g \in \text{Hom}_N(E, F) \) and \( e \in E \),

\[
(f + g)(e) = f(e) + g(e)
\]

\[
= g(e) + f(e)
\]

\[
= (g + f)(e), \text{ so } f + g = g + f.
\]

We are to show \( f + g \) is an N-homomorphism.

For \( e_1, e_2 \in E \), \( (f + g)(e_1 + e_2) = f(e_1 + e_2) + g(e_1 + e_2) \) [By given condition]

\[
= f(e_1) + f(e_2) + g(e_1) + g(e_2) \quad [ \because f, g \text{ are N-homomorphism} ]
\]

\[
= f(e_1) + g(e_1) + f(e_2) + g(e_2) \quad [ \because F \text{ is abelian} ]
\]

\[
= (f + g)(e_1) + (f + g)(e_2) \quad [ \text{By given condition} ]
\]

Next for \( e \in E \), \( n \in N \)

\[
(f + g)(ne) = f(ne) + g(ne) \quad [ \text{By given condition}]
\]

\[
= nf(e) + g(n(e)) \quad [ \because f, g \text{ are N-homomorphisms} ]
\]

\[
= (\sum_{i=1}^p s_i)f(e) + (\sum_{i=1}^p s_i)g(e) \quad [ \because N \text{ is dgnr} ]
\]
Thus \( f + g \) is an \( N \)-homomorphism.

**Proposition 3.2.3:** Let \( B, M \) be two \( N \)-groups and \( C \) an ideal of \( B \). For \( N \)-homomorphism \( f : B \rightarrow M \) \exists unique homomorphism \( \tilde{f} : \frac{B}{C} \rightarrow M \) such that \( \tilde{f}(\overline{b}) = f(b), \forall C \subseteq \text{Ker} f \).

**Proof:** Let \( \overline{b_1} = \overline{b_2} \)

\[
\Rightarrow \overline{b_1} - \overline{b_2} = 0
\]

\Rightarrow \overline{b_1} = \overline{b_2} + C = C

\Rightarrow \overline{b_1} - \overline{b_2} \in C \subseteq \text{Ker} f

\Rightarrow f(\overline{b_1} - \overline{b_2}) = 0

\Rightarrow f(\overline{b_1}) - f(\overline{b_2}) = 0

\Rightarrow \tilde{f}(\overline{b_1}) = \tilde{f}(\overline{b_2})

So \( \tilde{f} \) is well-defined.

Next \( \tilde{f}(\overline{b_1} + \overline{b_2}) \)

\[
= \tilde{f}(\overline{b_1} + \overline{b_2})
\]

\[
= f(\overline{b_1} + \overline{b_2})
\]
\[ = f(b_1) + f(b_2) \]
\[ = f(b_2) + f(b_2) \]

And \( \bar{f}(\tilde{n}b) \)
\[ = \bar{f}(n(b + C)) \]
\[ = \bar{f}(nb + C) \]
\[ = \bar{f}(\tilde{n}b) \]
\[ = f(nb) \]
\[ = nf(b) \]
\[ = nf(\tilde{b}) \]

So \( \bar{f} \) is an N-homomorphism and by definition obviously it is unique.

Thus we get if \( f \) is an epimorphism, then \( \bar{f} \) defined as above is also an epimorphism.

**Definition 3.2.4:** Let \( U \) be a commutative N-group and \( f: L \to M \) be an N-homomorphism. We can define a mapping

\[ f^* = \text{Hom}_N(f, U) : \text{Hom}_N(M, U) \to \text{Hom}_N(L, U) \]

by \( \text{Hom}_N(f, U) : \gamma \to \gamma f \) i.e. \( f^* \gamma = \gamma f \) then \( \text{Hom}_N(f, U) \) is an N-homomorphism.

**Proposition 3.2.5:** If \( U \) is a commutative N-group, then for every exact sequence

\[ 0 \to K \xrightarrow{f} E \xrightarrow{g} L \to 0 \]

the sequence \[ 0 \to \text{Hom}_N(L, U) \xrightarrow{g^*} \text{Hom}_N(E, U) \xrightarrow{f^*} \text{Hom}_N(K, U) \] is exact.

**Proof:** If \( \gamma \in \text{Hom}_N(L, U) \) and \( g^*(\gamma) = 0 \)
\[ \Rightarrow \gamma g = 0 \]
\[ \Rightarrow \gamma = 0 \quad [\because g \text{ is N-epimorphism}] \]
\[ \Rightarrow g^* \text{ is N-monomorphism.} \]

Next let \( \gamma \in \text{Hom}_N(L, U) \). Then \( f^* g^* (\gamma) = f^*(\gamma g) = (\gamma g)f = \gamma (gf) = \gamma 0 = 0^* = 0 = 0^* \gamma \)

So we get \( f^* g^* = 0 \Rightarrow \text{im } g^* \subseteq \text{Ker } f^* \).

Next let \( \beta \in \text{Ker } f^* \), then \( \beta f^* = f^* \beta = 0 \)

\[ \Rightarrow \beta(\text{im } f) = 0 \Rightarrow \beta(\text{Ker } g) = 0 \]

\[ \Rightarrow \text{Ker } g \subseteq \text{Ker } \beta. \]

Now \( \beta : E \to U \) is an N-homomorphism such that \( \text{Ker } g \subseteq \text{Ker } \beta \).

\[ \Rightarrow \exists \text{ a unique N-homomorphism } \bar{\beta} : \frac{E}{\text{Ker } g} \to U \text{ such that } \bar{\beta}(\bar{b}) = \beta(b). \]

Also \( g : E \to L \) is an N-epimorphism, so \( \exists \) an N-isomorphism \( \phi : \frac{E}{\text{Ker } g} \to L \) such that

\[ \phi(\bar{b}) = g(b). \]

We consider the following sequence of N-homomorphisms

\[ L \xrightarrow{\phi^{-1}} \frac{E}{\text{Ker } g} \xrightarrow{\bar{\beta}} U, \text{ which gives } \bar{\beta} \phi^{-1} \in \text{Hom}_N(L, U). \]

Now \( g^* (\bar{\beta} \phi^{-1}) = (\bar{\beta} \phi^{-1})g = \beta \)

\[ \Rightarrow \beta \in \text{im } g^*. \quad [\text{since } g^* (\bar{\beta} \phi^{-1})(b) = ((\bar{\beta} \phi^{-1})g)(b) = \bar{\beta}(\bar{b}) = \beta(b)]. \]

So \( \text{im } g^* = \text{Ker } f^*. \)
Proposition 3.2.6: A commutative N-group $U$ is $E$-injective if and only if $\text{Hom}_N(-, U)$ is exact.

Proof: We assume $U$ is $E$-injective.

We consider the exact sequence $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} C \to 0$.

Now exactness of $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} C \to 0$ implies

$$0 \to \text{Hom}_N(C, U) \xrightarrow{\beta^*} \text{Hom}_N(E, U) \xrightarrow{\alpha^*} \text{Hom}_N(A, U)$$

is exact.

So it is enough to show $\alpha^*$ is epic.

Let $f \in \text{Hom}_N(A, U)$. We consider the diagram

$$
\begin{array}{ccc}
0 & \to & A \\
& \alpha \downarrow & \downarrow \beta \\
& \gamma \downarrow & \\
& U & \to E
\end{array}
$$

Since $U$ is injective, $\exists \gamma \in \text{Hom}_N(E, U)$ such that $\gamma \alpha = f$

$$\Rightarrow \quad \alpha^* \gamma = f$$

$$\Rightarrow \quad \alpha^* \text{ is onto.}$$

Conversely, let $\text{Hom}_N(-, U)$ be exact. We consider the diagram with exact row

$$
\begin{array}{ccc}
U & \to & \\
\uparrow f & & \\
0 & \to A \xrightarrow{\alpha} & E
\end{array}
$$
0 → A \overset{a}{\rightarrow} E \overset{\beta}{\rightarrow} \text{Im}_a \rightarrow 0 \text{ is exact.}

\Rightarrow 0 \rightarrow \text{Hom}_N(\text{Im}_a, U) \overset{\beta^*}{\rightarrow} \text{Hom}_N(E, U) \overset{\alpha^*}{\rightarrow} \text{Hom}_N(A, U) \rightarrow 0 \text{ is exact.}

Since $\alpha^*$ is an epimorphism, for $f \in \text{Hom}_N(A, U)$ such that $\alpha^*\gamma = f$

$\Rightarrow \gamma \alpha = f.$

Thus $\exists \gamma : E \rightarrow U$ such that $\gamma \alpha = f \Rightarrow U \text{ is } E\text{-injective.}$

**Definitions 3.2.7:** An N-group $E$ is a WI-N-group if N-group $W$ is $E$-injective.

**Definition 3.2.8:** An N-group $E$ is a WC-I-N-group if a commutative N-group $W$ is $E$-injective.

**Definition 3.2.9:** An N-group $E$ is called a s-simple or a strict simple N-group if it has no proper normal N-subgroups.

Proposition 1.3.12 holds for normal N-subgroups also. Thus we get the following proposition:

**Proposition 3.2.10:** The following are equivalent

(a) Every normal N-subgroup of $E$ is a direct summand.

(b) $E$ is a sum of simple normal N-subgroups.

(c) $E$ is a direct sum of simple normal N-subgroups.

**Definitions 3.2.11:** We define $s\text{-Soc } E$ or strict socle of $E$ as direct sum of simple normal N-subgroups.

An N-group $E$ is called a strictly semisimple N-group if $s\text{-Soc}(E) = E$. In other words $E$ is strictly semisimple if one of the conditions of proposition 3.2.10 holds.
We observe that every semisimple N-group is strictly semisimple but the converse is not true. If N is a dgnr then every strictly semisimple N-group is semisimple.

The following is an example of strictly semisimple N-group which is not semisimple.

Example 3.2.12: We consider the near-ring \( N = \{ 0, a, b, x, y \} \) under the addition and multiplication defined as the following table

\[
\begin{array}{c|ccccccc}
+ & 0 & a & b & c & x & y \\
\hline
0 & 0 & a & b & c & x & y \\
a & a & 0 & y & x & c & b \\
b & b & x & 0 & y & a & c \\
c & c & y & x & 0 & b & a \\
x & x & b & c & a & y & 0 \\
y & y & c & a & b & 0 & x \\
\end{array}
\]

\[
\begin{array}{c|ccccccc}
. & 0 & a & b & c & x & y \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & a & b & c & 0 & 0 \\
b & 0 & a & b & c & 0 & 0 \\
c & 0 & a & b & c & 0 & 0 \\
x & 0 & 0 & 0 & 0 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Here \{0, a\}, \{0, b\}, \{0, c\}, \{0, x, y\} are simple left normal N-subgroups of N.

And \(N = \{0, a\} + \{0, b\} + \{0, c\} + \{0, x, y\}\). So N is strictly semisimple.

But \(N\) is not semisimple.

**Definitions 3.2.13:** An N-group \(E\) is called SI N-group if every singular N-group is \(E\)-injective.

An N-group \(E\) is called SWI N-group if every weak singular N-group is \(E\)-injective.

An N-group \(E\) is called V N-group if every simple N-group is \(E\)-injective.

An N-group \(E\) is called \(\text{V}_c\) N-group if every simple commutative N-group is \(E\)-injective.

An N-group \(E\) is called GV N-group if every simple singular N-group is \(E\)-injective.

An N-group \(E\) is called \(S^2\) I N-group if every strictly semi-simple N-group is \(E\)-injective.

An N-group \(E\) is called \(S^3\) I N-group if every strictly semi-simple singular N-group is \(E\)-injective.

An N-group \(E\) is called \(S^2\text{SWI}\) N-group if every strictly semi-simple weak singular N-group is \(E\)-injective.

**Definition 3.2.14:** A near-ring \(N\) is called V near-ring if \(NN\) is a V N-group and GV near-ring if \(NN\) is a GV N-group.

A near-ring \(N\) is called \(\text{V}_c\) near-ring if \(NN\) is a \(\text{V}_c\) N-group.

**Proposition 3.2.15:** N-subgroups of a WI N-group are again WI N-groups.
Proof: Let $E$ be a WI $N$-group.

$\Rightarrow W$ is $E$-injective.

And let $E'$ be any $N$-subgroup of $E$.

We show $E'$ is also a WI $N$-group.

That is we are to show $W$ is also $E'$-injective.

Let $h : E' \to E$ be an $N$-monomorphism and $K'$ be an $N$-subgroup of $E'$ and $f : K' \to E'$ be any $N$-monomorphism.

Then $hf$ is also an $N$-monomorphism, $hf : K' \to E$.

\[
\begin{array}{c}
\text{K'} \\
\downarrow hf \\
E
\end{array} \quad \begin{array}{c}
f \\
\downarrow h \\
E'
\end{array}
\]

Now $W$ is $E$-injective, so for any $N$-subgroup $K$ of $E$, the $N$-monomorphism $i : K \to E$ and any $N$-homomorphism $k : K \to W, \exists$ an $N$-homomorphism $\gamma : E \to W$ s.t. $k = \gamma i$.

i.e. the following diagram

\[
\begin{array}{c}
\text{K} \\
\downarrow k \\
W \\
\downarrow \gamma \\
\text{E}
\end{array}
\]

commutes.

Since $W$ is $E$-injective, so for $N$-monomorphism $hf : K' \to E$ and $p : K' \to W$ we get
\( \gamma : E \to W \) such that \( \gamma(hf) = p \).

That is the diagram

\[
\begin{array}{ccc}
K' & \xrightarrow{f} & E' \\
\downarrow p & & \downarrow \gamma \\
W & \xrightarrow{\gamma h} & E
\end{array}
\]

Now \( f : K' \to E' \) is an N-monomorphism and for any N-homomorphism \( p : K' \to W \), we get \( \gamma h : E' \to W \) such that the diagram

\[
\begin{array}{ccc}
K' & \xrightarrow{f} & E' \\
\downarrow p & & \downarrow \gamma h \\
W & & \\
\end{array}
\]

commutes. That is \( p = (\gamma h)f \).

Therefore \( W \) is \( E' \)-injective.

**Proposition 3.2.16:** Homomorphic images of a \( WCI \) N-groups are again \( WCI \) N-groups.

**Proof:** Given \( 0 \to E' \xrightarrow{h} E \xrightarrow{k} E'' \to 0 \) is exact and commutative N-group \( W \) is \( E' \)-injective.

We show \( W \) is \( E'' \)-injective.

Let \( E' \leq K \leq E \) and that \( E'' = E/E' \). Now we consider the canonical diagram
Now applying $\text{Hom}_N(-, W)$ we get the diagram

$$
0 \rightarrow \text{Hom}_N(E/E', W) \rightarrow \text{Hom}_N(E, W) \rightarrow \text{Hom}_N(E', W) \rightarrow 0
$$

Since $\text{Hom}_N(E/E', W) \rightarrow \text{Hom}_N(K/E', W)$ is epic, for all $\gamma \in \text{Hom}_N(K/E', W)$ there exists $\alpha \in \text{Hom}_N(E/E', W)$ such that $\phi(\alpha) = \gamma$.

$\Rightarrow \alpha f = \gamma$, where $f : K/E' \rightarrow E/E'$ is an $N$-monomorphism and $\phi = \text{Hom}_N(f, W)$.

Thus $W$ is $E/E'$-injective.

$\Rightarrow E''$ is $W \lhd N$-group of $E$. 

3.3. On direct sum of N-groups with Injectivity and E-injectivity:

In this section we study direct sums of relative injective N-groups and N-subgroups, direct product of relative injective N-groups. Using the notion of dominance of an element of an N-group by another N-group direct sums of relative injective N-groups several properties are established.

**Proposition 3.3.1:** Let \( N \) be a dgnr. If \( E_{\alpha} \) is a WI N-group for all \( \alpha \in A \) then \( E = \oplus_{\alpha \in A} E_{\alpha} \) is a WI N-group, where \( E \) is commutative.

**Proof:** Let \( E = \oplus_{\alpha \in A} E_{\alpha} \) and \( E_{\alpha} \) is WI N-group

\[ \Rightarrow W \text{ is } E_{\alpha}-\text{injective for all } \alpha \in A. \]

We consider an N-subgroup \( K \) of \( E \) and the N-homomorphism \( h : K \to W \).

Let \( \Omega = \{ f : L \to W / K \leq L \leq E \text{ and } (f\mid K) = h \} \).

Let \( g : A \to W, h : B \to W \in \Omega. g \leq h \text{ if } A \subseteq B \subseteq E. \)

Then \( \Omega \) is ordered set by set inclusion. \( \Omega \) is clearly inductive.

Let \( \overline{h} : M \to W \) be a maximal element in \( \Omega \).

To get the proof it is sufficient to show that each \( E_{\alpha} \) is contained in \( M \).

Let \( K_{\alpha} = E_{\alpha} \cap M \).

Then \( (\overline{h} \mid K_{\alpha}) : K_{\alpha} \to W, \) so since \( K_{\alpha} \leq E_{\alpha} \) and \( W \) is \( E_{\alpha}- \)injective, there is an N-homomorphism

\[ \overline{h_{\alpha}} : E_{\alpha} \to W \text{ with } (\overline{h_{\alpha}} \mid K_{\alpha}) = (\overline{h} \mid K_{\alpha}). \]

If \( e_{\alpha} \in E_{\alpha} \) and \( m \in M \) such that \( e_{\alpha} + m = 0 \), then \( e_{\alpha} = -m \in K_{\alpha} \) and \( \overline{h_{\alpha}} (e_{\alpha}) + \overline{h} (m) \)
\[\bar{h} (-m) + \bar{h} (m) = 0.\]

Thus \(f : e_a + m \mapsto \overline{h_a} (e_a) + \bar{h} (m)\) is a well defined \(N\)-homomorphism \(f : E_a + M \to W\).

But \((f \mid M) = \bar{h}\), so by maximality of \(\bar{h}\), \(E_a \subseteq M\).

**Proposition 3.3.2:** \(W\) is \(E\)-injective \(\Rightarrow\) \(W\) is \(N_e\)-injective for all \(e \in E\).

**Proof:** Since \(N_e\) is an \(N\)-subgroup of \(E\). As \(W\) is \(E\)-injective, proposition 3.2.15 implies \(W\) is \(N_e\)-injective.

**Proposition 3.3.3:** Let \(N\) be a dgnr. If \(W\) is a commutative \(N\)-group and \(\{N_e\} \subseteq N_e\) is an independent family of normal \(N\)-subgroups of \(N\)-group \(E\), \(W\) is \(N_e\)-injective for all \(e \in E\), then \(W\) is \(E\)-injective.

**Proof:** \(W\) is \(N_e\)-injective for all \(e \in E\).

So by proposition 3.3.1, \(W\) is \(\oplus_{e \in E} N_e\)-injective.

Since \(E\) is a homomorphic image of \(\oplus_{e \in E} N_e\) by proposition 3.1.8 and since homomorphic image of a \(WcI\) \(N\)-group is \(WcI\) \(N\)-group by proposition 3.2.16.

So \(W\) is \(E\)-injective.

**Proposition 3.3.4:** If a finite direct sum of injective normal \(N\)-subgroups (ideals) of \(E\), i.e.

\[Q = \oplus Q_\alpha\]

where \(Q_\alpha\) is normal \(N\)-subgroup (or ideal) of \(E\) is injective, then each \(Q_\alpha\) is injective.

**Proof:** Let \(Q = \oplus Q_\alpha\) be injective \(N\)-subgroup and consider the \(N\)-monomorphism \(f_\alpha : M \to Q_\alpha\), where \(M\) is some \(N\)-subgroup of \(E\).
\( Q \) is direct sum, for any \( \alpha = 1, 2, 3, \ldots, n \) there is the inclusion map \( i_\alpha : Q_\alpha \to Q \) and the projection on \( \Pi_\alpha : Q \to Q_\alpha \) such that \( \Pi_\alpha i_\alpha = 1_{Q_\alpha} \).

Consider a diagram

\[
\begin{array}{c}
O \xrightarrow{f_\alpha} M \xrightarrow{\Phi} N' \\
\downarrow \quad \downarrow i_\alpha \\
Q_\alpha \xrightarrow{i_\alpha} Q
\end{array}
\]

with top row exact.

Since \( Q \) is injective, there is an \( N \)-homomorphism \( h_\alpha : N' \to Q \), such that \( h_\alpha \Phi = i_\alpha f_\alpha \).

Now define \( \Psi : N' \to Q_\alpha \) by \( \Psi_\alpha = \Pi_\alpha h_\alpha \).

Since \( \Pi_\alpha i_\alpha = 1_{Q_\alpha} \), it follows that \( \Psi_\alpha \Phi = \Pi_\alpha h_\alpha \Phi = \Pi_\alpha i_\alpha f_\alpha = f_\alpha \).

So, the diagram

\[
\begin{array}{c}
O \xrightarrow{f_\alpha} M \xrightarrow{\Phi} N' \\
\downarrow \Psi_\alpha \downarrow \downarrow \Pi_\alpha \\
Q \xrightarrow{i_\alpha} Q_\alpha
\end{array}
\]

is commutative.

Thus \( Q_\alpha \) is injective.

**Proposition 3.3.5:** Let \( N \) be a dgmr. A finite direct sum of injective normal \( N \)-subgroups (ideals) of \( E \), i.e. \( Q = \bigoplus Q_\alpha \), where \( Q_\alpha \) is normal \( N \)-subgroup (or ideal) of \( E \), is injective if each \( Q_\alpha \) is injective.
**Proof:** Let \( Q = \oplus Q_\alpha \) with each \( Q_\alpha \) injective \( N \)-group.

Now consider a diagram

\[
\begin{array}{ccc}
O & \xrightarrow{f} & M \\
\downarrow & & \Phi \downarrow \\
Q & \rightarrow & N'
\end{array}
\]

where \( M, N' \) are \( N \) subgroups of \( E \) with the top row exact.

For any \( \alpha = 1, 2, 3, \ldots, n \), there is the canonical inclusion \( i_\alpha : Q_\alpha \rightarrow Q \) and the projection \( \Pi_\alpha : Q \rightarrow Q_\alpha \), so there are the \( N \)-homomorphisms \( \Pi_\alpha f : M \rightarrow Q_\alpha \).

Since \( Q_\alpha \) is injective there exists a \( N \)-homomorphism \( h_\alpha : N' \rightarrow Q_\alpha \) such that \( h_\alpha \Phi = \Pi_\alpha f \).

Now define a map \( h : N' \rightarrow Q \) by the formula

\[
h(x) = \sum_{\alpha=1}^{n} h_\alpha (x)
\]

\[= (h_1(x) + \ldots + h_n(x)) \quad \forall x \in N'.\]

Then \( h \) is \( N \)-homomorphism.

Since \( h(x_1 + x_2) = (h_1(x_1 + x_2) + \ldots + h_n(x_1 + x_2)) \)

\[= (h_1(x_1) + h_1(x_2) + \ldots + h_n(x_1) + h_n(x_2))\]

\[= h_1(x_1) + \ldots + h_n(x_1) + h_1(x_2) + \ldots + h_n(x_2) \quad [\text{since } Q \text{ is normal } N\text{-subgroup}]\]

\[= h(x_1) + \ldots + h(x_2)\]

\( h(n'x) = (h_1(n'x) + \ldots + h_n(n'x)) \)

\[= h_1(n'x) + \ldots + h_n(n'x)\]
\[= n' h_1(x) + \ldots \ldots + n' h_n(x)\]

\[= \sum_{i=1}^{n} s_i (h_i(x)) + \ldots \ldots + \sum_{i=1}^{n} s_i (h_n(x))\]

\[= s_1((h_1(x)) + \ldots \ldots + h_n(x))+ \ldots \ldots + s_n((h_1(x)) + \ldots \ldots + h_n(x))\]

\[= s_1 h(x) + \ldots \ldots + s_n h(x)\]

\[= (\sum_{i=1}^{n} s_i) h(x) = n' h(x)\].

We shall show the diagram

\[
\begin{array}{c}
\text{O} \\
\downarrow f \\
\text{Q} \\
\downarrow h \\
\text{M} \quad \phi \quad \downarrow \Phi \\
\downarrow f \\
\text{Q} \\
\downarrow \Pi_a \\
\text{N'} \\
\end{array}
\]

\[\text{commutes. i.e. } f = h\Phi.\]

Since Q is direct sum, for any \( x \in \text{N'} \)

\[h\phi(x) = (h_1\phi(x) + h_2\phi(x) + \ldots \ldots + h_n \phi(x))\]

\[= (\Pi_1 f (x) + \Pi_2 f (x) + \ldots \ldots + \Pi_n f (x))\]

\[= f(x)\]

\[\therefore h \phi = f.\]

Thus Q is injective.

**Corollary 3.3.6:** Let N be a dgnr. A finite direct sum of injective normal N-subgroups (ideals) of E, i.e. \( Q = \bigoplus Q_a \), where \( Q_a \) is normal N-subgroup (or ideal) of the group E, is injective if and only if each \( Q_a \) is injective.
Theorem 3.3.7: A finite direct sum of injective $\mathbb{N}$-groups, that is $Q = \oplus Q_\alpha$, where $Q_\alpha$ is $\mathbb{N}$-groups is injective if and only if each $Q_\alpha$ is injective.

Proof: Let $Q$ be injective, to show each $Q_\alpha$ is injective. Proof is same as theorem 3.3.4.

Conversely, let each $Q_\alpha$ be injective, to show $Q$ is injective.

Now consider a diagram

\[
\begin{array}{ccc}
O & \xrightarrow{f} & M \\
\downarrow & & \downarrow \Phi \\
& Q & \xrightarrow{h} \ N'
\end{array}
\]

where $M, N'$ are $\mathbb{N}$ groups with the top row exact.

For any $\alpha = 1, 2, 3, \ldots, n$, there is the canonical inclusion $i_\alpha : Q_\alpha \rightarrow Q$ and the projection $\Pi_\alpha : Q \rightarrow Q_\alpha$, so there are the $\mathbb{N}$-homomorphisms $\Pi_\alpha f : M \rightarrow Q_\alpha$.

Since $Q_\alpha$ is injective, there exists an $\mathbb{N}$-homomorphism $h_\alpha : N' \rightarrow Q_\alpha$ such that $h_\alpha \Phi = \Pi_\alpha f$.

Now define a map $h : N' \rightarrow Q$ by the formula

\[
h(x) = (h_1(x), \ldots, h_n(x)) \quad \forall x \in N'.
\]

Then $h$ is $\mathbb{N}$-homomorphism.

Since $h(x_1 + x_2) = (h_1(x_1 + x_2), \ldots, h_n(x_1 + x_2))$

\[
= (h_1(x_1) + h_1(x_2), \ldots, h_n(x_1) + h_n(x_2))
\]

\[
= (h_1(x_1), \ldots, h_n(x_1)) + (h_1(x_2), \ldots, h_n(x_2))
\]

\[
= h(x_1) + h(x_2)
\]

\[
h(n'x) = (h_1(n'x), \ldots, h_n(n'x))
\]
\[
(n' h_1(x), \ldots \ldots, n'h_n(x))
\]
\[
= n' (h_1(x), \ldots \ldots, h_n(x))
\]
\[
= n'h(x).
\]

We shall show the diagram commutes, i.e. \(f = h \circ \phi\).

Since \(Q\) is direct sum, for any \(x \in N'\)

\[
h \circ \phi(x) = (h_1 \circ \phi(x), h_2 \circ \phi(x), \ldots \ldots, h_n \circ \phi(x))
\]
\[
= (\Pi_1 f(x), \Pi_2 f(x), \ldots \ldots, \Pi_n f(x))
\]
\[
= f(x)
\]

\[\therefore h \circ \phi = f.\]

Thus \(Q\) is injective.

**Theorem 3.3.8:** Let \(N\) be a near-ring and \(\{Q_i\}_{i=1}^n\) a family of \(E\)-injective \(N\)-groups. Then the product \(Q = \Pi_{i=1}^n Q_i\) is \(E\)-injective.

**Proof:** Let \(A \subseteq E\) be an \(N\)-subgroup of \(E\) and \(f : A \rightarrow Q\) an \(N\)-homomorphism.

It is enough to show \(f\) can be extended to \(E\).
For $i \in I$ denote $\pi_i : Q \to Q_i$ the projection map.

Since $Q_i$ is $E$-injective for any $i \in I$, so the $N$-homomorphism $\pi_i f : A \to Q_i$ can be extended to $f'_i : E \to Q_i$. Then we have $f' : E \to Q$ by $f'(e) = (f'_i(e))_{i \in I}$.

If $a \in A$, then $f'(a) = f(a)$, so $f'$ is an extension of $f$.

Thus $Q$ is $E$-injective.

**Definition 3.3.9:** For an $N$-group $A$ an element $x \in A$ is said to be dominated by $N$-group $E$ if $\text{Ann}_N(x) \supset \text{Ann}_N(e)$ for some $e \in E$.

Given a family $\{A_\alpha\}_{\alpha \in J}$ of $N$-groups. Let $x$ be the element of $\prod_{\alpha \in J} A_\alpha$ whose $\alpha$-component is $x_\alpha$.

We define $I_x = \{n \in N/ nx \in \bigoplus_{\alpha \in J} A_\alpha\}$.

Then $x \in \prod_{\alpha \in J} A_\alpha$ is called a special element if $I_x x_\alpha = 0$ for almost all $\alpha$. In other words $\exists$ a finite subset $F$ of $J$ such that $nx_\alpha = 0$ for all $n \in I_x$ and for all $\alpha \in F$.

**Theorem 3.3.10:** If $\bigoplus_{\alpha \in J} A_\alpha$ is $E$-injective then each $A_\alpha$ is $E$-injective and every element of $\prod_{\alpha \in J} A_\alpha$ dominated by $E$ is special.

**Proof:** Let $A = \bigoplus_{\alpha \in J} A_\alpha$ be $E$-injective.

Consider the $N$-homomorphism $f_\alpha : N' \to A_\alpha$.

\[ \cdot \cdot \cdot A \text{ is direct sum, } N' \text{ some } N \text{-group of } N \text{ for any } \alpha \in J, \text{ there is the inclusion map} \]

\[ i_\alpha : A_\alpha \to A \text{ and the projection } \pi_\alpha : A \to A_\alpha \text{ such that } \pi_\alpha i_\alpha = 1_{A_\alpha}. \]

Consider a diagram,
Since $A$ is $E$-injective, there exists a homomorphism $h_a : E \to A$ such that $h_a \Phi = i_a f_a$.

Now define $\Psi_a : E \to A_a$ by $\Psi_a = \pi_a h_a$.

Since $\pi_a i_a = 1_{A_a}$, it follows that $\Psi_a \Phi = \pi_a h_a \Phi = \pi_a i_a f_a = f_a$

So the diagram

Thus $A_a$ is $E$-injective.

Let $x \in \Pi_a A_a$ be dominated by $E \Rightarrow$ there is an $e \in E$ such that $\text{Ann}_N(x) \supset \text{Ann}_N(e)$.

Then it gives an $N$-homomorphism $f : N \to \Pi A_a$ defined by $\lambda e \to \lambda x$ ($\lambda \in N$).

Let $(\lambda_1 e), (\lambda_2 e) \in N e$ and

$f(\lambda_1 e) \neq f(\lambda_2 e)$

$\Rightarrow (\lambda_1 x) \neq (\lambda_2 x)$

$\Rightarrow (\lambda_1 - \lambda_2) x \neq 0$
\[ \Rightarrow (\lambda_1 - \lambda_2) \notin \text{Ann}_N(x) \]

\[ \Rightarrow (\lambda_1 - \lambda_2) \notin \text{Ann}_N(e) \quad \text{[since \( \text{Ann}_N(x) \supseteq \text{Ann}_N(e) \)]} \]

\[ \Rightarrow (\lambda_1 - \lambda_2)e \neq 0 \]

\[ \Rightarrow (\lambda_1 \ e) \neq (\lambda_2 \ e) \]

.: the mapping is well defined.

\[
f(\lambda_1 \ e + \lambda_2 \ e) = f((\lambda_1 + \lambda_2) \ e)
\]

\[ = (\lambda_1 + \lambda_2)x \]

\[ = (\lambda_1 x + \lambda_2 x) \]

\[ = f(\lambda_1 \ e) + f(\lambda_2 \ e) \]

Next for \( n \in \mathbb{N} \), \( f(n(\lambda_1 \ e)) = f(n\lambda_1 \ e) \)

\[ = (n\lambda_1)x \]

\[ = n(\lambda_1 x) \]

\[ = n \ f(\lambda_1 \ e) \]

Thus \( f \) is an \( \mathbb{N} \)-homomorphism.

The image of the \( \mathbb{N} \)-subgroup \( I_x \ e \) by \( f \) is clearly \( I_x x \ (\subset \oplus A_\alpha) \).

Thus the restriction of \( f \) to \( I_x \ e \) is regarded as an \( \mathbb{N} \)-homomorphism \( I_x \ e \rightarrow \oplus A_\alpha \).

Since \( \oplus A_\alpha \) is \( \mathbb{E} \)-injective and so \( \text{Ne} \)-injective by proposition 3.3.2.

So, we get \( \mathbb{N} \)-homomorphism \( \text{Ne} \rightarrow \oplus A_\alpha \) which means that there exists a \( u \in \oplus A_\alpha \) such that \( \lambda x = \lambda u \) (for all \( \lambda \in I_x \)).

It follows that \( I_x x_\alpha = I_x u_\alpha \) for all \( \alpha \in J \).

But since \( u_\alpha = 0 \) for almost all \( \alpha \), it follows that \( I_x x_\alpha = 0 \) for almost all \( \alpha \) too.
\[ \Rightarrow x \text{ is special.} \]

**Theorem 3.3.11:** If \( \{N_e\}_{e \in E} \) is an independent family of normal \( N \)-subgroups of \( N \)-group \( E \) in a dgnr near-ring \( N \), \( \oplus_{\alpha \in J} A_{\alpha} \) is commutative \( N \)-group then each \( A_{\alpha} \) is \( E \)-injective and every element of \( \Pi_{\alpha \in J} A_{\alpha} \) dominated by \( E \) is special implies \( \oplus_{\alpha \in J} A_{\alpha} \) is \( E \)-injective.

**Proof:** let each \( A_{\alpha} \) is \( E \)-injective and every element of \( \Pi_{\alpha \in J} A_{\alpha} \) dominated by \( E \) is special.

Let \( e \in E \) and consider the \( N \)-subgroup \( N_e \) of \( E \).

Let \( J \) be an \( N \)-subgroup of \( N \).

Then \( J e \) is an \( N \)-subgroup of \( N_e \).

[Let \( s, t \in J e, s, t \in J, s + t e = (s + t)e \in J e \) and for \( n \in N, n(se) = (ns)e \in J e \), since \( ns \in J \) as \( J \) is \( N \)-subgroup of \( N \)]

Let there be given an \( N \)-homomorphism \( h : J e \rightarrow \oplus A_{\alpha} \).

Then since \( \oplus A_{\alpha} \subseteq \Pi A_{\alpha} \) and \( \Pi A_{\alpha} \) is \( E \)-injective (as each \( A_{\alpha} \) is \( E \)-injective, by proposition 3.3.8) whence \( N e \)-injective (by proposition 3.3.2), \( h \) can be extended to an \( N \)-homomorphism \( N e \rightarrow \Pi A_{\alpha} \).

Let \( x \in \Pi A_{\alpha} \) and we define the \( N \)-homomorphism as \( \lambda e \rightarrow \lambda x \) \((\lambda \in N)\)

Therefore it follows that \( J x = h (J e) \subseteq \oplus A_{\alpha} \), whence \( J \subseteq I_x \).

On the otherhand since clearly \( Ann_N (e) \subseteq Ann_N (x) \), \( x \) is dominated by \( E \) and thus \( x \) is special by assumption

\[ \Rightarrow I_x x_{\alpha} = 0 \text{ whence } J x_{\alpha} = 0 \text{ for almost all } \alpha. \]
Let $u$ be the element of $\Theta A_\alpha$, whose $\alpha$-component is $x_\alpha$ or 0 according as $J x_\alpha \neq 0$ or $J x_\alpha = 0$.

Then it is clear that $\lambda u = \lambda x$ for all $\lambda \in J$.

Further, it is also clear that $\operatorname{Ann}_M(e) \subset \operatorname{Ann}_N(x) \subset J$ and therefore the mapping gives an $N$-homomorphism $f : Ne \rightarrow \Theta A_\alpha$ which is an extension of $h$, because $f(\lambda e) = \lambda u = \lambda x \forall \lambda \in J$.

This implies that $\Theta A_\alpha$ is $N$-injective and so $E$-injective by proposition 3.3.3.

**Corollary 3.3.12:** Let $N$ be a dgnr. If $\{Ne\}_{e \in E}$ is an independent family of normal $N$-subgroups of $N$-group $E$, $\Theta_{aej} A_\alpha$ is commutative $N$-group then $\Theta_{aej} A_\alpha$ is $E$-injective if and only if each $A_\alpha$ is $E$-injective and every element of $\Pi_{aej} A_\alpha$ dominated by $E$ is special implies $\Theta_{aej} A_\alpha$ is $E$-injective

**Theorem 3.3.13:** Suppose $\{ A_\alpha \}_{aej}$ is a family of $E$-injective $N$-groups such that for every countable subset $k$ of $J$, $\Theta_{aej} A_\alpha$ is $E$-injective. Then $\Theta_{aej} A_\alpha$ is itself $E$-injective.

**Proof:** Assume that $\Theta_{aej} A_\alpha$ is not $E$-injective.

Then by theorem 3.3.10, there exists an $x \in \Pi_{aej} A_\alpha$ which is dominated by $E$ but is not special $\Rightarrow I_\alpha x_\alpha \neq 0$ for infinitely many $\alpha \in J$.

Let $k$ be an infinite countable subset of the infinite set $\{ \alpha \in J / I_\alpha x_\alpha \neq 0 \}$.

Let $y$ be element of $\Pi_{aej} A_\alpha$, whose $\alpha$-component $y_\alpha$ is equal to $x_\alpha$ for all $\alpha \in K$.

Then clearly $I_\alpha \subset I_y$, so that it follows that $y$ is dominated by $E$ and $I_\alpha y_\alpha = I_y x_\alpha \neq 0 \forall \alpha \in K$.

This implies again by theorem 3.3.10, that $\Theta_{aej} A_\alpha$ is not $E$-injective (because each $A_\alpha$ is $E$-injective by our assumption). This is a contradiction and so the proof is complete.
3.4: E-injective and injective N-groups with chain conditions:

In this section we study E-injective N-groups with chain conditions. In particular, E-injective N-groups with descending chain condition are investigated. It is shown that the singular and semi-simple characters play a vital role in characterization of E-injective N-groups.

**Theorem 3.4.1:** Let N be dgnr. If \( \{ N_e \}_{e \in E} \) is an independent family of normal N-subgroups of N-group E, \( \Theta_{\alpha \in J} A_\alpha \) is commutative N-group then direct sum of any family \( \{ A_\alpha \} \) of E-injective N-groups is E-injective if E is Noetherian.

**Proof:** let \( \{ A_\alpha \} \) be a family of E-injective N-group.

Let x be an element of \( \Pi A_\alpha \), dominated by e.

Then there is an \( e \in E \) such that \( \text{Ann}_N(x) \subset \text{Ann}_N(e) \).

Consider \( I_x(e) \).

Since clearly \( \text{Ann}_N(x) \subset I_x \), whence \( \text{Ann}_N(e) \subset I_x \), it follows that \( I_x / \text{Ann}_N(x) \cong I_x(e) \).

On the other hand \( I_x(e) \) is a N-subgroup of \( N_e \), so N subgroup of Noetherian N-group E.

Hence, \( I_x / \text{Ann}_N(x) \) is finitely generated

\( \Rightarrow \) there exists a finite number of elements \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of \( I_x \) such that

\[ I_x = N\lambda_1 + N\lambda_2 + \cdots + N\lambda_n + \text{Ann}_N(x) \]

It follows therefore

\[ I_x x_\alpha = N\lambda_1 x_\alpha + N\lambda_2 x_\alpha + \cdots + N\lambda_n x_\alpha \] for all components \( x_\alpha \).

Since however for each i, \( \lambda_1 x_\alpha = 0 \), for almost all \( \alpha \), it follows that \( I_x x_\alpha = 0 \) for almost all \( \alpha \).
$\Rightarrow \text{x is special.}$

Thus $\oplus A_\alpha$ is $\mathcal{E}$-injective by theorem 3.3.11.

**Proposition 3.4.2:** If $\{N_e\}_{e \in \mathcal{E}}$ is an independent family of normal $\mathcal{N}$-subgroups of $\mathcal{N}$-group $E$ in a dgnr near-ring $\mathcal{N}$, direct sum of $\mathcal{E}$-injective $\mathcal{N}$-groups is commutative $\mathcal{N}$-group then $E$ is Noetherian $V \mathcal{N}$-group($V_c \mathcal{N}$-group) implies every strictly semi- simple $\mathcal{N}$-group is $\mathcal{E}$-injective.

**Proof:** $E$ is Noetherian $V$- $\mathcal{N}$-group

$\Rightarrow E$ is Noetherian and every simple $\mathcal{N}$-group is $\mathcal{E}$- injective.

Again direct sum of $\mathcal{E}$-injective $\mathcal{N}$-groups is $\mathcal{E}$- injective as $E$ is Noetherian

(by theorem 3.4.1).

Let $K$ be any strictly semi simple $\mathcal{N}$-group

$\Rightarrow K$ is direct sum of simple normal $\mathcal{N}$-subgroups.

So $K$ is $\mathcal{E}$- injective.

**Proposition 3.4.3:** For a finitely generated $\mathcal{N}$-group $E$ every countably generated strictly semi- simple $\mathcal{N}$-group is $\mathcal{E}$- injective implies $E$ is weakly Noetherian $V_c \mathcal{N}$-group.

**Proof:** Suppose $\{A_\alpha\}_{\alpha \in J}$ is a family of $\mathcal{N}$-groups such that for every countable subset $K$ of $J$, $\bigoplus_{\alpha \in K} A_\alpha$ is $\mathcal{E}$- injective. Then by theorem 3.3.13 $\bigoplus_{\alpha \in J} A_\alpha$ itself $\mathcal{E}$-injective.

Now given that every countably generated strictly semi simple $\mathcal{N}$-group is $\mathcal{E}$-injective.

To show $E$ is weakly Noetherian and every simple commutative $\mathcal{N}$-group is $\mathcal{E}$-injective.

Let $U$ be a countably generated strictly semi- simple $\mathcal{N}$-group.
Then \( U = \bigoplus U_\alpha \), where \( U_\alpha \) is simple normal \( N \)-subgroups, so \( U_\alpha \)'s can be taken as commutative \( N \)-groups and \( \alpha \in K \), \( K \) is countable subset of \( J \) (as \( U \) countably generated).

Given \( U \) is \( E \)-injective. So we have \( \bigoplus U_\alpha \), \( \alpha \in J \) is also \( E \)-injective (By theorem 3.3.13).

So by theorem 3.3.10, we get every \( U_\alpha \) is \( E \)-injective

\[ \Rightarrow E \text{ is } V_c \text{ } N \text{-group.} \]

Next to show \( E \) is weakly Noetherian.

Given \( E \) is finitely generated and \( W \) countably generated semi-simple \( N \)-group & \( W \) is \( E \)-injective.

Let \( N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \ldots \) be an ascending chain of distinct ideals of \( E \).

Let \( f_k : N_k \rightarrow W \) \( (k = 1, 2, 3, \ldots \ldots \infty) \)

As \( W \) is \( E \)-injective, for inclusion map \( i_k : N_k \rightarrow E \), \( \exists \) a map \( \gamma_k : E \rightarrow W \) s.t. \( f_k = \gamma_k i_k \)

Let \( N' = \Sigma_{k=1}^\infty N_k \)

Define the map \( f : N' \rightarrow W \) by

\[ f(x) = \Sigma_{k=1}^\infty f_k(x) = \Sigma_{k=1}^\infty \gamma_k i_k(x) \]

\( f \) is well defined.

\( \because W \) is \( E \)-injective, \( \exists \) a map \( g : E \rightarrow W \) extending \( f \).

But \( E \) is finitely generated & \( g(E) \subseteq W \), \( W \) countably generated. So \( g \) can be defined as

\[ g(x) = \sum_{k=1}^m \gamma_k i_k(x) \]

for some positive integer \( m \), which gives chain of ideals must be finite.
**Corollary 3.4.4:** For a finitely generated $N$-group $E$, every strictly semi-simple $N$-group is $E$-injective implies $E$ is weakly Noetherian $Vc$ $N$-group.

**Proposition 3.4.5:** For $dgnr$ $N$, if $E$ is a finitely generated $S^3I$-$N$-group, then $\frac{E}{\text{Soc}(E)}$ is a weakly Noetherian $Vc$ $N$-group.

**Proof:** From the above corollary 3.4.4, it is enough to show that every strictly semi-simple $N$-group is $\frac{E}{\text{Soc}(E)}$ injective.

Let $L$ be a strictly semi-simple $N$-group.

So as $N$ $dgnr$, $L$ is a semi-simple $N$-group.

Let $\frac{M}{\text{Soc}(E)}$ be an ideal of $\frac{E}{\text{Soc}(E)}$. $f: \frac{M}{\text{Soc}(E)} \rightarrow L$ is a non-zero $N$-homomorphism.

Let $\frac{K}{\text{Soc}(E)} = \text{Ker} f$.

We claim $K$ is essential ideal in $M$.

For if $K \cap I = 0$ for some non-zero ideal $I$ of $M$ then $I \equiv \frac{I+K}{K}$ and since the latter is isomorphic to an ideal of $L$, it follows that for some ideal $I_1 \neq 0$ and contained in $I$ that $I_1 \subseteq L$, hence $I_1 \subseteq \text{Soc}(E) \subseteq K$, a contradiction.

Now $\frac{M}{K}$ singular, we may take $L$ singular, since $f(\frac{M}{K}) \subseteq Z(L)$.

Let $\eta: M \rightarrow \frac{M}{\text{Soc}(E)}$ denote the quotient map and consider the map $f.\eta: M \rightarrow L$.

$L$ is $E$-injective $f.\eta$ extends to a map of $E$ into $L$.

$\triangleright$ $\text{Soc}(E) \subseteq K$. This yields a map of $\frac{E}{\text{Soc}(E)}$ into $L$ by proposition 3.2.3.
Proposition 3.4.6: Let $N$ be a dgnr If $E$ is an $N$-group satisfying the following conditions

(i) $\{Ne\}_{e \in E}$ is an independent family of normal $N$-subgroups of $E$,
(ii) direct sum of $E$-injective $N$-groups is a commutative $N$-group
(iii) No non-zero homomorphic image of $Nx$, $\forall x(\neq 0) \in Soc(E)$, is semi-simple,
singular
(iv) $\frac{E}{Soc(E)}$ is Noetherian $V$ $N$-group,

then $E$ is an $S^3I$-$N$-group.

Proof: Let $L$ be a strictly semi-simple singular $N$-group.

Let $M$ be an $N$-subgroup of $E$.

$f: M \rightarrow L$ a non-zero map with $\ker f = K$.

Then by given condition $Soc(E) \cap M$ is contained in $K$.

[For $x \in Soc(E) \cap M \Rightarrow x \in Soc(E)$, $x \in M \Rightarrow Nx \subseteq Soc(E)$, $Nx \subseteq M \Rightarrow Nx \in Soc(E) \cap M$].

So by proposition 3.2.3, $\exists$ an $N$-homomorphism $f': \frac{M}{Soc(E) \cap M} \rightarrow L$.

Since $\frac{M}{Soc(E) \cap M} \cong \frac{Soc(E) + M}{Soc(E)}$, so $f': \frac{Soc(E) + M}{Soc(E)} \rightarrow L$.

As $\frac{E}{Soc(E)}$ is Noetherian $V$ $N$-group and $L$ semi-simple singular by proposition 3.4.2, $L$ is

$\frac{E}{Soc(E)}$-injective, that is $f'$ is extended to $g': \frac{E}{Soc(E)} \rightarrow L$.

If we define $g: E \rightarrow L$ by $g(e) = g'(\bar{e} + Soc(E))$. $g$ is extension of $f$. 
Proposition 3.4.7: Let $E$ be an $N$-group. Then $E/M$ is weakly Noetherian for every essential ideal $M$ of $E$ if and only if $E$ has A.C.C. on essential ideals.

Proof: Let $M$ be an essential ideal of $E$.

Then $E/M$ weakly Noetherian

We show $E$ has A.C.C. on essential ideals.

Let $M_1 \subset M_2 \subset M_3 \subset \ldots \ldots$ \rightarrow (1) be a chain of ideals of $E$ where $M_i \leq E$.

Considering an essential $N$-subgroup $M \subseteq M_i \forall i$, we can construct another chain

$M_i/M \subset M_2/M \subset M_3/M \subset \ldots \ldots$ of $E/M$.

Since $E/M$ is weakly Noetherian we get $M_i/M = M_{i+1}/M$ for some $i$.

Now $M_i \subset M_{i+1}$. Our aim is to show $M_{i+1} \subset M_i$.

Let $x_{i+1} \in M_{i+1}$ but $x_{i+1} \notin M$.

Then $x_{i+1} + M \in M_{i+1}/M \Rightarrow x_{i+1} + M \in M_i/M \Rightarrow x_{i+1} \in M_i$ (since $x_{i+1} \notin M$).

So $M_i = M_{i+1}$.

$\Rightarrow E$ has A.C.C. on essential ideals.

Converse is clear.

Proposition 3.4.8: $N$-group $E$ is almost weakly Noetherian if and only if $E/M$ is weakly Noetherian for every essential ideal $M$ of $E$.

Proof: Let $E/SocE$ be weakly Noetherian.

We know if $N$ ideal of $M$, $M$ weakly Noetherian $\Leftrightarrow N \& M/N$ weakly Noetherian, by proposition 4.1.7.
M is essential ideal of E and SocE is the intersection of all essential ideals \( \Rightarrow \text{Soc} \ E \subseteq M \).

\[ \Rightarrow \frac{E}{\text{Soc} \ E} \text{ is weakly Noetherian } \iff \frac{M}{\text{Soc} \ E} \text{ and } \frac{E}{\text{Soc} \ E} \cong \frac{E}{M} \text{ weakly Noetherian.} \]

Conversely, \( \frac{E}{M} \) is weakly Noetherian for every essential ideal M of E.

We show \( \frac{E}{\text{Soc} \ E} \) is weakly Noetherian. It is enough to show that every essential ideal of \( \frac{E}{\text{Soc} \ E} \)
is finitely generated by proposition 3.4.7.

Let \( \frac{M}{\text{Soc} \ E} \) be an essential ideal of \( \frac{E}{\text{Soc} \ E} \).

Let \( k \) be an ideal of \( M \) maximal with respect to \( K \cap \text{Soc} \ E = 0 \).

Then \( K \oplus \text{Soc} \ E \) is essential in \( M \) and hence essential in \( E \).

\( [K \oplus \text{Soc} \ E \text{ ideal of } M, \text{ let } M' \text{ ideal of } M \text{ such that } M' \cap (K \oplus \text{Soc} \ E) = 0. \text{Then } M' \oplus (K \oplus \text{Soc} \ E) \text{ is a direct sum } \Rightarrow M' \oplus K \oplus \text{Soc} \ E \text{ is a direct sum.} \text{Whence } (M' \oplus K) \cap \text{Soc} \ E ' = 0. \text{By maximality of } K, (M' \oplus K) = K, \text{ i.e } M' = 0.] \)

Then \( \frac{E}{K \oplus \text{Soc} \ E} \) is weakly Noetherian. So \( \frac{M}{K \oplus \text{Soc} \ E} \) is finitely generated.

From the exactness of the sequence \( 0 \rightarrow K \rightarrow \frac{M}{\text{Soc} \ E} \rightarrow \frac{M}{K \oplus \text{Soc} \ E} \rightarrow 0 \), it suffices to show \( K \) is finitely generated.

We claim that \( K \) is finite dimensional.

For, if not \( \exists \) an infinite direct sum of non-zero ideals \( \oplus_{i \in I} K_i \) which is essential in \( K \).

Since \( K_i \cap \text{Soc} \ E = 0 \), each \( K_i \) has a proper essential ideal \( T_i \).

[since \( K_i \cap \text{Soc} \ E = \text{Soc} \ K_i = 0 \).]
Let $T = \bigoplus_{i \in I} T_i$.

Then $T$ is an essential ideal of $K$.

Let $K'$ be an ideal of $K$, $T = \bigoplus_{i \in I} T_i$, where $T_i$ are essential ideals of $K$.

Now $K' = \bigoplus_{i \in I} K'_i$, $K'_i \subseteq K_i$. Then $T_i \cap K'_i \neq 0$

$\Rightarrow \bigoplus_{i \in I} T_i \cap K'_i \neq 0$

$\Rightarrow T \cap \bigoplus_{i \in I} K'_i \neq 0$.

$\Rightarrow T \cap K' \neq 0$.

Again $\text{Soc}E$ is an essential ideal of $\text{Soc}E$ and $T \cap \text{Soc}E = 0$.

So $T \oplus \text{Soc}E \leq K \oplus \text{Soc}E \Rightarrow T \oplus \text{Soc}E$ is an essential ideal of $E$.

Hence $E / T \oplus \text{Soc}E$ is weakly Noetherian,

As ideal of a weakly Noetherian $N$-group is weakly Noetherian, $\frac{\bigoplus_{i \in I} K_i}{T \oplus \text{Soc}E}$ is weakly Noetherian.

$\Rightarrow \frac{\bigoplus_{i \in I} T_i}{T \oplus \text{Soc}E}$ is weakly Noetherian.

$\frac{\bigoplus_{i \in I} T_i}{T \oplus \text{Soc}E} \leq \frac{\bigoplus_{i \in I} K_i}{T \oplus \text{Soc}E}$ and $\frac{\bigoplus_{i \in I} K_i}{T \oplus \text{Soc}E}$ weakly Noetherian imply $\frac{\bigoplus_{i \in I} K_i}{T \oplus \text{Soc}E} \cong \frac{\bigoplus_{i \in I} K_i}{\bigoplus_{i \in I} T_i}$ weakly Noetherian, a contradiction, since it is an infinite direct sum of non-zero $N$-groups.

Thus $K$ is finite dimensional.

Let $(K_i)_{i=1}^n$ be a family of non-zero ideals of $K$ such that $\bigoplus_{i=1}^n K_i$ is essential in $K$.

$\Rightarrow \bigoplus_{i=1}^n K_i \leq K$, so $\bigoplus_{i=1}^n K_i \oplus \text{Soc}E \leq K \oplus \text{Soc}E \leq E$. 
$\Rightarrow \oplus_{i=1}^{n} K_i \oplus \text{Soc}E \leq E.$

$\Rightarrow \frac{E}{\oplus_{i=1}^{n} K_i \oplus \text{Soc}E}$ is weakly Noetherian.

We define $f : \frac{K}{\oplus_{i=1}^{n} K_i} \rightarrow \frac{K}{\oplus_{i=1}^{n} K_i \oplus \text{Soc}E}$ by $f(k + \oplus_{i=1}^{n} K_i) = f(k + \oplus_{i=1}^{n} K_i \oplus \text{Soc}E)$

Now $f(k_1 + \oplus_{i=1}^{n} K_i) \neq f(k_2 + \oplus_{i=1}^{n} K_i)$

$\Rightarrow (k_1 + \oplus_{i=1}^{n} K_i \oplus \text{Soc}E) \neq (k_2 + \oplus_{i=1}^{n} K_i \oplus \text{Soc}E)$

Next, let $\bar{k} \in \frac{K}{\oplus_{i=1}^{n} K_i \oplus \text{Soc}E}$.

If $\bar{k} = k_1 + \oplus_{i=1}^{n} K_i \oplus \text{Soc}E$, $\exists k_1 + (\oplus_{i=1}^{n} K_i) \in \frac{K}{\oplus_{i=1}^{n} K_i}$ such that

$f(k_1 + (\oplus_{i=1}^{n} K_i)) = k_1 + (\oplus_{i=1}^{n} K_i \oplus L).$

So $f$ is onto, that is $f$ is isomorphism.

Thus $\frac{K}{\oplus_{i=1}^{n} K_i}$ is isomorphic to the ideal $\frac{K}{\oplus_{i=1}^{n} K_i \oplus \text{Soc}E}$ of weakly noetherian $N$-group

$\frac{E}{\oplus_{i=1}^{n} K_i \oplus \text{Soc}E}$. So we have that $\frac{K}{\oplus_{i=1}^{n} K_i}$ is finitely generated, whence $K$ is finitely generated.

Thus $\frac{E}{\text{Soc}E}$ is weakly Noetherian.

**Proposition 3.4.9:** If $N$-group $E$ is almost weakly Noetherian then $E$ has A.C.C. on essential ideals.

**Proof:** Given $\frac{E}{\text{Soc}E}$ is weakly Noetherian.

To show $E$ has A.C.C. on essential ideals.
Soc E is the intersection of all essential ideals of E.

Hence if \( \frac{E}{\text{Soc} E} \) is weakly Noetherian, E has A.C.C. on essential ideals.

**Proposition 3.4.10:** Let N be a dgnr. If N-group E has A.C.C. on essential ideals then E is almost weakly Noetherian.

**Proof:** We assume that E has A.C.C. on essential ideals.

Let \( A \subseteq B \) be ideals of M such that A is essential in B.

By Zorn's lemma there is a maximal ideal L of E such that \( L \cap A = 0 \).

And \( A \oplus L \) is essential in E.

Since \( A + L = A \oplus L \), so that \( A \oplus L \) is an ideal of E. Let C ideal of E with \( C \cap (A \oplus L) = 0 \). Then \( (A \oplus L) \oplus C \) is direct \( \Rightarrow (A \oplus L) + C = (A \oplus L \oplus C) \) whence \( A \cap (L \oplus C) = 0 \). By maximality of L we obtain \( L \oplus C = L \) Thus \( C = 0 \). \( A \oplus L \) essential ideal of E.

Hence \( E/(A \oplus L) \) satisfies ACC on its ideals.

We consider the map \( \phi : B \oplus L \rightarrow B/A \) by \( b + l \rightarrow b + A \). [N dgnr]

Now \( \phi(b_1 + l_1 + b_2 + l_2) \)

\[ = \phi(b_1 + b_2 + l_1 + l_2) \]

\[ = (b_1 + b_2) + A \]

\[ = b_1 + A + b_2 + A \]

\[ = \phi(b_1 + l_1) + \phi(b_2 + l_2) \]

Again, \( \phi(n + l) \)

\[ = \phi(n_1 + n_2 + n_3 + \ldots + n_k)(b + l) \]
\[ \phi \{ n_1(b+1) + n_2(b+1) + \ldots + n_k(b+1) \} \]
\[ = \phi \{ (n_1 b + n_1 l) + (n_2 b + n_2 l) + \ldots + (n_k b + n_k l) \} \]
\[ = (n_1 b + A) + (n_2 b + A) + \ldots + (n_k b + A) \]
\[ = (n_1 b + n_2 b + \ldots + n_k b) + A \]
\[ = nb + A \]
\[ = n(b + A) \]
\[ = n\phi(b + l) \]

So \( \phi \) is an \( N \)-homomorphism.

\[ \text{Ker} \phi = \{ x / \phi(x) = A \} \]
\[ = \{ a + l / \phi(a + l) = A \} \]
\[ = A + L \]

As \( A \leq B \) and \( B \cap L = 0 \), \( A \cap L = 0 \).

\[ \therefore \text{Ker} \phi = A \oplus L \]

So \( B/A \cong (B \oplus L)/(A \oplus L) \).

Hence we get \( B/A \) also satisfies acc on its ideals.

In particular, every uniform ideal of \( E \) satisfies acc on its ideals.

Since if \( I \) is uniform ideal of \( E \) and \( J_1 \subseteq J_2 \subseteq \ldots \) an ascending chain of ideals of \( I \). As \( I \) is uniform, each \( J_i \leq I \).

\[ \Rightarrow I/J_i \text{ satisfies acc on its ideals.} \]

\[ \Rightarrow I \text{ satisfies acc on essential ideals. (by proposition 3.4.7)} \]
As each $J_i \leq J_j$, $\exists t$ such that $J_t = J_{t+1} \Rightarrow I$ satisfies acc on its ideals.

Now, let $H$ be an ideal of $E$ which is maximal with respect to the condition $H \cap \text{Soc}(E) = 0$.

Then $H \oplus \text{Soc}(E)$ is essential in $E$ and $E/H \oplus \text{Soc}(E)$ satisfies acc on its ideals.

Hence for proving that $E/\text{Soc}(E)$ satisfies acc on its ideals it is enough to prove that $H$ satisfies acc on its ideals.

We first show that $H$ has finite Goldie dimension.

Assume that $H$ contains an infinite direct sum $X = X_1 \oplus X_2 \oplus \ldots \ldots$ of non-zero ideals $X_i$.

Since, $\text{Soc}(X_i) = X_i \cap \text{Soc}(E)$, each $X_i$ contains a proper essential ideal $Y_i$ and

$Y = Y_1 \oplus Y_2 \oplus \ldots \ldots$ is an essential ideal of $X$.

By the above $X/Y$ satisfies acc on its ideals.

But this is impossible because

$X/Y = X_1/Y_1 \oplus X_2/Y_2 \oplus \ldots \ldots$ with each $X_i/Y_i$ non-zero.

This contradiction shows that $H$ has finite Goldie dimension $k$ (say). Then $H$ contains $k$ independent uniform ideals $U_i$ such that $U = U_1 \oplus U_2 \oplus \ldots \ldots \oplus U_k$ is essential in $H$.

By the above $U$ and $H/U$ satisfies acc on ideals.

Hence $H$ satisfies acc on ideals.

**Proposition 3.4.11**: if $E$ is non-singular and Every singular homomorphic image of $E$ is weakly Noetherian then $E$ is almost weakly Noetherian.
Proof: As $M$ is essential ideal of $E$ and $E$ is non-singular, $E/M$ is singular. Again $E/M$ is homomorphic image of $E$, by given condition $E/M$ is weakly Noetherian.

**Proposition 3.4.12:** $E$ is non-singular and almost weakly Noetherian and in $E$ every weakly essential $N$-subgroup is essential then every singular homomorphic image of $E$ is weakly Noetherian.

**Proof:** Let $f: E 	o L$ be an $N$-epimorphism and $L$ is singular.

Now $E$ is non-singular and $\ker f \leq E$, $L \cong E/\ker f$ singular,
so $\ker f \leq \text{soc} E$ by proposition 3.1.7.

Then $\text{Soc}(E) \subseteq \ker f$.

So by proposition 3.2.3 we get $L \cong E/\text{Soc}(E)$.

As $E$ is almost weakly Noetherian, $L$ is weakly Noetherian.

**Corollary 3.4.13:** The following conditions on an $N$-group $E$ of a dgnr near-ring $N$ are equivalent:

i. $E$ is almost weakly Noetherian.

ii. $E/M$ is weakly Noetherian for every essential ideal $M$ of $E$.

iii. $E$ has A.C.C. on essential ideals.

Moreover if $E$ is non-singular, every weakly essential $N$-subgroup is essential
then above conditions are equivalent to

iv. Every singular homomorphic image of $E$ is weakly Noetherian.

**Proposition 3.4.14:** Near-ring $N$ is weakly Noetherian if $\bigoplus_{i \in I} E_i$ of injective $N$-groups is injective.
Proof: Let $\oplus_{i=1}^{\infty} E_i$ of commutative $N$-groups is injective and that

$I_1 \leq I_2 \leq \ldots$ be an ascending chain of left ideals in $N$.

Let $I = \bigcup_{i=1}^{\infty} I_i$.

If $a \in I$, then $a \in I_i$ for all but finitely many $I_i \in N$.

So there is an

$$f : I \to \oplus_{i=1}^{\infty} E(N/I_i)$$

defined via $\Pi_i f(a) = a + I_i \quad (a \in I)$.

By theorem 4.1.9, there is an $x \in \oplus_{i=1}^{\infty} E(N/I_i)$ such that $f(a) = ax$ for all $a \in I$. Now choose $n$ such that $\Pi_{n+k} I(x) = 0$, $k = 0, 1, \ldots$.

So $I/ I_{n+k} = \Pi_{n+k}(f(I)) = \Pi_{n+k}(I_x) = L\Pi_{n+k}(x) = 0$

or, equivalently, $I_n = I_{n+k}$ for all $k = 0, 1, 2, \ldots$.

So, $N$ is weakly Noetherian.

Definition 3.4.15: An $N$-subgroup $U$ of $N$-group $E$ is called pure in $E$ if $IU = U \cap IE$ for each ideal $I$ of $N$.

Example 3.4.16: $N = \{0, a, b, c\}$ is the Klein’s four group with multiplication

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Then \((N, +, \cdot)\) is a near-ring. Here \(A = \{0, c\}\) is \(N\)-subgroup of \(N\) and \(B = \{0, b\}\) is ideal of \(N\).

Now \(BA = \{0\}\) and \(A \cap BN = \{0, c\} \cap \{0, b\} = \{0\}\). So \(BA = A \cap BN\). So, \(A\) is pure in \(N\).

**Proposition 3.4.17:** If \(N\) is non-singular, \(\text{Soc}N\) is pure and every injective right \(N\)-group is injective as an \(N/K\)-group for ideal \(K\) of \(N\) then direct sum of (countably many) injective hulls of simple weak singular left \(N\)-groups is injective implies \(N\) is an almost weakly Noetherian near-ring.

**Proof:** Let \(\{S_i\}_{i \in I}\) be a family of simple weak singular \(N/\text{Soc}(N)\)-groups.

Since a simple \(N\)-group is weak singular if and only if it is annihilated by \(\text{Soc}(N)\).

For let \(E\) is simple and weak singular. So \(Z_{\text{soc}}(E) = \{x \in E / Ix = 0, I \leq N\} = E\).

So \(x \in E \Rightarrow \exists I \leq N\) such that \(Ix = 0 \Rightarrow \text{Soc}(N)x = 0\). Thus \(E\) is annihilated by \(\text{Soc}(N)\).

Again let \(E\) is annihilated by \(\text{Soc}(N)\), we get \(\text{Soc}(N)E = 0\).

\[\Rightarrow \text{Soc}(N) \subseteq \text{Ann}(E).\]

Now we show \(\text{Ann}(E) = \{x \in N / xE = 0\}\) is essential ideal in \(N\).

If possible \(\text{Ann}(E)\) is not essential ideal in \(N\).

Then \(\text{Ann}(E) \cap J = 0\) for some non-zero ideal \(J\) of \(N\).

If \(\forall x \in E\) \(f : J \to Jx\), defined by \(f(j) = jx\), it is a well defined \(N\)-homomorphism.

\[f(j_1) \neq f(j_2) \Rightarrow (j_1)x \neq (j_2)x \Rightarrow (j_1 - j_2)x \neq 0 \Rightarrow (j_1 - j_2) \neq 0 \Rightarrow j_1 \neq j_2.\] So \(f\) is well-defined.
Next let \( j_1 \neq j_2 \Rightarrow (j_1 - j_2) \neq 0 \Rightarrow (j_1 - j_2)x \neq 0 \Rightarrow (j_1x) \neq (j_2x) \Rightarrow f(j_1) \neq f(j_2) \).

So \( f \) is one-one.

Again for every \( jx \in Jx \), \( \exists j \in J \) such that \( f(j) = jx \). So \( f \) is onto.

\[
f(j_1 + j_2) = (j_1 + j_2)x = (j_1x + j_2x) = f(j_1) + f(j_2),
\]

\[
f(nj) = (nj)x = n(jx) = nf(j).
\]

So \( f \) is \( N \)-isomorphism.

\( \Rightarrow \forall x \in E, J \ni Jx. \)

Again \( Z(N) = 0 \Rightarrow Z(J) = 0 \Rightarrow Z(Jx) = 0 \)

\( \Rightarrow \forall I \leq N, I(Jx) \neq 0 \Rightarrow \text{Soc}_{N}(Jx) \neq 0. \)

But \( Jx \subseteq E \) and \( \text{Soc}_{N}E = 0 \Rightarrow \text{Soc}_{N}(Jx) = 0 \), a contradiction.

So \( \text{Ann}(E) \) is essential ideal of \( N \), so \( E \) is weak singular.

It follows that each \( N_{S_i} \) is weak singular as \( N \)-group.

Since \( \text{Soc}_{N} \) is pure we get \( \text{Soc}_{(N_{S_i})}E_{(N_{S_i})} \cap N_{S_i} = \text{Soc}_{N}S_i, \forall i \in I. \)

As each \( N_{S_i} \) is annihilated by \( \text{Soc}(N), \)

\( \text{Soc}_{N}S_i = 0. \) So \( \text{Soc}_{(N_{S_i})}E_{(N_{S_i})} \cap N_{S_i} = 0. \) i.e. \( \forall x \in E_{(N_{S_i})}, \text{Soc}_{(N_{S_i})}x \cap N_{S_i} = 0. \)

\( E_{(N_{S_i})} \) is an essential extension of \( N_{S_i} \) and since \( \text{Soc}_{(N_{S_i})}x \) is \( N \)-subgroup of \( E_{(N_{S_i})} \) we get

\( \forall x \in E_{(N_{S_i})}, \text{Soc}_{(N_{S_i})}x = 0. \)

Thus \( E_{(N_{S_i})} \) is annihilated by \( \text{Soc}(N), \forall i \in I. \)

We claim that \( \forall i \in I, E_{(N_{S_i})} \) is weak singular as \( N \)-group.
For $x \in E_{(N)}$ with $x \in Z(E_{(N)})$ then $\forall I \leq N$, $Ix \neq 0 \Rightarrow \text{Ann}_N(x)$ is not essential in $N$.

So $\text{Ann}_N(x) \cap J = 0$ for some non-zero ideal $J$ of $N$.

Since $J \cong Jx$ and $Z(N) = 0$, we infer that $Z(Jx) = 0$, whence $Jx \cap S_i = 0$

[Let $Jx \cap S_i \neq 0$.

$Z(Jx \cap S_i) = 0 \Rightarrow \forall I \leq W N, I(Jx \cap S_i) \neq 0 \Rightarrow \text{Soc} N(Jx \cap S_i) \neq 0$.

But $(Jx \cap S_i) \subseteq E_{(N)}$ and $\text{Soc} N.E_{(N)} = 0$, a contradiction].

This implies that $Jx = 0$.

So $J \subseteq \text{Ann}_N(x)$, a contradiction.

Now $E_{(N/\text{Soc}(N) S_i)} = \{ x \in E_{(N)} : \text{Soc}(N)x = 0 \} = E_{(N)}$ is injective as $N$-group.

By given condition $\oplus_{i \in I} E_i$ is injective as an $N$-group and hence injective as $N/\text{Soc}(N)$-group. This implies that $N/\text{Soc}(N)$ is weakly Noetherian by proposition 3.4.14.

For a distributively generated near-ring we get the following definition, note and three results.

**Definition 3.4.18 [Pliz]:** The Jacobson-radical of $N$-group $E$ is the intersection of maximal ideals of $E$ which is maximal as $N$-subgroup. We denote it by $J_2(E)$

**Note 3.4.19 [Pliz]:** The Jacobson-radical, $J_2(E)$ of $N$-group $E$ contains all nilpotent $N$-subgroups of $E$.

**Lemma 3.4.20:** Let $N$ be a GV- near-ring, then $Z(E) \cap J_2(E) = 0$, for every $N$-group $E$.

**Proof:** If $Z(E) = 0$, we are done.

Otherwise let $(0 \neq) x \in Z(E)$.

By Zorn's lemma, the set of all ideals $M$ of $E$ with $x \in M$, has a maximal member $L$.

The quotient $N$-group $S = (N + L)/L$ is simple and singular, therefore $E$-injective.
Let $\bar{y} \in (Nx + L)/L$ such that $\bar{y} = nx + 1 + L$.

Now for some essential $N$-subgroup $I$ in $N$,

$Iy = \left\{ \frac{n'y}{n' \in I} \right\}$

$= \left\{ \Sigma_{i=1}^{k} s_i (nx + L) / n' = (\Sigma_{i=1}^{k} s_i) \in I \right\}$

$= \left\{ s_1(nx + L) + s_2(nx + L) + \ldots + s_k(nx + L) / n' \in I \right\}$

$= \left\{ s_1 nx + L + s_2 nx + L + \ldots + s_k nx + L / n' \in I \right\}$

$= \left\{ (s_1 nx + s_2 nx + \ldots + s_k nx) + L / n' \in I \right\}$ [since $s_i nx \in L$ as $s_i \in N$]

$= \left\{ L \right\} = \bar{0}$.

So $\bar{y} \in Z((Nx + L)/L)$.

This means that the natural map of $Nx$ onto $S$ extends to all of $E$.

The kernel of this extension map is a maximal ideal of $E$ which does not contain $x$. Whence $x$ can not be in $J_2E$.

So $Z(E) \cap J_2(E) = 0$

**Theorem 3.4.21:** If $N$ is a GV near-ring with A.C.C. on essential ideals and if finite intersection of essential $N$-subgroups of $N$ is distributively generated, then $Z(N) = 0$. In particular, if $N$ is $S^3$ I near-ring with unity then it is non-singular.

**Proof:** Let $x \in Z(N)$.

Then $\text{Ann}_N(x) \subseteq \text{Ann}_N(x^2) \subseteq \ldots$ is an ascending chain of essential left ideals in $N$,

since $\text{Ann}_N(x) \leq N$.

So for some $t \in I^+$, $\text{Ann}_N(x^{t+1}) \leq N$ by proposition 1.3.3.

We claim $x^t = 0$.

Suppose $x^t \neq 0$. 

Then we get $\text{Ann}_N(x^{t+1}) \cap Nx^t \neq 0$.

As $N$ has A.C.C. on essential left ideals $\exists t \in I^+$ such that $\text{Ann}_N(x^t) = \text{Ann}_N(x^{t+1})$, whence we get $\text{Ann}_N(x^{t+k}) = \text{Ann}_N(x^t)$ for all $k \in I^+$.

Let $y = n x^t (\neq 0) \in \text{Ann}_N(x^{t+1}) \cap Nx^t$ for $n \in N$.

Now $y \in \text{Ann}_N(x^t) \Rightarrow y x^t = 0 \Rightarrow n x^{2t} = 0 \Rightarrow n \in \text{Ann}_N(x^{2t}) = \text{Ann}_N(x^t) \Rightarrow y = n x^t = 0$, a contradiction.

i.e. $y \in \text{Ann}_N(x^{t+1}) \Rightarrow y \not\in \text{Ann}_N(x^t) \Rightarrow \text{Ann}_N(x^t) \neq \text{Ann}_N(x^{t+1})$, a contradiction.

Thus $Z(N)$ contains nilpotent elements.

As finite intersection of essential $N$-subgroups of $N$ is distributively generated, $Z(N)$ is $N$-subgroup of $N$. [ by proposition 2.1.14]

So $J_2(N)$ contains $Z(N)$.

By lemma 3.4.20, $Z(N) = 0$.

For $S^1$ near-ring $N$, $N/\text{Soc}(N)$ is weakly Noetherian by proposition 3.4.5. Again from proposition 3.4.9, (considering $N$ as $N$-group) it follows that $N$ has acc on essential ideals when we get $N$ is non singular.

**Theorem 3.4.22:** If $\{N^i\}_{i \in \text{Soc}(N)}$ is an independent family of normal $N$-subgroups of $N/\text{Soc}(N)$-group $E$, direct sum of $E$-injective $N/\text{Soc}(N)$-groups is commutative $N$-group, then $N/1$ is weakly Noetherian $V_e N$-group for every essential ideal $I$ of $N$ implies $N/\text{Soc}(N)$ is weakly Noetherian $V_e$ near-ring.

**Proof:** $N/1$ is weakly Noetherian for every essential ideal $I$ of $N$ implies $N/\text{Soc}(N)$ is weakly Noetherian as proposition 3.4.8.
Let $L$ be a strictly semi-simple $N/Soc(N)$-group.

Then as $N$ dgnr, $L$ is a semi-simple $N/Soc(N)$-group.

$I/ Soc(N)$ an ideal of $N/Soc(N)$ and $f: I/ Soc(N) \rightarrow L$ a non-zero $N$-homomorphism.

Let $Ker f = K/ Soc(N)$.

Now $K$ is essential in $N$. For if $K \cap J = 0$ for some non-zero ideal $J$ of $N$ then $J \cong \frac{J + K}{K}$ and since the latter is isomorphic to an ideal of $L$, it follows that for some ideal $I_1 \neq 0$ and contained in $J$ that $I_1 \subseteq L$, hence $I_1 \subseteq Soc(N) \subseteq K$, a contradiction.

Thus $N/K$ is a weakly Noetherian $Vc$ $N$-group.

If $N \rightarrow N/ Soc(N)$ is canonical quotient map, then $(N/ Soc(N))/( K/ Soc(N))$ is a weakly Noetherian $Vc$ $N$-group. Proposition 3.4.2, yields a map of $\frac{N}{Soc(N)}$ into $L$. So, $L$ is $\frac{N}{Soc(N)}$ injective.

Thus by corollary 3.4.4, $\frac{N}{Soc(N)}$ is weakly Noetherian $Vc$ near-ring.

\textit{If every injective right $N/K$-group is injective as an $N$-group we get the following result.}

**Theorem 3.4.23:** For a near-ring $N$ with unity the following conditions are equivalent:

i. $N$ is $S^2S_0I$-near-ring.

ii. $\frac{N}{Soc(N)}$ is weakly Noetherian $Vc$ near-ring.

**Proof:** i. $\Rightarrow$ ii. By corollary 3.4.4, we have to show that every strictly semi-simple $N/Soc(N)$-group $E$ is injective.

If $E$ is $N/Soc(N)$-group then $SocN.E = 0$.

Now $Ann(E) = \{ x \in N / xE = 0 \}$ is essential in $N$.

Again as $Soc(N).E = 0$, $Soc(N) \subseteq Ann(E)$. Thus $Soc(N) = Ann(E)$, that is $E$ is annihilated by $Soc(N)$. Again $Z_w(E) = \{ x \in E / lx = 0, I \leq_w N \}$ and we get $E$ is weak singular.
For if not for some $x \in E$, $\forall I \leq E N$, $Ix \neq 0$, that is $\text{Soc}N.x \neq 0$, a contradiction.

By (i.) $E$ is injective as an $N$-group and hence injective as an $N/\text{Soc}(N)$-group.

(ii. $\Rightarrow$ i.) Let $L$ be a semi-simple weak singular $N$-group.

Then $L$ can be regarded as $N/\text{Soc}(N)$-group and hence injective as $N/\text{Soc}(N)$-group by (ii).

So $L$ is injective as $N$-group.

For near-ring $N$ with identity and $M$ unital $N$-group if for every right ideal $U$ of $N$ and every $N$-homomorphism $f : U \rightarrow M$, there exists an element $m$ in $M$ such that $f(a) = ma$ for all $a$ in $U$ implies $M$ is injective then we get the following results.

**Proposition 3.4.24:** $\bigoplus_{i \in I} E_i$ of injective $N$-groups is injective if near-ring $N$ is weakly Noetherian.

**Proof:** Let $N$ be weakly Noetherian, $I$ be an ideal of $N$ and $f : I \rightarrow \bigoplus_A E_\alpha$.

Then since $I$ is finitely generated, $\text{Im}f$ is contained in $\bigoplus_{F \subseteq A} E_\alpha$ for some finite subset $F \subseteq A$.

So $\bigoplus_{F \subseteq A} E_\alpha$ is injective since finite direct sum is injective by theorem 3.3.7.

By theorem 4.1.9, as $\bigoplus_{F \subseteq A} E_\alpha$ is injective then for every right ideal $U$ of $N$ and every $N$-homomorphism $f : U \rightarrow \bigoplus_{F \subseteq A} E_\alpha$, there exists an element $m$ in $\bigoplus_{F \subseteq A} E_\alpha$ such that $f(a) = ma$ for all $a$ in $U$. But $m \in \bigoplus_A E_\alpha$ also. So for every right ideal $U$ of $N$ and every $N$-homomorphism $f : U \rightarrow \bigoplus_A E_\alpha$, there exists an element $m$ in $\bigoplus_A E_\alpha$ such that $f(a) = ma$ for all $a$ in $U$.

Then $\bigoplus_A E_\alpha$ is injective.

**Proposition 3.4.25:** For any near-ring $N$ the following conditions are equivalent:

i. $N$ is an almost weakly Noetherian near-ring.

ii. $N/I$ is weakly Noetherian for every essential left ideal $I$ of $N$. 
iii. N has A.C.C. on essential left ideals.

Moreover if $Z(nN) = 0$, N dgnr and every injective right N/K-group is injective as an N-group for ideal K of N we get

iv. Direct sum of (countably many) weak singular injective left N-groups is injective.

Again if $Z(nN) = 0$ and every injective right N-group is injective as an N/K-group for ideal K of N where SocN is pure we get

v. Direct sum of (Countably many) injective hulls of simple weak singular left N-groups is injective.

Proof: Equivalence between (i), (ii), (iii) is clear from above corollary 3.4.13, considering N as N-group.

(i) $\Rightarrow$ (iv). Let $\{E_i\}_{i \in I}$ be a family of weak singular left N-groups. Since $Z_w(E_i) = \{x \in E_i / \exists x = 0 \text{ for } x <_{eiN}E_i\} = E_i$, we get $\text{SocN}.E_i = 0$. So each $E_i$ can be regarded as an N/Soc(N)-groups. Since N/Soc(N) is weakly Noetherian, $\bigoplus_{i \in I} E_i$ is injective as an N/Soc(N)-group by proposition 3.4.24, hence $\bigoplus_{i \in I} E_i$ is injective as an N-group.

(iv) $\Rightarrow$ (v). clear.

(v) $\Rightarrow$ (i). Proposition 3.4.17

If every injective right N/K-group is injective as an N-group we get the following results:

Theorem 3.4.26: For a dgnr near-ring N, then the following conditions are equivalent:

i. N is $S^2S_{\text{weak}}$-near-ring.

ii. $N/\text{Soc}(N)$ is weakly Noetherian $V_c$ near-ring.

iii. N is GV-near-ring and direct sum of weak singular injective N-groups is injective.
iv. N is GV-near-ring and N has A.C.C. on essential left ideals.

**Proof:** i. $\Leftrightarrow$ ii. From theorem 3.4.23

ii. $\Rightarrow$ iii. From equivalence between (i) and (ii) clearly N is a GV-near-ring.

$N/\text{Soc}(N)$ is weakly Noetherian.

Let $\{E_i\}_{i \in I}$ be a family of weak singular left $N$-groups. Clearly each $E_i$ can be regarded as an $N/\text{Soc}(N)$-groups.

Since $N/\text{Soc}(N)$ is weakly Noetherian, so by proposition 3.4.24, $\bigoplus_{i \in I} E_i$ is injective as an $N/\text{Soc}(N)$-group. So $\bigoplus_{i \in I} E_i$ is injective as an $N$-group.

iii. $\Rightarrow$ i. is obvious.

**Theorem 3.4.27:** For a dgnr GV near-ring $N$ direct sum of weak singular injective $N$-groups is injective implies $N$ has A.C.C. on essential left ideals.

**Proof:** Since (iii) is equivalent to (ii) in theorem 3.4.26, we can conclude that $N$ has A.C.C. on essential ideals.

**Theorem 3.4.28:** For a dgnr GV near-ring $N$ if every injective right $N/K$-group is injective as an $N$-group for ideal $K$ of $N$ and $N$ has A.C.C. on essential left ideals then direct sum of weak singular injective $N$-groups is injective.

**Proof:** From theorem 3.4.21, $Z(N) = 0$.

From proposition 3.4.25 direct sum of weak singular injective $N$-groups is injective.