4.1 INTRODUCTION

The notion of statistical convergence was studied at the initial stage by Fast [33] and Schoenberg [131] independently. Later on it was further investigated by Šalát [137], Fridy [34] and many others.

A subset $E$ of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

exists, where $\chi_E$ is the characteristic function of $E$.

A sequence $(x_k)$ is said to be statistically convergent to $L$ if for every $\varepsilon > 0$,

$$\delta\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} = 0.$$ 

For $L=0$, we say this is statistically null.

By a lacunary sequence $\theta = (k_i)$; $r = 1, 2, 3, \ldots$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. We denote $I_r = (k_{r-1}, k_r]$ and $\eta_r = \frac{k_r}{k_{r-1}}$ for $r = 1, 2, 3, \ldots$. The space of lacunary strongly convergent sequence $N_\theta$ was defined by Freedman, Sember and Raphael [40] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{for some } L \right\}.$$ 

This chapter is published in *Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica*, 20(1) (2012), 417-430 (see for instance reference [152]).
The space $N_\theta$ is a $BK$-space with the norm

$$
\|x\|_\theta = \sup_r \frac{1}{h_r} \sum_{k=1}^r |x_k|.
$$

$N^*_\theta$ denotes the subset of those sequences in $N_\theta$ for which $L = 0$. ($N^*_\theta, \|\cdot\|_\theta$) is also a $BK$-space. Freedman, Sember and Raphael [40] also defined the space $|\sigma|$ of strongly Cesàro summable sequences as follows:

$$
|\sigma| = \left\{ x = (x_k) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n |x_k - L| = 0, \text{for some } L \right\}.
$$

In the special case when $\theta = (2^r)$, $N_\theta = |\sigma|$.  

**REMARK 4.1.1.** An Orlicz function $M$ satisfies the inequality $M(\lambda x) < \lambda M(x)$ for all $x \geq 0$ and $\lambda$ with $0 < \lambda < 1$. 

The following inequality will be used throughout the article. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup p_k = G$, $D = \max(1, 2^{G-1})$. Then for all $a_k, b_k \in C$ for all $k \in N$, we have

$$
|a_k + b_k|^{p_k} \leq D \left[ |a_k|^{p_k} + |b_k|^{p_k} \right].
$$

### 4.2 DEFINITIONS AND PRELIMINARIES

**LEMMA 4.2.1.** (Isik, Et and Tripathy [65], Lemma 1.1) Let $p$ and $q$ be seminorms on a linear space $X$. Then $p$ is stronger than $q$ if and only if there exists a constants $M$ such that $q(x) \leq Mp(x)$ for all $x \in X$.

Let $M = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $X$ be a seminormed space over the field $C$ of complex numbers with the seminorm $q$. $w(X)$ denotes the space of all sequences $x = (x_k)$, where $x_k \in X$. We define the following sequence spaces:

$$
w_0(M, \theta, \Delta^n, p, q) = \left\{ x \in w(X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=1}^r \left[ M_k \left( q \left( \frac{\Delta^n x_k}{\rho} \right) \right) \right]^{p_k} = 0, \text{for some } \rho > 0 \right\},
$$
\[ w_1(M, \theta, \Delta^a_m, p, q) = \left\{ x \in w(X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^a_m x_k - L}{\rho} \right) \right)^{p_k} = 0, \text{for some } \rho > 0 \text{ and } L \in X \right\}, \]

\[ w_\infty(M, \theta, \Delta^a_m, p, q) = \left\{ x \in w(X) : \sup_{r} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^a_m x_k}{\rho} \right) \right)^{p_k} < \infty, \text{for some } \rho > 0 \right\}, \]

where \( \Delta^a_m x = (\Delta^a_m x_k) = (\Delta^{a-1}_m x_k - \Delta^{a-1}_m x_{k+m}) \) and \( \Delta^0_m x_k = x_k \) for all \( k \in N \).

If \( M_\delta(x) = x \), for all \( x \in [0, \infty) \), \( p_k = 1 \), for all \( k \in N \), \( X = C \), \( q(x) = |x| \), for all \( x \in X \) and \( n = 0 \), then \( w_1(M, \theta, \Delta^a_m, p, q) = N_\theta \) and \( w_\infty(M, \theta, \Delta^a_m, p, q) = N_\delta^0 \). If in addition, we take \( \theta = (2') \), then \( w_1(M, \theta, \Delta^a_m, p, q) = |\sigma| \).

**DEFINITION 4.2.1.** Two non-negative functions \( f \) and \( g \) are called equivalent, whenever \( C_f \leq g \leq C_f \) \( j = 1, 2 \) and in this case we will use \( f \sim g \).

### 4.3 MAIN RESULTS

**ON LACUNARY DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS**

**THEOREM 4.3.1.** Let \( M = (M_k) \) be a sequence of Orlicz functions. Then \( w_0(M, \theta, \Delta^a_m, p, q) \subset w_1(M, \theta, \Delta^a_m, p, q) \subset w_\infty(M, \theta, \Delta^a_m, p, q) \).

**THEOREM 4.3.2.** The spaces \( w_0(M, \theta, \Delta^a_m, p, q), w_1(M, \theta, \Delta^a_m, p, q) \) and \( w_\infty(M, \theta, \Delta^a_m, p, q) \) are linear.

**THEOREM 4.3.3.** The spaces \( w_0(M, \theta, \Delta^a_m, p, q), w_1(M, \theta, \Delta^a_m, p, q) \) and \( w_\infty(M, \theta, \Delta^a_m, p, q) \) are paranormed spaces paranormed by

\[
g(x) = \sum_{i=1}^{\infty} q(x_i) + \inf \left\{ \frac{L}{\rho} : \sup_{r} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^a_m x_k}{\rho} \right) \right) \leq 1, \rho > 0, r \in N \right\}, \]
where $H = \max \{1, \sup_r p, \}$. 

**THEOREM 4.3.4.** Let $M = \{M_k\}$ and $T = \{T_k\}$ be sequences of Orlicz functions and $Z = w_1, w_0$ and $w_\infty$. Then for any two sequences $p = (p_k)$ and $t = (t_k)$ of bounded positive real numbers and for any two seminorms $q_1$ and $q_2$ we have 

(i) If $q_1$ is stronger than $q_2$, then $Z(M, \theta, \Delta^*_n, p, q_1) \subset Z(M, \theta, \Delta^*_m, p, q_2)$,

(ii) $Z(M, \theta, \Delta^*_n, p, q_1) \cap Z(M, \theta, \Delta^*_m, p, q_2) \subset Z(M, \theta, \Delta^*_m, p, q_1+q_2)$,

(iii) $Z(M, \theta, \Delta^*_n, p, q_1) \cap Z(T, \theta, \Delta^*_m, p, q_1) \subset Z(M+T, \theta, \Delta^*_m, p, q_1)$,

(iv) $Z(M, \theta, \Delta^*_n, p, q_1) \cap Z(M, \theta, \Delta^*_m, t, q_2) \neq \emptyset$,

(v) The inclusions $Z(M, \theta, \Delta^*_n, p, q_1) \subset Z(M, \theta, \Delta^*_m, p, q_1)$ are strict. In general $Z(M, \theta, \Delta^*_m, p, q) \subset Z(M, \theta, \Delta^*_n, p, q)$ for $i = 1, 2, \ldots, n-1$ and the inclusion is strict.

**THEOREM 4.3.5.** Let $Z = w_1, w_0$ and $w_\infty$.

(i) Let $0 < \inf p_k < p_k \leq 1$. Then $Z(M, \theta, \Delta^*_n, p, q) \subset Z(M, \theta, \Delta^*_m, q)$,

(ii) Let $1 < p_k \leq \sup p_k < \infty$. Then $Z(M, \theta, \Delta^*_n, q) \subset Z(M, \theta, \Delta^*_m, p, q)$,

(iii) Let $0 < p_k \leq t_k$ and \[ \left( \frac{p_k}{t_k} \right) \] be bounded. Then $Z(M, \theta, \Delta^*_n, t, q) \subset Z(M, \theta, \Delta^*_m, p, q)$.

**THEOREM 4.3.6.** Let $M = \{M_k\}$ and $T = \{T_k\}$ be two sequences of Orlicz functions such that $M_k \approx T_k$ for each $k \in N$. Then $Z(M, \theta, \Delta^*_n, p, q) = Z(T, \theta, \Delta^*_m, p, q)$, for $Z = w_1, w_0$ and $w_\infty$.

**THEOREM 4.3.7.** Let $M = \{M_k\}$ be a sequence of Orlicz functions and $Z = w_1, w_0$ and $w_\infty$. Then $Z(M, \theta, \Delta^*_n, p, q) = Z(\theta, \Delta^*_n, p, q)$, if the following conditions hold

\[ \lim_{t \to 0} \frac{M_k(t)}{t} > 0 \text{ and } \lim_{t \to 0} \frac{M_k(t)}{t} < \infty, \text{ for each } k \in N. \]
4.4 PROOF OF THE RESULTS OF SECTION 4.3

PROOF OF THEOREM 4.3.1. It is obvious that \( w_0(M, \theta, \Delta^*_m, p, q) \subset w_1(M, \theta, \Delta^*_m, p, q) \) and \( w_0(M, \theta, \Delta^*_m, p, q) \subset w_0(M, \theta, \Delta^*_m, p, q) \). We shall prove that \( w_1(M, \theta, \Delta^*_m, p, q) \subset w_0(M, \theta, \Delta^*_m, p, q) \).

Let \((x_k) \in w_1(M, \theta, \Delta^*_m, p, q)\). Then there exist some \( \rho > 0 \) and \( L \in X \) such that

\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left( \frac{\Delta_m x_k - L}{\rho} \right) \right) \right]^{p_k} = 0.
\]

On taking \( \rho_1 = 2 \rho \), we have

\[
\frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left( \frac{\Delta_m x_k - L}{\rho} \right) \right) \right]^{p_k} \\
\leq \frac{D}{h_r} \sum_{k \in I_r} \left[ \frac{1}{2} M_k \left( \left( \frac{\Delta_m x_k - L}{\rho} \right) \right) \right]^{p_k} + \frac{D}{h_r} \sum_{k \in I_r} \left[ \frac{1}{2} M_k \left( \frac{L}{\rho} \right) \right]^{p_k} \\
\leq \frac{D}{h_r} \sum_{k \in I_r} \left[ \frac{1}{2} M_k \left( \left( \frac{\Delta_m x_k - L}{\rho} \right) \right) \right]^{p_k} + D \max \left[ 1, \sup_k \left[ \frac{1}{2} M_k \left( \frac{L}{\rho} \right) \right] \right]^{H},
\]

where \( \sup_k p_k = G, H = \max(1, G) \) and \( D = \max \{1, 2^{G-1}\} \).

Hence we get \((x_k) \in w_0(M, \theta, \Delta^*_m, p, q)\).

The inclusions are strict follows from the following examples.

**EXAMPLE 4.4.1.** Let \( m = n = 2, \theta = (3^r), p_k = 1 \), for all \( k \in N, X = C^2, q(x) = \max \{|x^1|, |x^2|\} \), for \( x = (x^1, x^2) \in C^2 \) and \( M_k(x) = x^2 \), for all \( x \in [0, \infty) \) and \( k \in N \). We consider the sequence \((x_k)\) defined by \( x_k = (k^2, k^2) \) for each fixed \( k \in N \). Then \((x_k) \in w_1(M, \theta, \Delta^*_m, p, q)\) but \((x_k) \notin w_0(M, \theta, \Delta^*_m, p, q)\).

**EXAMPLE 4.4.2.** Let \( m = n = 2, \theta = (2^r), p_k = 2 \), for all \( k \) odd and \( p_k = 3 \), for all \( k \) even, \( X = C^3, q(x) = \max \{|x^1|, |x^2|, |x^3|\} \), where \( x = (x^1, x^2, x^3) \in C^3 \) and \( M_k(x) = x^4 \), for all \( x \in [0, \infty) \) and \( k \in N \). We consider the sequence \((x_k)\) defined by \( x_k = (k, k, k) \) for each fixed \( k \in N \). Then \((x_k) \in w_0(M, \theta, \Delta^*_m, p, q)\) but \((x_k) \notin w_1(M, \theta, \Delta^*_m, p, q)\).
PROOF OF THEOREM 4.3.2. Linearity is easy to check and so omitted.

PROOF OF THEOREM 4.3.3. Clearly \( g(x) = g(-x) \). Since \( M_\theta(0) = 0 \), for all \( k \in N \) we get \( \inf \left\{ \frac{\rho x}{\rho^H} \right\} = 0 \) for \( x = \theta \). Now let \( x, y \in \mathbb{w_0}(\mathbb{L}, \theta, \Delta^\theta_m, p, q) \) and let us choose \( \rho_1 > 0 \) and \( \rho_2 > 0 \) such that

\[
\sup_{r, k} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^\theta_m x_k}{\rho_1} \right) \right) \leq 1
\]

and

\[
\sup_{r, k} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^\theta_m y_k}{\rho_2} \right) \right) \leq 1
\]

Let \( \rho = \rho_1 + \rho_2 \). Then we have

\[
\sup_{r, k} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^\theta_m (x_k + y_k)}{\rho} \right) \right) \leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_{r, k} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^\theta_m x_k}{\rho_1} \right) \right)
\]

\[
+ \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_{r, k} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^\theta_m y_k}{\rho_2} \right) \right) \leq 1.
\]

Hence

\[ g(x+y) \leq g(x) + g(y). \]

Finally let \( \lambda \) be a given non-zero scalar, then the continuity of the scalar multiplication follows from the following equality:

\[
g(\lambda x) = \sum_{i=1}^n q(\lambda x_i) + \inf \left\{ \frac{p_x}{\rho^H} : \sup_{r, k} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^\theta_m (\lambda x_k)}{\rho} \right) \right) \leq 1 \right\}
\]

\[
= |\lambda| \sum_{i=1}^n q(\lambda x_i) + \inf \left\{ \left| \lambda \right| \frac{p_x}{\rho^H} : \sup_{r, k} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^\theta_m x_k}{s} \right) \right) \leq 1 \right\}, \text{ where } s = \left| \frac{s}{\lambda} \right|
\]

This completes the proof.

PROOF OF THEOREM 4.3.4. The proof is easy and so omitted.
PROOF OF THEOREM 4.3.5. Proof of (i) and (ii) is easy and so omitted. We give the proof of (iii) for \( Z = \omega \) only and for other cases it will follow on applying similar arguments.

We write \( S_k = \left[ M_k \left( q \left( \frac{\triangle_m x_k - L}{\rho} \right) \right) \right]^n \) and \( \mu_k = \frac{p_k}{t_k} \) so that \( 0 < \mu \leq \mu_k \leq 1 \).

Define \( S'_k = S_k \) if \( S_k \geq 1 \), \( S'_k = 0 \) if \( S_k < 1 \)

Then \( S_k = S'_k + S''_k \), \( S''_k = S''_k + S''_k \).

Now it follows that \( S''_k \leq S_k \leq S'_k \), \( S''_k \leq S'_k \).

We have the following inequality

\[
\frac{1}{h_r} \sum_{k \in I_r} S''_k \leq \frac{1}{h_r} \sum_{k \in I_r} S_k + \frac{1}{h_r} \sum_{k \in I_r} S''_k.
\]

Therefore if \( (x_k) \in w_1(M, \theta, \Delta^m, t, q) \), then \( (x_k) \in w_1(M, \theta, \Delta^m, p, q) \).

PROOF OF THEOREM 4.3.6. The proof is trivial and so omitted.

PROOF OF THEOREM 4.3.7. If the given conditions are satisfied, we have \( M_k(t) = t \) for each \( k \in N \) and the proof follows.

4.5 MAIN RESULTS

ON \( q \)-LACUNARY \( \Delta^m \)-STATISTICAL CONVERGENCE

DEFINITION 4.5.1. Let \( \theta \) be a lacunary sequence, then the sequence \( x = (x_k) \) is said to be \( q \)-lacunary \( \Delta^m \)-statistical convergent to the number \( L \) provided that for every \( \varepsilon > 0 \),

\[
\lim_{r \to +\infty} \frac{1}{h_r} \text{card}\{k \in I_r : q(\Delta^m x_k - L) \geq \varepsilon\} = 0.
\]
In this case we write \( x_k \to L \left( S^a_m \right) \) or \( S^q_m \left( \Delta^a_m \right) \) - \( \lim x_k = L \) and we define
\[
S^q_m \left( \Delta^a_m \right) = \{ x \in \mathcal{W}(X) : S^q_m \left( \Delta^a_m \right) - \lim x_k = L, \text{ for some } L \}.
\]

In the case \( \theta = (2^r) \), we shall write \( S^q \left( \Delta^a_m \right) \) instead of \( S^q_m \left( \Delta^a_m \right) \).

If \( X = C \), \( q(x) = |x| \), we shall write \( S^q \left( \Delta^a_m \right) \) instead of \( S^q_m \left( \Delta^a_m \right) \) and if \( \theta = (2^r) \) we shall write \( S \left( \Delta^a_m \right) \) instead of \( S^q \left( \Delta^a_m \right) \).

In the special case \( L = 0 \), we denote it by \( S^q \left( \Delta^a_m \right) \).

**THEOREM 4.5.1.** Let \( \theta \) be a lacunary sequence and \( 0 < p < \infty \). Then

(i) If \( x_k \to L \left( w^q_m \left( \Delta^a_m \right) \right) \), then \( x_k \to L \left( S^q_m \left( \Delta^a_m \right) \right) \),

(ii) If \( x \in \ell(p, \Delta^a_m) \) and \( x_k \to L \left( S^q_m \left( \Delta^a_m \right) \right) \), then \( x_k \to L \left( w^q_m \left( \Delta^a_m \right) \right) \),

where \( \ell(p, \Delta^a_m) = \{ x \in \mathcal{W}(X) : \sup_{k \geq 1} q \left( \Delta^a_m x_k \right) < \infty \} \),

\[
w^q_m \left( \Delta^a_m \right) = \left\{ x \in \mathcal{W}(X) : \lim_{r \to \infty} \frac{1}{r} \sum_{k=1}^{r} q \left( \Delta^a_m x_k - L \right) = 0, \text{ for some } L \right\}.
\]

(iii) \( \ell(p, \Delta^a_m) \cap S^q_m \left( \Delta^a_m \right) = \ell(p, \Delta^a_m) \cap w^q_m \left( \Delta^a_m \right) \).

**THEOREM 4.5.2.** Let \( \theta \) be any lacunary sequence. Then

(i) If \( \lim \inf \eta_r > 1 \), then \( S^q \left( \Delta^a_m \right) \subseteq S^q_m \left( \Delta^a_m \right) \),

(ii) If \( \lim \sup \eta_r < \infty \), then \( S^q_m \left( \Delta^a_m \right) \subseteq S^q \left( \Delta^a_m \right) \),

(iii) If \( 1 < \lim \inf \eta_r \leq \lim \sup \eta_r < \infty \), then \( S^q_m \left( \Delta^a_m \right) = S^q \left( \Delta^a_m \right) \).

**THEOREM 4.5.3.** (i) \( w_1 \left( M, \theta, \Delta^a_m, p, q \right) \subseteq S^q_m \left( \Delta^a_m \right) \),

(ii) \( w_0 \left( M, \theta, \Delta^a_m, p, q \right) \subseteq S^q_0 \left( \Delta^a_m \right) \).

**THEOREM 4.5.4.** (i) \( \ell(p, \Delta^a_m) \cap S^q_m \left( \Delta^a_m \right) = \ell(p, \Delta^a_m) \cap w_1 \left( M, \theta, \Delta^a_m, p, q \right) \),

(ii) \( \ell(p, \Delta^a_m) \cap S^q_0 \left( \Delta^a_m \right) = \ell(p, \Delta^a_m) \cap w_0 \left( M, \theta, \Delta^a_m, p, q \right) \).
4.6 PROOF OF THE RESULTS OF SECTION 4.5

**PROOF OF THEOREM 4.5.1.** (i) Let \( x_k \to L(w^\Phi(A^\pi_m)) \) and \( \varepsilon > 0 \). Then we have

\[
\sum_{k \in I_r} (q(A^\pi_m x_k - L))^p \geq \varepsilon^p \text{card}\{k \in I_r : q(A^\pi_m x_k - L) \geq \varepsilon\}.
\]

Hence \( x_k \to L(S^q_\Phi(A^\pi_m)) \).

(ii) Suppose that \( x \in \ell_\infty(q, A^\pi_m) \) and \( x_k \to L(S^q_\Phi(A^\pi_m)) \). Let \( \varepsilon > 0 \) be given and \( n_0(\varepsilon) \in N \) such that

\[
\frac{1}{h_r} \text{card}\left\{k \in I_r : q(A^\pi_m x_k - L) \geq \left( \frac{\varepsilon}{2} \right)^p \right\} < \frac{\varepsilon}{2K^p} \quad \text{for all } r > n_0(\varepsilon),
\]

where \( K = \sup_k (q(A^\pi_m x_k - L)) \) and we set \( L_r = \left\{k \in I_r : q(A^\pi_m x_k - L) \geq \left( \frac{\varepsilon}{2} \right)^p \right\} \).

Now for all \( r > n_0 \), we have

\[
\frac{1}{h_r} \sum_{k \in I_r} (q(A^\pi_m x_k - L))^p = \frac{1}{h_r} \sum_{k \in I_{r-1}} (q(A^\pi_m x_k - L))^p + \frac{1}{h_r} \sum_{k \in I_r} (q(A^\pi_m x_k - L))^p
\]

\[
\leq \frac{1}{h_r} \left( \frac{h_r \varepsilon}{2K^p} \right) K^p + \frac{1}{h_r} \frac{\varepsilon}{2} = \varepsilon.
\]

Hence \( x_k \to L(w^\Phi(A^\pi_m)) \).

(iii) The proof follows from (i) and (ii).

**PROOF OF THEOREM 4.5.2.** (i) If \( \liminf \eta_r > 1 \), then there exists a \( \delta > 0 \) such that \( 1 + \delta \leq \eta_r \) for sufficiently large \( r \). Since \( h_r = k_r - k_{r-1} \), we have \( \frac{k_r}{h_r} \leq \frac{1 + \delta}{\delta} \). Let \( x_k \to L(S^q(A^\pi_m)) \).

Then for every \( \varepsilon > 0 \), we have

\[
\frac{1}{k_r} \text{card}\{k \leq k_r : q(A^\pi_m x_k - L) \geq \varepsilon\} \geq \frac{1}{k_r} \text{card}\{k \in I_r : q(A^\pi_m x_k - L) \geq \varepsilon\}
\]

\[
\geq \left( \frac{\delta}{1 + \delta} \right) \frac{1}{h_r} \text{card}\{k \in I_r : q(A^\pi_m x_k - L) \geq \varepsilon\}.
\]
Thus \( x^* \rightarrow L(S^q_\theta(\Delta^*_m)) \). Hence \( S^q(\Delta^*_m) \subseteq S^q_\theta(\Delta^*_m) \).

**(ii)** Suppose that \( \lim \sup r \eta_r < \infty \). Then there exists \( M > 0 \) such that \( \eta_r < M \) for all \( r \geq 1 \). Let \( x^* \rightarrow L(S^q_\theta(\Delta^*_m)) \) and \( \varepsilon > 0 \). We set \( E_r = \text{card} \{ k \in I : q(\Delta^*_m x_k - L) \geq \varepsilon \} \). Then there exists \( n_0 \in N \) such that \( \frac{1}{h_j} E_r < \varepsilon \) for all \( r > n_0 \). Let \( K = \max_{1 \leq s \leq r} E_r \) and choose \( n \) such that \( k_{r-1} < n \leq k_r \), then we have

\[
\frac{1}{n} \text{card} \{ k \leq n : q(\Delta^*_m x_k - L) \geq \varepsilon \} \leq \frac{1}{k_{r-1}} \text{card} \{ k \leq k_r : q(\Delta^*_m x_k - L) \geq \varepsilon \}
\]

\[
\leq \frac{1}{k_{r-1}} \{ E_1 + E_2 + \ldots + E_n \} + \frac{1}{k_{r-1}} \sup_{r \geq n} \{ h_1 + \ldots + h_r \}
\]

\[
\leq \frac{K}{k_{r-1}} n_0 + \frac{1}{k_{r-1}} \sup_{r \geq n} \{ h_1 + \ldots + h_r \}
\]

\[
\leq \frac{K}{k_{r-1}} n_0 + \frac{k_r - k_{r-1}}{k_{r-1}} \eta_r
\]

\[
\leq \frac{K}{k_{r-1}} n_0 + \varepsilon \eta_r
\]

\[
\leq \frac{K}{k_{r-1}} n_0 + \varepsilon M.
\]

Since \( k_{r-1} \to \infty \) as \( n \to \infty \), it follows that \( x^* \rightarrow L(S^q_\theta(\Delta^*_m)) \). Hence \( S^q_\theta(\Delta^*_m) \subseteq S^q(\Delta^*_m) \).

**(iii)** The proof follows from (i) and (ii).

**PROOF OF THEOREM 4.5.3.** (i) Let \( (x_k) \in w_1(M, \theta, \Delta^*_m, p, q) \). Then there exist some \( \rho > 0 \) and \( L \in X \) such that

\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^*_m x_k - L}{\rho} \right) \right)^{p_k} = 0.
\]

Let \( \varepsilon > 0 \) be given and \( \sum_1 \) denote the sum over \( k \in I_r \) such that \( q(\Delta^*_m x_k - L) \geq \varepsilon \) and \( \sum_2 \) denote the sum over \( k \in I_r \) such that \( q(\Delta^*_m x_k - L) < \varepsilon \). Then

\[
\frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^*_m x_k - L}{\rho} \right) \right)^{p_k}
\]

\[
= \left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^*_m x_k - L}{\rho} \right) \right)^{p_k} \right) \left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q \left( \frac{\Delta^*_m x_k - L}{\rho} \right) \right)^{p_k} \right)^{-1}.
\]
\[
\frac{1}{h_r} \sum_{k=1}^{A} \left[ M_k \left( q \left( \frac{\Delta_m x_k - L}{\rho} \right) \right) \right] + \frac{1}{h_r} \sum_{k=2}^{B} M_k \left( q \left( \frac{\Delta_m x_k - L}{\rho} \right) \right)
\]
\[
\geq \frac{1}{h_r} \sum_{k=1}^{A} \left[ M_k \left( \varepsilon_1 \right) \right] + \frac{1}{h_r} \sum_{k=1}^{B} \min \left\{ \left( M_k \left( \varepsilon_1 \right) \right)^{\text{def} p_k}, \left( M_k \left( \varepsilon_1 \right) \right)^{\text{def} q_k} \right\}
\]
\[
\geq \frac{1}{h_r} \text{card} \left\{ k \in I, : q \left( \Delta_m x_k - L \right) \geq \varepsilon \right\} \min \left\{ \left( M_k \left( \varepsilon_1 \right) \right)^{\text{def} p_k}, \left( M_k \left( \varepsilon_1 \right) \right)^{\text{def} q_k} \right\}
\]

Hence \((x_k) \in S^g \left( \Delta_m \right)\).

(ii) The Proof is similar to that of part (i).

**PROOF OF THEOREM 4.5.4.** (i) By Theorem 4.5.3, we need only show that
\[
S^g \left( \Delta_m \right) \subseteq \ell_\infty \left( \Delta^*_m \right) \cap w_1 (M, \theta, \Delta^*_m, p, q).
\]
and \( t_k = (\Delta_m x_k - L) \rightarrow 0 \left( S^g \left( \Delta_m \right) \right) \). Let \((x_k) \in \ell_\infty \left( \Delta^*_m \right) \cap S^g \left( \Delta_m \right)\) and \((x_k) \in \ell_\infty \left( \Delta^*_m \right) \cap w_1 (M, \theta, \Delta^*_m, p, q)\).

Since \((x_k) \in \ell_\infty \left( \Delta^*_m \right) \), there exists \( K > 0 \) such that \( M_k \left( q \left( \frac{f_k}{\rho} \right) \right) \leq K \) for all \( k \). Then given \( \varepsilon > 0 \), we have
\[
\frac{1}{h_r} \sum_{k=1}^{A} M_k \left( q \left( \frac{f_k}{\rho} \right) \right) + \frac{1}{h_r} \sum_{k=2}^{B} M_k \left( q \left( \frac{f_k}{\rho} \right) \right)
\]
\[
\leq \frac{K}{h_r} \text{card} \left\{ k \in I, : q (t_k) \geq \varepsilon \rho \right\} + \frac{1}{h_r} \sum_{k=1}^{A} M_k \left( \frac{\varepsilon}{\rho} \right).
\]

Hence \((x_k) \in \ell_\infty \left( \Delta^*_m \right) \cap w_1 (M, \theta, \Delta^*_m, p, q)\).

(ii) The proof is similar to that of part (i).