Chapter 4

Rings with chain condition on fuzzy annihilators

4.1: Basic results and definitions
4.2: Singular fuzzy ideals
4.3: Chain condition on fuzzy annihilators
Chapter 4
Rings with chain condition on fuzzy annihilators

Introduction:

In Chapter 2 we have discussed basic concepts of annihilator of a fuzzy subset of a module. The concepts of essential fuzzy submodule are discussed in chapter 3. With the help of these concepts singular fuzzy ideal of a ring and singular fuzzy submodule of a module can be defined.

4.1. Basic definitions and Results:

In this section r(a) stands for right annihilator of a and l(a) stands for left annihilator of a.

Lemma 4.1.1[6]. Let M be a right R-module. Let a be a non zero element of M and let K be an essential submodule of M. Then there is an essential right ideal L of R such that aL ≠ 0 and aL ⊆ K.

Proof: Let L = {r ∈ R | ar ∈ K}. Then L is a right ideal of R and from definition of L: aL ⊆ K. Since K is an essential submodule of M, we have aR ∩ K ≠ 0. Therefore there exists a non zero element ar of K, for some r ∈ R. Hence r ∈ L and aL ≠ 0. Now let I be a non zero right ideal of R. We wish to show that I ∩ L ≠ 0. If aI = 0 then I ⊆ L. So in this case I ∩ L = I ≠ 0.

Now suppose that aI ≠ 0. Then aI ∩ K ≠ 0. Let ax be a non zero element of K, where x ∈ I ⊆ R. Then x ∈ L. Consequently I ∩ L ≠ 0.

The following is a useful way of constructing essential submodule of a module M. Let A and B be submodules of M with A ∩ B = 0. Zorn's lemma can be applied to the set of submodules of M which contain B and have zero intersection with A to obtain a submodule C of M such that A ∩ C = 0, B ⊆ C and A ⊕ C is an essential submodule of M. The next result is an illustration of how this method can be used.
We define $\text{soc}(M)$ the socle of right $R$-Module $M$, to be the sum of all the simple submodules of $M$, with $\text{soc}(M) = 0$ if $M$ has no simple submodules. A zorn's lemma argument shows that $\text{soc}(M)$ is a direct sum of simple submodules.

Lemma 4.1.2[6]: Let $E$ be the intersection of all essential submodules of a module $M$, then $\text{soc}(M) = E$.

The right singular ideal $Z(R)$ of a ring $R$ is defined by:

$$Z(R) = \{ r \in R \mid rK = 0 \text{ for some essential right ideal } K \text{ of } R \}.$$ 

In other words, if $x \in R$ then $x \in Z(R)$ if and only if $\tau(x)$ is an essential right real of $R$. It is clear that $Z(R)$ is a two sided ideal of $R$. Let $z \in Z(R)$ and $a \in R$. Because $\tau(z)$ is essential there is an essential right ideal $L$ of $R$ such that $aL \subseteq \tau(z)$. (If $a = 0$ we can take $L = R$). Thus $zaL = 0$ so that $za \in Z(R)$. The singular submodule of a module is defined similarly.

Lemma 4.1.3[6]: Let $R$ be a commutative ring. Then the singular ideal $Z(R)$ of $R$ is zero if and only if $R$ is semiprime.

Proof: suppose that $R$ is semiprime and let $z \in Z(R)$. Set $I = zR \cap \tau(z)$. Since $R$ is a commutative, $zR \tau(z) = 0$ also $F \subseteq zR \tau(z) = 0$, from which it follows that $I = 0$. But $\tau(z)$ is an essential ideal of $R$. Hence $zR = 0$, i.e., $z = 0$. Therefore $Z(R) = 0$.

Conversely suppose that $Z(R) = 0$ and let $a$ be an element of $R$ such that $a^2 = 0$. We shall show that $a = 0$, from which it follows that $R$ has no non zero nilpotent elements. Let $x$ be a non zero element of $R$. Then either $ax = 0$, in which case $x \in \tau(a)$ or $ax \neq 0$ in which case $ax$ is a non zero element of $\tau(a)$. Thus $xR \cap \tau(a) \neq 0$. Therefore $\tau(a)$ is an essential ideal of $R$. Hence $a \in Z(R)$. So $a = 0$. Therefore $R$ is semiprime.

A ring $R$ which has $Z(R) = 0$ is called right non-singular. Thus a commutative ring is non singular if and only if it is semiprime.

Theorem 4.1.4[6]: Let $R$ be a ring with the ascending chain condition (acc) for right annihilators, then the right singular ideal of $R$ is nilpotent.
Proof: In this proof we shall write \( Z \) rather than \( Z(R) \) for right singular ideal of \( R \). Since \( Z \supseteq Z^2 \supseteq \ldots \ldots \) we have

\[
\mathcal{r}(Z) \subseteq \mathcal{r}(Z^2) \subseteq \mathcal{r}(Z^3) \subseteq \ldots \ldots 
\]

Therefore there is a positive integer \( n \) such that \( \mathcal{r}(Z^n) = \mathcal{r}(Z^{n+1}) \). Suppose that \( Z^{n+1} \neq 0 \). There is an element \( a \) of \( Z \) such that \( Z^a \neq 0 \). We choose such an element \( a \) with \( \mathcal{r}(a) \) as large as possible. Let \( b \in Z \). Then \( \mathcal{r}(b) \) is an essential right ideal. Therefore \( \mathcal{r}(b) \cap aR \neq 0 \).

Thus there is an element \( r \in R \) such that \( ar \neq 0 \) and \( ar \in \mathcal{r}(b) \).

Now \( b \in Z \Rightarrow ba \in Z \)

Let \( x \in \mathcal{r}(a) \). Then \( ax = 0 \).

\[
\Rightarrow (ba)x = b(ax) = 0
\]

\[
\Rightarrow x \in \mathcal{r}(ba)
\]

Therefore \( \mathcal{r}(a) \subseteq \mathcal{r}(ba) \).

But \( ar \neq 0 \) and \( bar = 0 \).

i.e. \( r \notin \mathcal{r}(a) \) but \( r \in \mathcal{r}(ba) \).

Therefore \( \mathcal{r}(a) \) is strictly contained in \( \mathcal{r}(ba) \).

From our choice of \( a \), \( Z^a ba = 0 \). But \( b \) is an arbitrary element of \( Z \).

So \( Z^{a+1} a = 0 \Rightarrow a \in \mathcal{r}(Z^{a+1}) = \mathcal{r}(Z^a) \)

\[
\Rightarrow Z^a a = 0
\]

Which is a contradiction.

Therefore \( Z^{a+1} = 0 \).

i.e. \( Z \) is nilpotent.

Theorem 4.1.5[6]: Let \( R \) be a semiprime ring with the acc for right annihilators, then \( R \) has no non zero nil one-sided ideals.

Proof: Let \( I \) be a non zero one sided ideal of \( R \).

Let \( a \) be a non zero element of \( I \) with \( \mathcal{r}(a) \) as large as possible. Since \( R \) is semiprime, there
exists $x \in R$ such that $axa \neq 0$.

Let $y \in r(a)$. Then $ay = 0$.

Now $(axa)y = ax(ay) = 0 \Rightarrow y \in r(axa)$.

Therefore $r(a) \subseteq r(axa)$ where $axa \neq 0$.

$\Rightarrow r(a) = r(axa)$

We have $ax \neq 0$ (Because if $ax = 0$ then $axa = 0$).

$\Rightarrow x \notin r(a) = r(axa)$

$\Rightarrow (ax)x \neq 0$

$\Rightarrow (ax)^2 \neq 0$ and $xax \notin r(a) = r(axa)$

$\Rightarrow (ax)(xax) \neq 0$

$\Rightarrow (ax)^3 \neq 0$ and $xaxax \notin r(a) = r(axa)$

and so on.

Therefore $ax$ and hence also $xa$ is not nilpotent and $ax$ or $xa \in I$.

So $I$ is not nil.

Let $A, B$ be two nilpotent right Ideals of a ring $R$. Then $A^k = B^n = 0$ for some integers $k$ and $n$. It is easy to see that $(A + B)^{k+n} = 0$ since $(A + B)^{k+n}$ is a sum of products in which either $A$ occurs at least $k$ times or $B$ occurs at least $n$ times. It follows that the sum of a finite number of nilpotent right ideals is nilpotent.

**Corollary: 4.1.6[6]**: Let $R$ be a right Noetherian ring then each nil one-sided ideal of $R$ is nilpotent.

An element $c$ of a ring $R$ is right regular if $r(c) = 0$, left regular if $l(c) = 0$, and regular if $l(c) = r(c) = 0$. For example, every non zero element of an integral domain is regular. If $I$ is an ideal of $R$ we set $c(I) = \{ c \in R | c + I$ is a regular element of $R/I \}$. We write $c(0)$ for the set of regular elements of $R$.

We next introduce the concept of Goldie dimension (also known as uniform dimension).
A non zero module $U$ is said to be uniform if any two non-zero submodules of $U$ have non-zero intersection i.e. if each non-zero submodule of $U$ is essential in $U$. Let $M$ be an $R$-module. We say that $M$ has finite Goldie dimension if $M$ does not contain a direct sum of an infinite number of non-zero submodules. It is easy to show that $M$ has finite Goldie dimension if $M$ is Noetherian or Artinian. A ring $R$ is said to have finite right Goldie dimension if $R$ has finite Goldie dimension as a right $R$-module. We call $R$ a right Goldie ring if it has finite right Goldie dimension and satisfies the acc for right annihilators. A right Noetherian ring is right Goldie, but the converse is not true because any commutative integral domain is trivially a Goldie ring.

**Lemma 4.1.7** [6]: Let $M$ be a non zero right $R$-module.

(a) If $M$ has finite Goldie dimension then each non-zero submodule of $M$ contains a uniform submodule, and there is a finite number of uniform submodules of $M$ whose sum is direct and is an essential submodule of $M$.

(b) Suppose that $M$ has uniform submodules $U_1, \ldots, U_n$ such that the sum $U_1 + \ldots + U_n$ is direct and is an essential submodule of $M$, then $M$ has finite Goldie dimension and the positive integer $n$ is independent of the choice of the $U_i$'s. We call $n$ the Goldie dimension of $M$.

**Theorem (Goldie) 4.1.8** [6]: Let $R$ be a semiprime right Goldie ring and let $I$ be an essential right ideal of $R$, then $I$ contains a regular element of $R$.

**Lemma 4.1.9** [6]: Let $R$ be a ring with finite right Goldie dimension and let $c$ be a right regular element of $R$, then $cR$ is an essential right ideal of $R$.

**Lemma 4.1.10** [6]: Let $R$ be a right non-singular ring with finite right Goldie dimension then right regular elements of $R$ are regular.

**Corollary 4.1.11** [6]: Let $R$ be a semiprime right Goldie ring then right regular elements of $R$ are regular.

### 4.2 Singular Fuzzy Ideal:

Let $a \in M$, $a \neq 0$ and $\gamma$ be an essential fuzzy submodule of $M$. We define a fuzzy set
\( \sigma \in [0,1]^R \) by:

\[
\sigma = \bigcup \{ \delta \mid \delta \in [0,1]^R, \delta a_x \subseteq \gamma \}.
\]

**Lemma 4.2.1:** \( \sigma = \bigcup \{ r_a \mid r \in R, \alpha \in [0,1], r_a a_x \subseteq \gamma \} \).

**Proof:** \( \{ r_a \mid r \in R, \alpha \in [0,1] \} \subseteq [0,1]^R \).

Therefore \( \{ r_a \mid r \in R, \alpha \in [0,1], r_a a_x \subseteq \gamma \} \subseteq \{ \delta \mid \delta \in [0,1]^R, \delta a_x \subseteq \gamma \} \).

This implies \( \bigcup \{ r_a \mid r \in R, \alpha \in [0,1], r_a a_x \subseteq \gamma \} \subseteq \bigcup \{ \delta \mid \delta \in [0,1]^R, \delta a_x \subseteq \gamma \} = \sigma \).

Let \( \delta \in [0,1]^R \) such that \( \delta a_x \subseteq \gamma \).

Let \( r \in R \) and \( \delta(r) = \alpha \).

Now \( (r_a a_x)(x) = \bigvee \{ r_a(s) \land a_x(y) \mid s \in R, y \in M, sy = x \} \)

\[
= \bigvee \{ \delta(r) \land a_x(y) \mid y \in M, ry = x \}
\]

\[
\leq \bigvee \{ \delta(s) \land a_x(y) \mid s \in R, y \in M, sy = x \}
\]

\[
= (\delta a_x)(x)
\]

\[
\leq \gamma(x) \quad \forall x \in M.
\]

Thus \( r_a a_x \subseteq \gamma \).

So \( \sigma \subseteq \bigcup \{ r_a \mid r \in R, \alpha \in [0,1], r_a a_x \subseteq \gamma \} \).

Therefore \( \sigma = \bigcup \{ r_a \mid r \in R, \alpha \in [0,1], r_a a_x \subseteq \gamma \} \).

**Lemma 4.2.2:** \( \sigma = \bigcup \{ \delta \mid \delta \in \text{Fl}(R), \delta a_x \subseteq \gamma \} \).

**Proof:** Clearly \( \bigcup \{ \delta \mid \delta \in \text{Fl}(R), \delta a_x \subseteq \gamma \} \subseteq \bigcup \{ \delta \mid \delta \in [0,1]^R, \delta a_x \subseteq \gamma \} = \sigma \).

Let \( r \in R, \alpha \in [0,1] \) and \( r_a a_x \subseteq \gamma \).

Let \( \delta = < r_a > \)

Now \( < r_a > a_x = (\chi_0 \cup \alpha_{cr}) a_x \)

\[
= \chi_0 a_x \cup \alpha_{cr} a_x
\]

\[
\subseteq \chi_0 \cup \alpha_{cr} a_x
\]

Again \( (\alpha_{cr} a_x)(x) = \bigvee \{ \alpha_{cr}(s) \land a_x(y) \mid s \in R, y \in M, sy = x \} \)

\[
= \bigvee \{ \alpha \land a_x(y) \mid s \in < r >, y \in M, sy = x \}
\]

99
\[ \leq \bigvee \{ (r_t^a_p)(r)| t \in R, y \in M, t(r(y)) = x \} \]
\[ \leq \bigvee \{ (r^a_p)| t \in R, y \in M, t(r(y)) = x \} \]
\[ \leq \bigvee \{ (t^a_f)r)| t \in R, y \in M, t(r(y)) = x \} \]
\[ = y(x) \quad \forall x \in M. \]

Therefore \( \alpha_{r^a_p} \subseteq \gamma \).

So \( r_a > a_p \subseteq \chi_0 \cup \gamma = \gamma \).

Hence \( \bigcup \{ \delta | \delta \in \text{FI}(R), \delta a_p \subseteq \gamma \} \supseteq \bigcup \{ \delta r \in R, \alpha \in [0,1], r_a, a_p \subseteq \gamma \} = \sigma. \)

Therefore \( \sigma = \bigcup \{ \delta | \delta \in \text{FI}(R), \delta a_p \subseteq \gamma \}. \)

Lemma 4.2.3: \( \sigma \in \text{FI}(R) \) and \( \sigma(0) = 1. \)

Proof: Clearly \( \chi_0 a_p = \chi_0 \subseteq \gamma. \)

So \( \chi_0 \subseteq \sigma. \)

Now

\[ \sigma(r_1) \land \sigma(r_2) = \left( \bigvee \{ \delta | \delta \in \text{FI}(R), \delta a_p \subseteq \gamma \} \right) \land \left( \bigvee \{ \delta | \delta \in \text{FI}(R), \delta a_p \subseteq \gamma \} \right) \]
\[ = \left( \bigvee \{ \delta | \delta \in \text{FI}(R), \delta a_p \subseteq \gamma \} \right) \land \left( \bigvee \{ \delta | \delta \in \text{FI}(R), \delta a_p \subseteq \gamma \} \right) \]
\[ \leq \bigvee \{ (\delta_1 \land \delta_2)(r_1) \land (\delta_1 \land \delta_2)(r_2) | \delta_1, \delta_2 \in \text{FI}(R), \delta_1 a_p \subseteq \gamma, \delta_2 a_p \subseteq \gamma \} \]
\[ \leq \bigvee \{ (\delta_1 \land \delta_2)(r_1 - r_2) | \delta_1, \delta_2 \in \text{FI}(R), \delta_1 a_p \subseteq \gamma, \delta_2 a_p \subseteq \gamma \} \]
\[ = \sigma(r_1 - r_2) \quad \forall r_1, r_2 \in R \]

Now \( \sigma(s) = \bigvee \{ \delta(s) | \delta \in \text{FI}(R), \delta a_p \subseteq \gamma \} \)
\[ \geq \bigvee \{ \delta(r) | \delta \in \text{FI}(R), \delta a_p \subseteq \gamma \} \]
\[ = \sigma(r) \quad \forall s, r \in R \]

Therefore \( \sigma \in \text{FI}(R). \)

Theorem 4.2.4: If \( \gamma \) is an essential fuzzy submodule of \( M \) and \( a \neq 0 \) then there exists an essential fuzzy ideal \( \sigma \) such that \( a_p \sigma \neq \chi_0 \) and \( a_p \sigma \subseteq \gamma \) where \( p \in (0,1]. \)
Proof: We set \( \sigma = \cup \{ r_{a} | r \in R, \alpha \in [0,1], r_{a}a_{p} \subseteq \gamma \} \).

By previous lemmas \( \sigma \in \text{FI}(R) \).

Now \( \gamma \) being an essential fuzzy submodule, \( \gamma' \) is an essential submodule of \( M \). So for \( a \neq \emptyset \) and \( \gamma' \), there exists an essential ideal \( L = \{ r \in R | ra \in \gamma' \} \)

such that \( aL \neq \{ \emptyset \} \) and \( aL \subseteq \gamma' \).

Now \( (r_{a}a_{p})(m) = \bigvee \{ r_{a}(x) \land a_{p}(y) | x \in R, y \in M, xy = m \} \) = \[
\begin{cases} 
0 & \text{if } m \neq ra \\
\alpha \land p & \text{if } m = ra
\end{cases}
\]

Therefore \( r_{a}a_{p} = (ra)_{\alpha \land p} \).

So \( \sigma = \cup \{ r_{a} | r \in R, \alpha \in [0,1], (ra)_{\alpha \land p} \in \gamma \} \).

\( x \in \sigma^* \) implies that there exists \( \alpha \in (0,1] \) such that \( (xa)_{\alpha \land p} \in \gamma \).

i.e. \( \gamma(xa) \geq \alpha \land p > 0 \)

i.e. \( xa \in \gamma' \)

i.e. \( x \in L \).

Therefore \( \sigma^* \subseteq L \).

Conversely, \( x \in L \) implies \( xa \in \gamma' \).

i.e. \( \gamma(xa) = \alpha \), say, where \( \alpha \neq 0 \)

i.e. \( \gamma(xa) \geq \alpha \land p \)

i.e. \( (xa)_{\alpha \land p} \in \gamma \)

i.e. \( x_{a} \in \{ x_{a} | x \in R, \alpha \in [0,1], (xa)_{\alpha \land p} \in \gamma \} \)

i.e. \( x_{a} \in \sigma \)

i.e. \( \sigma(x) \geq \alpha > 0 \)

i.e. \( x \in \sigma^* \)

Therefore \( L \subseteq \sigma^* \).

So \( \sigma^* = L \).
Therefore from above $a\sigma \neq \{\theta\}$, $a\sigma^* \subseteq \gamma^*$ and $\sigma^*$ is essential and hence $\sigma$ is essential.

From the definition of $\sigma$, $\sigma_{a_p} \subseteq \gamma$.

Since $a\sigma \neq \{\theta\}$, there exists $r \in \sigma^*$ such that $ra \neq \{\theta\}$.

Now $(\sigma_{a_p})(ra) \geq \sigma(r) \wedge a_p(a) > 0$.

So $\sigma_{a_p} \neq \chi_0$.

Hence the result.

Definition: 4.2.5: We define a fuzzy set $Z(\chi_R)$ by:

$$Z(\chi_R) = \cup \{\gamma \mid \gamma \in [0,1]^R, \gamma \sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\}.$$  

Lemma 4.2.6: $Z(\chi_R) = \cup \{r_\alpha \mid r \in R, \alpha \in [0,1], R_\alpha \sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\}$.

Proof: Clearly $\{r_\alpha \mid r \in R, \alpha \in [0,1], R_\alpha \sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\} 
\subseteq \{\gamma \mid \gamma \in [0,1]^R, \gamma \sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\}$.

Therefore $\cup \{r_\alpha \mid r \in R, \alpha \in [0,1], R_\alpha \sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\} 
\subseteq \cup \{\gamma \mid \gamma \in [0,1]^R, \gamma \sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\} 
= Z(\chi_R)$.

Let $\gamma \in [0,1]^R$ such that $\gamma \sigma \subseteq \chi_0$ for some essential fuzzy ideal $\sigma$.

Let $r \in R$ such that $\gamma(r) = \alpha$.

Now $(r_\alpha \sigma)(x) = \vee \{r_\alpha(s) \wedge \sigma(y) \mid x = sy; s,y \in R\} 
= \vee \{r_\alpha(s) \wedge \sigma(y) \mid x = rz\}$
$$\leq \vee \{r_\alpha(s) \wedge \sigma(y) \mid x = sy; s,y \in R\} 
= (\gamma \sigma)(x) 
\subseteq \chi_0(x) \ \forall x \in R.$$  

Thus $R_\alpha \sigma \subseteq \chi_0$.

So $Z(\chi_R) \subseteq \cup \{r_\alpha \mid r \in R, \alpha \in [0,1], R_\alpha \sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\}.$

Hence the result.

Lemma 4.2.7: $Z(\chi_R) = \cup \{\gamma \mid \gamma \in \text{FI}(R), \gamma \sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma \text{ of } R\}$.
Proof: Clearly \( \gamma \in \text{FI}(R), \gamma \sigma \subseteq \chi_0 \), for some essential fuzzy ideal \( \sigma \)

\[ \leq \{ \gamma \in [0,1]^R, \gamma \sigma \subseteq \chi_0 \}, \text{ for some essential fuzzy ideal } \sigma \} . \]

Therefore \( \bigcup \{ \gamma \in \text{FI}(R), \gamma \sigma \subseteq \chi_0 \}, \text{ for some essential fuzzy ideal } \sigma \} \subseteq Z(\chi_R) . \)

Let \( r \in R, \alpha \in [0,1] \) and \( r_\alpha \sigma \subseteq \chi_0 \) for some essential fuzzy ideal \( \sigma \).

Let \( \gamma = \langle r_\alpha \rangle \).

Now \( \langle r_\alpha \rangle \sigma = (\chi_0 \cup \alpha_{\sigma < r}) \sigma \)

\[ = \chi_0 \sigma \cup \alpha_{\sigma < r} \sigma \]

\[ \leq \chi_0 \cup \alpha_{\sigma < r} \sigma \]

Again \( (\alpha_{\sigma < r} \sigma)(x) = v[\alpha_{\sigma < r} \sigma(y) | s, y \in R, sy = x] \)

\[ = v[\alpha \wedge \sigma(y) | s < r >, y \in R, sy = x] \]

\[ \leq v[(r_\alpha \sigma)(t, r, y \in R, x = t(ry))] \]

\[ \leq v[\chi_0(ry) | t, r, y \in R, x = t(ry)] \]

\[ \leq v[\chi_0(t(ry)) | t, r, y \in R, x = t(ry)] \]

\[ = \chi_0(x) \quad \forall x . \]

So \( \alpha_{\sigma < r} \sigma \subseteq \chi_0 \cup \chi_0 = \chi_0 \).

Hence \( \bigcup \{ \gamma \in \text{FI}(R), \gamma \sigma \subseteq \chi_0 \}, \text{ for some essential fuzzy ideal } \sigma \}

\[ \geq \bigcup \{ r_\alpha | r \in R, \alpha \in [0,1], r_\alpha \sigma \subseteq \chi_0 \}, \text{ for some essential fuzzy ideal } \sigma \} \]

\[ = Z(\chi_R) . \]

Therefore \( Z(\chi_R) = \bigcup \{ \gamma \in \text{FI}(R), \gamma \sigma \subseteq \chi_0 \}, \text{ for some essential fuzzy ideal } \sigma \text{ of } R \} . \)

Theorem 4.2.8: \( Z(\chi_R) \subseteq \text{FI}(R) \).

Proof: Clearly \( \chi_R \) is an essential fuzzy ideal of \( R \).

Since \( \chi_0 \chi_R = \chi_0 \), so \( \chi_0 \subseteq Z(\chi_R) \).

Now \( Z(\chi_R)(r_1) \wedge Z(\chi_R)(r_2) = (\vee \{ \gamma_1 | \gamma_1 \in \text{FI}(R), \gamma_1 \sigma_1 \subseteq \chi_0 \}, \text{ for some essential fuzzy ideal } \sigma_1 \}) \wedge (\vee \{ \gamma_2 | \gamma_2 \in \text{FI}(R), \gamma_2 \sigma_2 \subseteq \chi_0 \}, \text{ for some essential fuzzy ideal } \sigma_2 \} . \)
= \vee \{ \gamma_1(r_1) \land \gamma_2(r_2) \mid \gamma_1, \gamma_2 \in \text{FI}(R), \gamma_1 \sigma_1 \subseteq \chi_0, \gamma_2 \sigma_2 \subseteq \chi_0, \\
for some essential fuzzy ideals \sigma_1 and \sigma_2 \}
\leq \vee \{ (\gamma_1 + \gamma_2)(r_1) \land (\gamma_1 + \gamma_2)(r_2) \mid \gamma_1, \gamma_2 \in \text{FI}(R), \\
\gamma_1 \sigma_1 \subseteq \chi_0, \gamma_2 \sigma_2 \subseteq \chi_0 \}
\leq \vee \{ (\gamma_1 + \gamma_2)(r_1 - r_2) \mid (\gamma_1 + \gamma_2)(\sigma_1 \cap \sigma_2) \\
\subseteq \gamma_1 (\sigma_1 \cap \sigma_2) + \gamma_2 (\sigma_1 \cap \sigma_2) \\
\subseteq \gamma_1 \sigma_1 + \gamma_2 \sigma_2 \\
\subseteq \chi_0 + \chi_0 \subseteq \chi_0 \}
= \vee \{ (\gamma_1 + \gamma_2)(r_1 - r_2) \mid (\gamma_1 + \gamma_2)\sigma \subseteq \chi_0, \\
for some essential fuzzy ideal \sigma \}
= Z(\chi_R)(r_1 - r_2) \quad \forall r_1, r_2 \in R.

Now \ Z(\chi_R)(sr) = \vee \{ \gamma(sr) \mid \gamma \in \text{FI}(R), \gamma \sigma \subseteq \chi_0 \text{ for some essential fuzzy ideal } \sigma \}
\geq \vee \{ \gamma(r) \mid \gamma \in \text{FI}(R), \gamma \sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma \}
= Z(\chi_R)(r) \quad \forall s, r \in R

Therefore \ Z(\chi_R) \in \text{FI}(R).

Note: The fuzzy set \ Z(\chi_R) \ defined in Def 4.2.5, also defined in Lemma 4.2.6 and Lemma 4.2.7 is called the singular fuzzy ideal of \ R.

Theorem 4.2.9: For \ \gamma \in \text{FI}(R), \ \gamma \subseteq \chi_R \ if and only if \ \text{ann}(\gamma) \ is essential.
Proof: Let \ \gamma \subseteq \chi_R. Then \ \gamma \sigma \subseteq \chi_0 \ for some essential fuzzy ideal \sigma \ of \ R.

So \ \sigma \subseteq \text{ann}(\gamma).

Now \ \sigma \ being essential, \ \text{ann}(\gamma) \ is also essential.

Conversely, let \ \text{ann}(\gamma) \ be essential.

Then \ \gamma \text{ann}(\gamma) \subseteq \chi_0.

By definition of \ Z(\chi_R), \ \gamma \subseteq Z(\chi_R).

Theorem 4.2.10: For \ \gamma_a \in [0,1]^R, \ \gamma_a \in Z(\chi_R) \ if and only if \ \text{ann}(\gamma_a) \ is essential.
Proof: Let \( r_a \in Z(\chi_R) \). Then \( r_a \sigma \subseteq \chi_0 \) for some essential fuzzy ideal \( \sigma \).

So \( \sigma \subseteq \text{ann}(r_a) \).

Now \( \sigma \) being essential, \( \text{ann}(r_a) \) is also essential.

Conversely, let \( \text{ann}(r_a) \) be essential.

Now \( r_a \text{ann}(r_a) \subseteq \chi_0 \)

By definition of \( Z(\chi_R) \), \( r_a \in Z(\chi_R) \).

Lemma 4.2.11: Let \( \gamma_1, \gamma_2, \sigma \in \mathcal{F}(R) \) be such that \( \gamma_1 \subseteq \sigma \) and \( \gamma_2 \subseteq \sigma \).

Then \( \gamma_1 + \gamma_2 \subseteq \sigma \).

Proof: \( \gamma_1 \subseteq \sigma \) and \( \gamma_2 \subseteq \sigma \) imply \( \gamma_1 + \gamma_2 \subseteq \sigma = \gamma_1 \cup \gamma_2 \subseteq \sigma \).

Note: Let \( \delta \) be a fuzzy submodule of \( M \).

Let us define a fuzzy subset \( Z(\delta) \) of \( M \) by:

\[
Z(\delta) = \cup \{ \mu | \mu \in [0,1]^M, \mu \subseteq \delta, \mu \sigma \subseteq \chi_0, \text{ for some essential fuzzy ideal } \sigma \text{ of } R \}.
\]

Lemma 4.2.12: \( Z(\delta) = \cup \{ m_a | m_a \in \delta, m_a \sigma \subseteq \chi_0, \text{ for some essential fuzzy ideal } \sigma \text{ of } R \} \).

Lemma 4.2.13: \( Z(\delta) = \cup \{ \mu | \mu \in \mathcal{F}(M), \mu \sigma \subseteq \chi_0, \text{ for some essential fuzzy ideal } \sigma \text{ of } R \} \).

Theorem 4.2.14: \( Z(\delta) \) is a fuzzy submodule of \( M \).

Proof: This can be easily proved using lemmas: 4.2.11, 4.2.12 and 4.2.13.

Let \( \delta \) be a fuzzy ideal of \( R \). We define a fuzzy subset \( Z(\delta) \) by:

\[
Z(\delta) = \cup \{ \gamma | \gamma \in [0,1]^R, \gamma \subseteq \delta, \gamma \sigma \subseteq \chi_0, \text{ for some essential fuzzy ideal } \sigma \text{ of } R \}.
\]

Lemma 4.2.15: \( Z(\delta) = \cup \{ t_a | t_a \in \delta, t_a \sigma \subseteq \chi_0, \text{ for some essential fuzzy ideal } \sigma \} \).

Lemma 4.2.16: \( Z(\delta) = \cup \{ \gamma | \gamma \in \mathcal{F}(R), \gamma \subseteq \delta, \gamma \sigma \subseteq \chi_0, \text{ for some essential fuzzy ideal } \sigma \} \).

Lemma 4.2.17: \( Z(\delta) \) is a fuzzy ideal of \( \delta \).

Lemma 4.2.18: If \( \delta_1 \) and \( \delta_2 \) are fuzzy submodules of \( M \) with \( \delta_2 \subseteq \delta_1 \) then \( Z(\delta_2) = \delta_2 \cap Z(\delta_1) \).

Proof: Let \( m_a \in Z(\delta_2) \). Then \( m_a \in \delta_2 \) and \( m_a \sigma \subseteq \chi_0, \text{ for some essential fuzzy ideal } \sigma \text{ of } R \). So \( m_a \in \delta_1 \) and \( m_a \sigma \subseteq \chi_0 \) imply \( m_a \in Z(\delta_1) \). Therefore \( m_a \in \delta_2 \cap Z(\delta_1) \).
Conversely let $m_a \in Z(\delta_1) \cap \delta_2$. Then $m_a \in \delta_2$ and $m_a \sigma \subseteq \chi_\delta$ for some essential fuzzy ideal $\sigma$. So $m_a \in Z(\delta_2)$. Consequently $Z(\delta_2) = \delta_2 \cap Z(\delta_1)$.

**Definition 4.2.19:** A fuzzy submodule $\delta$ of $M$ is singular or non singular according as $Z(\delta) = \delta$ or $\chi_\delta$.

**Lemma 4.2.20:** If $\delta \in F(M)$, then $Z(\delta)$ is singular.

**Proof:**
\[ Z(Z(\delta)) = \cup \{ m_a | m_a \in Z(\delta), m_a \sigma \subseteq \chi_\delta, \text{for some essential fuzzy ideal } \sigma \} \]
\[ = \cup \{ m_a | m_a \in Z(\delta) \} \]
\[ = Z(\delta) \]

**Theorem 4.2.21:** (1) Let $\delta_2$ be any fuzzy submodule of a non singular (singular) fuzzy submodule $\delta_1$. Then $\delta_2$ is also non singular (singular).

(2) If $\delta_1$ is an essential extension of a non singular fuzzy submodule $\delta_2$, then $\delta_1$ is non singular.

**Proof:** (1) By lemma:4.2.18, $Z(\delta_2) = \delta_2 \cap Z(\delta_1)$.

$\delta_1$ non singular implies $Z(\delta_1) = \chi_\delta$.

So $Z(\delta_2) = \delta_2 \cap \chi_\delta = \chi_\delta$.

Therefore $\delta_2$ is also non singular.

Again $\delta_1$ singular implies $Z(\delta_1) = \delta_1$.

So $Z(\delta_2) = \delta_2 \cap \delta_1 = \delta_2$.

Therefore $\delta_2$ is singular.

(2) $\delta_2$ non singular implies $Z(\delta_2) = \chi_\delta$.

Now $Z(\delta_2) = \delta_2 \cap Z(\delta_1)$ implies $\delta_2 \cap Z(\delta_1) = \chi_\delta$.

But $\delta_1$ is an essential extension of $\delta_2$. So $Z(\delta_1) = \chi_\delta$.

Therefore $\delta_1$ is non singular.

**Theorem 4.2.22:** The sum (direct or not) of two singular fuzzy submodules of a fuzzy submodule is again singular.

**Proof:** Let $\delta_1$ and $\delta_2$ be two singular fuzzy submodules such that $\delta_1 \subseteq \delta$ and $\delta_2 \subseteq \delta$. 

106
Then $Z(\delta_1) = \delta_1$, $Z(\delta_2) = \delta_2$.

Now $\delta_1 \subseteq \delta \Rightarrow Z(\delta_1) \subseteq Z(\delta)$

$\Rightarrow \delta_1 \subseteq Z(\delta)$

and $\delta_2 \subseteq \delta \Rightarrow Z(\delta_2) \subseteq Z(\delta)$

$\Rightarrow \delta_2 \subseteq Z(\delta)$

$\therefore \delta_1 + \delta_2 \subseteq Z(\delta)$

By Lemma 4.2.20, $Z(\delta)$ is singular.

So $\delta_1 + \delta_2$ is a fuzzy submodule of a singular fuzzy submodule $Z(\delta)$.

Hence by Theorem 4.2.21, $\delta_1 + \delta_2$ is singular.

**Lemma 4.2.23:** For $r \in R$, $\alpha \in [0,1]$, $r_\alpha x_R$ is a fuzzy ideal.

**Proof:**

\[
(r_\alpha x_R)(x) = \vee \{r_\alpha(y) \land \chi_R(z) | y, z \in R, x = yz\}
\]

\[
= \vee \{r_\alpha(y) | y \in R, x = yz\}
\]

\[
= \begin{cases} 
0 & \text{if } x \notin <r> \\
\alpha & \text{if } x \in <r>
\end{cases}
\]

Therefore $r_\alpha x_R = <r>_{\alpha}$.

Let $x, y \in R$.

If $<r>_{\alpha}(x) = 0$ then $<r>_{\alpha}(x) \land <r>_{\alpha}(y) \leq <r>_{\alpha}(x - y)$.

If $<r>_{\alpha}(x) \neq 0$ (i.e. $= \alpha$) and $<r>_{\alpha}(y) \neq 0$ (i.e. $= \alpha$)

then $x = rz_1$ and $y = rz_2$ for some $z_1, z_2 \in R$.

So $x - y = r(z_1 - z_2)$.

Hence $<r>_{\alpha}(x - y) = \alpha = <r>_{\alpha}(x) \land <r>_{\alpha}(y)$.

Similarly $<r>_{\alpha}(xy) \geq <r>_{\alpha}(x)$ $\forall x, y \in R$.

Therefore $<r>_{\alpha}$ i.e. $r_\alpha x_R$ is a fuzzy ideal.

**Lemma 4.2.24:** $\chi_\alpha \cup r_\alpha x_R$ is a fuzzy ideal, where $r \in R, \alpha \in [0,1]$.

**Proof:** Let $\sigma = r_\alpha x_R$ and $\gamma = \chi_\alpha \cup \sigma$.

Let $t \in [0,1]$ and $t \leq \alpha$. Then $\gamma_t = \sigma_t$ which is an ideal of $R$.  

107
Let \( t > \alpha \). Then \( \gamma_i = \{0\} \) which is an ideal of \( R \).

Hence \( \gamma \) is a fuzzy ideal of \( R \).

**Theorem 4.2.25:** \( \chi_\alpha \) is semiprime if and only if \( Z(\chi_R) = \chi_\alpha \).

**Proof:** We suppose that \( Z(\chi_R) = \chi_\alpha \).

Let \( r_\alpha \in \chi_R \) such that \( r_\alpha^2 \subseteq \chi_\alpha \).

Let \( \sigma \) be a fuzzy ideal and \( x_{\alpha'} \in \sigma \) where \( x_{\alpha'} \neq \chi_\alpha \).

Now \( (x_{\alpha'}, \chi_R)(y) = \vee \{x_{\alpha'}(a) \land \chi_R(b) \mid y = ab\} \)

\[
= \begin{cases} 
\alpha' & \text{if } y = xb, \text{ for some } b \\
0 & \text{if } y \neq xb, \text{ for all } y, b
\end{cases}
\]

Therefore \( x_{\alpha'} \chi_R = < x >_{\alpha'} \).

If \( y \in < x > \) then \( < x >_{\alpha'}(y) \subseteq \sigma(y) \).

If \( y \in < x > \) then \( y = xz \) for some \( z \in R \).

Now \( \sigma(y) = \sigma(xz) \geq \sigma(x) \geq \alpha' = < x >_{\alpha'}(y) \).

Therefore \( < x >_{\alpha'} \subseteq \sigma \).

That is \( x_{\alpha'} \chi_R \subseteq \sigma \).

Now, either \( r_\alpha x_{\alpha'} \subseteq \chi_\alpha \) or \( r_\alpha x_{\alpha'} \not\subseteq \chi_\alpha \).

\( r_\alpha x_{\alpha'} \subseteq \chi_\alpha \) implies \( x_{\alpha'} \in \text{ann}(r_\alpha) \).

\( r_\alpha x_{\alpha'} \not\subseteq \chi_\alpha \) implies \( r_\alpha (r_\alpha x_{\alpha'}) \subseteq \chi_0 \),

that is \( r_\alpha x_{\alpha'} \in \text{ann}(r_\alpha) \).

In both the cases \( x_{\alpha'} \chi_R \cap \text{ann}(r_\alpha) \neq \chi_\alpha \) where \( x_{\alpha'} \in \sigma \).

Therefore \( \sigma \cap \text{ann}(r_\alpha) \neq \chi_\alpha \).

So \( \text{ann}(r_\alpha) \) is essential.

This implies \( r_\alpha \in Z(\chi_R) = \chi_\alpha \).
Hence $\chi_0$ is semiprime.

Conversely we suppose that $\chi_0$ is semiprime.

Clearly $\chi_0 \subseteq Z(\chi_R)$.

Let $r_\alpha \in Z(\chi_R)$ where $r \neq 0$.

We set $\mu = r_\alpha \chi_R \cap \text{ann}(r_\alpha)$ with $\mu(0) = 1$.

Now $(r_\alpha \chi_R) \text{ann}(r_\alpha) \subseteq \chi_0$.

Therefore $\mu^2 = \mu \mu \subseteq (r_\alpha \chi_R) \text{ann}(r_\alpha) \subseteq \chi_0$.

This implies $\mu \subseteq \chi_0$, as $\chi_0$ is semiprime.

Again, $r_\alpha \in Z(\chi_R)$ implies $\text{ann}(r_\alpha)$ is essential.

Therefore $(r_\alpha \chi_R)(x) = 0$ for $x \neq 0$.

That is $(< r >_\alpha)(x) = 0$ for $x \neq 0$.

In particular $(< r >_\alpha)(r) = 0$.

This implies $\alpha = 0$.

Hence $Z(\chi_R) = \chi_0$.

### 4.3. Chain condition on fuzzy annihilators:

In this section we attempt to investigate fuzzy concepts of rings with chain condition on annihilators. The finiteness conditions play a key role in various structure theorems of rings and modules. Imposing ascending chain condition (acc) on fuzzy annihilators we establish some results which may lead to fuzzy aspects of Goldie theorems.

**Definition 4.3.1:** A fuzzy ideal $\mu$ of a ring $R$ is called nilpotent if $\mu^n \subseteq \chi_0$, for some $n \in \mathbb{Z}^+$.

**Theorem 4.3.2:** Let $R$ be a ring with ascending chain condition for fuzzy annihilators, then the singular fuzzy ideal of $R$ is nilpotent.

**Proof:** For simplicity let us denote the singular fuzzy ideal $Z(\chi_R)$ of $R$ by $Z$. 

109
Clearly \( Z \supseteq Z^2 \supseteq Z^3 \supseteq \ldots \), where \( Z^n = Z^{n-1} \).

This implies \( \text{ann}(Z) \subseteq \text{ann}(Z^2) \subseteq \text{ann}(Z^3) \subseteq \ldots \).

Therefore there exists \( n \in \mathbb{Z}^+ \) such that \( \text{ann}(Z^n) = \text{ann}(Z^{n+1}) \).

We suppose that \( Z^{n+1} \neq \mathcal{X}_0 \).

That is \( Z^n \neq \mathcal{X}_0 \).

Therefore there exists \( b (\neq 0) \in \mathbb{R} \) such that \( (Z^n Z)(b) \neq 0 \).

So \( Z(\alpha) \wedge Z^n(t) \neq 0 \) for some \( r, t \in \mathbb{R} \) such that \( b = rt \).

This implies \( Z(\alpha) \neq 0 \) and \( Z^n(t) \neq 0 \).

Now \( Z(b) = Z(\alpha t) \geq Z(\alpha) > 0 \).

And \( Z^n(b) = Z^n(\alpha t) \geq Z^n(\alpha) > 0 \).

Let \( Z(\alpha) = \alpha \). Then \( Z(b) \geq \alpha \).

So \( (Z^n \tau_\alpha)(b) \geq Z^n(t) \wedge \tau_\alpha(t) > 0 \).

Therefore there exists \( \tau_\alpha \in Z \) such that \( (Z^n \tau_\alpha)(b) \neq 0 \) for some \( b (\neq 0) \in \mathbb{R} \). We choose such an \( \tau_\alpha \in Z \) so that \( \text{ann}(\tau_\alpha) \) is as large as possible.

Let \( x_\alpha \in Z \). Then \( \text{ann}(x_\alpha) \) is an essential fuzzy ideal of \( \mathbb{R} \). By considering the fuzzy ideal \( \mathcal{X}_0 \cup \tau_\alpha \mathcal{X}_R \) we have \( \text{ann}(x_\alpha) \cap (\mathcal{X}_0 \cup \tau_\alpha \mathcal{X}_R) \neq \mathcal{X}_0 \).

Thus there exists an element \( y \in \mathcal{X}_R \) and \( \tau_\alpha y_k \in \text{ann}(x_\alpha) \) such that \( (\tau_\alpha y_k)(\alpha) \neq 0 \) for some \( \alpha (\neq 0) \in \mathbb{R} \).

For any \( d \in \mathbb{R} \),

\[
(r_\alpha x_\alpha)(d) = \vee \{ r_\alpha(e) \wedge x_\alpha(f) \mid d = ef \}
\]

\[
= \begin{cases} 
0 & \text{if } d \neq \alpha x \\
\alpha \wedge t & \text{if } d = \alpha x
\end{cases}
\]

i.e. \( r_\alpha x_\alpha = (\alpha x)_{\alpha \wedge t} \) is a fuzzy point of \( \mathbb{R} \).

If \( d = \alpha x \), then

\[
(r_\alpha x_\alpha)(d) = \alpha \wedge t \leq t \leq Z(x) \leq Z(\alpha x) = Z(d).
\]
If \( d \neq r x_i \), then

\[ (r_a x_i)(d) = 0 \leq Z(d). \]

So \( r_a x_i \in Z \) if \( x_i \in Z \).

Let \( w_p \in \text{ann}(r_a) \). Then \( r_a w_p \subseteq \chi_0 \).

This implies \( x_i(r_a w_p) \subseteq x_i \chi_0 \subseteq \chi_0 \).

Therefore \( w_p \in \text{ann}(x_i r_a) \).

So \( \text{ann}(r_a) \subseteq \text{ann}(x_i r_a) \).

But \( (r_a y_k)(a) \neq 0 \) that is \( r_a y_k \not\subseteq \chi_0 \).

This implies \( y_k \in \text{ann}(r_a) \).

And \( y_k(r_a x_i) \subseteq \chi_0 \) implies \( y_k \in \text{ann}(r_a x_i) \).

Therefore \( \text{ann}(r_a) \subseteq \text{ann}(r_a x_i) \).

If \( (Z^n(r_a x_i))(d) \neq 0 \) for some \( d(\neq 0) \in R \), then this together with \( \text{ann}(r_a) \subseteq \text{ann}(r_a x_i) \) will give a contradiction. Because we have chosen \( r_a \) so that \( \text{ann}(r_a) \) is as large as possible with \( (Z^n r_a)(b) \neq 0 \) for some \( b(\neq 0) \in R \).

Therefore \( (Z^n(r_a x_i))(d) = 0 \ \forall d(\neq 0) \in R \).

But \( x_i \) is an arbitrary element of \( Z \), so \( Z^{n+1} r_a \subseteq \chi_0 \).

This implies \( r_a \in \text{ann}(Z^{n+1}) = \text{ann}(Z^n) \).

Therefore \( (Z^n r_a)(d) = 0 \ \forall d(\neq 0) \in R \).

And this is a contradiction as \( (Z^n r_a)(b) \neq 0 \).

Hence \( Z^{n+1} = \chi_0 \).

**Definition 4.3.3:** Let \( \mu \) be a fuzzy ideal of \( R \). A fuzzy point \( r_a \in \mu \) is called nilpotent if \( (r_a)^n \subseteq \chi_0 \) for some \( n \in Z^+ \). \( \mu \) is called nil ideal if each of its fuzzy points is nilpotent.

**Theorem 4.3.4:** Let \( R \) be with ascending chain condition for fuzzy annihilators. If \( \chi_0 \) is semiprime then \( R \) has no non zero nil fuzzy ideal.
Proof: Let $\mu$ be a fuzzy ideal of $R$ and $\mu \neq \chi_0$. Let $r_a$ be a fuzzy point with $r_a \not\subseteq \chi_0$ such that $\text{ann}(r_a)$ is as large as possible. If possible let $r_a x_r r_a \subseteq \chi_0$ for all fuzzy points $x_r$ of $R$. Then $r_a x_r r_a \subseteq \chi_0$ where $\chi_r(1) = 1$ and $\chi_r(a) = 0$ if $a \neq 1$. This implies $(r_a)^2 \subseteq \chi_0$, so $r_a \subseteq \chi_0$.

This is a contradiction. Hence there exists a fuzzy point $x_r$ such that $r_a x_r r_a \not\subseteq \chi_0$.

Let $y_k \in \text{ann}(r_a)$. Then $y_k r_a \subseteq \chi_0$.

Now $y_k (r_a x_r r_a) = (y_k r_a) (x_r r_a) \subseteq \chi_0$.

So $y_k \in \text{ann}(r_a x_r r_a)$.

Therefore $\text{ann}(r_a) \subseteq \text{ann}(r_a x_r r_a)$.

This implies that $\text{ann}(r_a) = \text{ann}(r_a x_r r_a)$.

If $r_a x_r \subseteq \chi_0$ then $r_a x_r x_r \subseteq \chi_0$. So $r_a x_r \not\subseteq \chi_0$.

This implies $x_r \not\in \text{ann}(r_a) = \text{ann}(r_a x_r r_a)$.

So $x_r (r_a x_r r_a) \not\subseteq \chi_0$.

Hence $(x_r r_a)^2 \not\subseteq \chi_0$ and $x_r r_a x_r \not\in \text{ann}(r_a) = \text{ann}(r_a x_r r_a)$.

This implies $(x_r r_a x_r)(r_a x_r r_a) \not\subseteq \chi_0$.

So $(x_r r_a)^3 \not\subseteq \chi_0$ and so on.

Hence $x_r r_a$ is not nilpotent and $x_r r_a \in \mu$.

So $\mu$ is not a nil ideal.

Lemma 4.3.5: Sum of a finite number of nilpotent fuzzy ideals is again a nilpotent fuzzy ideal.

Proof: We have, if $\mu_i$, (i=1,2,...,n) are fuzzy ideals then $\sum_{i=1}^{n} \mu_i$ is also a fuzzy ideal.

Let $\mu$ and $\sigma$ be two nilpotent fuzzy ideals.

Then $\mu^m \subseteq \chi_0$ and $\sigma^n \subseteq \chi_0$ for some $m,n \in Z^+$.

Now $(\mu + \sigma)^2 = (\mu + \sigma)(\mu + \sigma) \subseteq (\mu + \sigma)\mu + (\mu + \sigma)\sigma$

$\subseteq \mu\mu + \sigma\mu + \mu\sigma + \sigma\sigma$

$= \mu^2 + 2\mu\sigma + \sigma^2$.

We can show by induction that $(\mu + \sigma)^{m+n}$ is contained in a sum of products in which either $\mu$
Definition 4.3.6: Let $\delta$ be a fuzzy subring of $R$. A fuzzy point $r_a$ of $\delta$ is called regular if $\text{ann}(r_a) = \chi_0$.

Notation: $C(\chi_0)$ denotes the set of all regular fuzzy points.

Lemma 4.3.7: Let $\delta$ be a fuzzy subring of an integral domain $R$. Every fuzzy point $r_a$, with $r \neq 0$, $\alpha \neq 0$, of $\delta$ is regular.

Proof: Obviously $\chi_0 \subseteq \text{ann}(r_a)$.

Let $s_k \subseteq \text{ann}(r_a)$ where $s \neq 0$.

Then $r_a s_k \subseteq \chi_0$.

This implies $(rs)_{\alpha k} \subseteq \chi_0$.

So $\alpha k = 0$ as $rs \neq 0$.

Hence $k = 0$.

This gives $s_k \subseteq \chi_0$.

Hence $\text{ann}(r_a) = \chi_0$. So $r_a$ is regular.

Definition 4.3.8: A fuzzy submodule $\delta(\neq \chi_0)$ is said to be uniform if any two nonzero fuzzy submodules of $\delta$ have non zero intersection, that is, if each nonzero fuzzy submodule of $\delta$ is essential in $\delta$. $\delta$ has finite Goldie dimension if $\delta$ does not contain a direct sum of infinite number of nonzero fuzzy submodules. A fuzzy subring $\delta$ with $\delta(0) = 1$ is said to have finite Goldie dimension if it has finite Goldie dimension as a fuzzy submodule of $R$. We call $\delta \in \text{F}(R)$, a Goldie fuzzy subring if it has finite Goldie dimension and satisfies ascending chain condition for fuzzy annihilators.

Theorem 4.3.9: If $\delta$ has finite Goldie dimension then each nonzero fuzzy submodule of $\delta$ contains a uniform fuzzy submodule, and there is a finite number of uniform fuzzy submodules of $\delta$ whose sum is direct and is an essential fuzzy submodule of $\delta$. 
Proof: We suppose that $\delta$ has finite Goldie dimension and let $\sigma$ be a fuzzy submodule of $\delta$ and $\sigma \neq \chi_\delta$. We shall show that $\sigma$ has a uniform fuzzy submodule. If $\sigma$ is itself uniform then the result is trivial. Suppose that $\sigma$ is not uniform. Then there are two fuzzy submodules $\mu_1, \sigma_1$ of $\sigma$ such that $\mu_1 \cap \sigma_1 = \chi_\delta$. Thus $\mu_1 + \sigma_1$ is a direct sum of two non zero fuzzy submodules. If $\mu_1$ or $\sigma_1$ is uniform, we stop. Otherwise, there are non zero fuzzy submodules $\mu_2, \sigma_2$ of $\sigma_1$ such that $\mu_2 \cap \sigma_2 = \chi_\delta$. Thus $\mu_1 + \mu_2 + \sigma_2$ is a direct sum of three non zero fuzzy submodules of $\delta$. If $\mu_2$ or $\sigma_2$ is uniform, we stop. Otherwise, we continue the process. Since $\delta$ has finite Goldie dimension, this process must stop after a finite number of steps and then we will have a uniform fuzzy submodule of $\sigma$.

Let $\gamma_1$ be a uniform fuzzy submodule of $\delta$. Suppose that $\gamma_1$ is not essential in $\delta$. Then there is a non zero fuzzy submodule $\sigma_1$ of $\delta$ such that $\gamma_1 \cap \sigma_1 = \chi_\delta$. Let $\gamma_2$ be a uniform fuzzy submodule of $\sigma_1$, then the sum $\gamma_1 + \gamma_2$ is direct. If $\gamma_1 + \gamma_2$ is not essential in $\delta$, then there is a non zero fuzzy submodule $\sigma_2$ of $\delta$ such that $(\gamma_1 \oplus \gamma_2) \cap \sigma_2 = \chi_\delta$. This process also must stop after a finite number of steps. Consequently we have a direct sum of finite number uniform fuzzy submodules which is essential in $\delta$. 

114