CHAPTER II

A POISSON INPUT QUEUE WITH TWO PHASES OF HETEROGENEOUS SERVICE UNDER BERNOULLI VACATION SCHEDULE

2.1 Introduction

Most of the queueing systems encountered in real life situations can be studied with service interruptions in which the servers may well be subject to lengthy and unpredictable breakdowns while serving a customer. For instance, in manufacturing systems the machine may breakdown due to machine or job related problems; in computer systems, the machine may be subject to scheduled backups and unpredictable failures. The queueing models with service interruptions have been extensively discussed by a number of researchers notable among them are Gaver (1962), Avi-Itzhak and Naor (1963), Thirurengadan (1963) and Mitrany and Avi-Itzhak (1968) for some fundamental works. While Sengupta (1990), Ibe and Trivedi (1990), Li et al. [1997(a)], Tang (1997), Takin and Sengupta (1998), Madan (2003), Li and Lin (2006), Fieems et al. (2008), Krishnamoorthy et al. (2009), among others have studied some queueing systems with interruptions wherein one of the underlying assumptions is that, the service channel undergoes repair instantaneously, whenever it fails.

The classical vacation scheme with Bernoulli service discipline was originated and developed significantly by Keilson and Servi (1986) and co-workers. Kella (1990)

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suggested a generalized Bernoulli scheme according to which a single server goes on $k$ consecutive vacations with probability $p$, if the queue is empty upon his return. Recently, considerable efforts have been made to study $M/G/1$ type queueing system with two phases of service under Bernoulli vacation schedule under different vacation policies by Choudhury and Madan [2004, 2005], Choudhury and Paul (2006), Choudhury et al. (2007). The motivations for these types of models come from some computer networks and telecommunication systems, where messages are processed in two stages by a single server. As modern telecommunication system become more complicated and processing power of microprocessors becomes more expensive, the advantages of more sophisticated scheduling becomes more apparent. The need for scheduling the allocation of resource among two or more heterogeneous type of tasks arises from many applications. In most of the previous studies, it is assumed that the server is available in the service station on a permanent basis and service station never fails. However, these assumptions are practically unrealistic. In practice we often meet the case where service stations may fail and can be repaired. Hence Li et al. [1997(a)] considered reliability analysis of such a model under Bernoulli vacation schedule with the assumption that the server is subject to breakdowns and repairs. However, no work has been done in the queueing model taking into account two phases of service, Bernoulli vacation and server breakdown during the service together. Thus in this study, we proposed to investigate such a type of $M/G/1$ unreliable server queue with two phases of service and Bernoulli vacation schedule, where concepts of repair times for both the phases of service are also introduced. The supplementary variable technique has been applied to obtain the probability generating functions of the queue size distributions at different stages of service as well as reliability function of the
model under study. This may lead us to remarkable simplification while solving other similar types of queueing problems.

The following results have been obtained under the present study of this chapter

(i) The stationary queue size distribution.
(ii) The queue size distribution at departure epoch
(iii) Particular cases
(iv) The busy period distribution
(v) The waiting time distribution
(vi) The Reliability analysis
(vii) The numerical illustration

2.2 The model description

We consider an \( M/G/1 \) queueing system, where arrivals occur according to a Poisson process with arrival rate \( \lambda \) and the server provides two phases of heterogeneous service in succession: first phase of service (FPS) denoted by \( B_1 \) followed by a second phase of service (SPS) denoted by \( B_2 \). The service discipline is assumed to be first come, first served (FCFS). The service times for \( i^{th} \) phases of service are independent random variables follow general law of distribution with probability distribution function (d.f) \( B_i(x) \), \( i=1,2 \), Laplace-Stieltjes transform (LST) \( \beta_i^r(\theta) = E[e^{-\theta B_i}] \) and finite moments \( \beta_i^{(k)} \), \( k \geq 1 \) for \( i = 1,2 \). As soon as the SPS of a unit is completed, the server may go for a vacation of random length \( V \) with probability \( p(0 \leq p \leq 1) \) or it may continue to serve the next unit, if any, with probability
\[ q(=1 - p), \text{ otherwise, it remains in the system and waits for a new arrival i.e. the server takes a Bernoulli vacation. The vacation time random variable of the server follows a general law of distribution with} \ d.f. \ \mathcal{V}(\gamma), \ LST \ \mathcal{G}^* (\theta) = E[e^{-\theta \mathcal{V}}] \text{ and finite moments } \mathcal{G}^{(k)}, \ k \geq 1 \text{ independent of the service time random variables. While the server is working with any phase of service, it may breakdown at any time and the service channel will fail for a short interval of time. The breakdowns are generated by exogenous Poisson process with rates } \alpha_1 \text{ for } FPS \text{ and } \alpha_2 \text{ for } SPS \text{ respectively. As soon as breakdown occurs it is sent for repair during which the server stops providing service to the arriving customers till service channel is repaired. The customers which were just being served before server breakdown wait for the service to complete its remaining service. The repair time (denoted by } R_1 \text{ for } FPS \text{ and } R_2 \text{ for } SPS \text{) distributions of the server for both the phases of service are assumed to be arbitrarily distributed with } d.f. \ G_1(\gamma) \text{ and } G_2(\gamma), \ LSTs \ G_1^* (\theta) = E[e^{-\theta R_1}] \text{ and } G_2^* (\theta) = E[e^{-\theta R_2}] \text{ and finite } k^{th} \text{ moments } g_1^{(k)} \text{ and } g_2^{(k)} \text{ respectively. Immediately after the server is fixed i.e. , repaired, the server is ready to start its remaining service to customers in both phases of service and in this case the service times are cumulative, which may be referred to as generalized service times. Further we assume that input process, server’s life time, server’s repair time, service time and vacation time random variables are mutually independent of each other.}

Thus the time required by a unit to complete the service cycle, which may be called as modified service time is given by,

\[ B = \begin{cases} B_1 + B_2 + V, & \text{with probability } p \\ B_1 + B_2, & \text{with probability } q(=1 - p) \end{cases} \]
By $H_i$, we denote the generalized service time for $i^{th}$ phase of service and

$H_i^*(\theta) = E[e^{-\theta x}]$ as its LST then

$$H_i^*(\theta) = \sum_{n=0}^{\infty} e^{-\alpha x} \left[ \frac{(\alpha x)^n}{n!} \right] G_i^*(\theta) \ dx = B_i^*(\theta + \alpha_i (1 - G_i^*(\theta))) \text{ for } i = 1, 2$$

Hence the first two moments are found to be

$$h_i^{(1)} = \frac{dH_i^*(\theta)}{d\theta} \bigg|_{\theta=0} = \beta_i^{(0)} \left( 1 + \alpha_i g_i^{(0)} \right)$$

$$h_i^{(2)} = (-1)^2 \frac{d^2 H_i^*(\theta)}{d\theta^2} \bigg|_{\theta=0} = \beta_i^{(2)} \left( 1 + \alpha_i g_i^{(0)} \right)^2 + \alpha_i \beta_i^{(1)} g_i^{(1)}$$

Where $h_i^{(k)}$ is the $k^{th}$ moment of the $i^{th}$ phase of generalized service time distribution.

2.3 Stationary queue size distribution:

In this section, we first set up the system state equations for its stationary queue size distribution by treating the elapsed FPS time, the elapsed SPS time, the elapsed vacation time, the elapsed repair time of the server for both the phases of service as supplementary variables. Then we solve the equations and derive the probability generating function (PGF) of it.

We now define:

$N_q(t)$ – the queue size (including the one being served, if any) at time $t$

$B_0^\circ(t)$ – the elapsed FPS time at time $t$.

$B_1^\circ(t)$ – the elapsed SPS time at time $t$.

$V^\circ(t)$ – the elapsed vacation time at time $t$. 
Further, let us introduce the following random variable:

$$Y(t) = \begin{cases} 0, & \text{if the system is idle at time } t \\ 1, & \text{if the server is busy with FPS at time } t \\ 2, & \text{if the server is busy with SPS at time } t \\ 3, & \text{if the server is on vacation at time } t \\ 4, & \text{if the server is under repair during FPS at time } t \\ 5, & \text{if the server is under repair during SPS at time } t \end{cases}$$

Thus the supplementary variables $V^0(t)$, $B^0(t)$, and $R^0(t)$ for $i = 1, 2$ are introduced in order to obtain a bivariate Markov process $\{N_Q(t), X(t)\}$, where $X(t) = 0$

if $Y(t) = 0$, $X(t) = B^0_1(t)$ if $Y(t) = 1$, $X(t) = B^0_2(t)$ if $Y(t) = 2$, $X(t) = V^0(t)$ if $Y(t) = 3$, $X(t) = R^0_i(t)$ if $Y(t) = 4$ and $X(t) = R^0_2(t)$ if $Y(t) = 5$

Next we define the following probabilities:

$$U_0(t) = P_{\{N_Q(t) = 0, X(t) = 0\}}$$

$$Q_n(y, t) = P_{\{N_Q(t) = n, X(t) = V^0(t); y < V^0(t) \leq y + dy\}; y > 0, n \geq 0}$$

and for $i = 1, 2$ and $n \geq 0$

$$P_{i,n}(x; t) = P_{\{N_Q(t) = n, X(t) = B^0_i(t); x < B^0_i(t) \leq x + dx\}; x > 0}$$

$$R_{i,n}(x, y; t) = P_{\{N_Q(t) = n, X(t) = R^0_i(t); y < R^0_i(t) \leq y + dy\}; B^0_i(t) = x}; (x, y) > 0$$

So that analysis of the limiting behaviour of this queueing process a stationary point of time can be performed with the help of Kolmogorov forward equations provided limiting probabilities

$$U_0 = \lim_{t \to \infty} U_0(t), \quad Q_n(y)dy = \lim_{t \to \infty} Q_n(y; t)dy,$$
for $i = 1,2$ and $n \geq 0$

$$P_{i,n}(x)dx = \lim_{t \to \infty} P_{i}(x;t)dx, \text{ and } R_{i,n}(x,y)dy = \lim_{t \to \infty} R_{i}(x,y;t)dy$$

exist and positive under the condition that they independent of the initial state.

Further, it is assumed that $V(0) = 0, V(\infty) = 1, B_i(0) = 0, B_i(\infty) = 1,$ $G_i(0) = 0, G_i(\infty) = 1$ for $i = 1,2$ and that $V(y)$ is continuous at $y = 0$ for $i = 1,2$; $B_i(x)$ is continuous at $x = 0$ and $G_i(y)$ are continuous at $y = 0$ for $i = 1,2$ respectively, so that

$$\gamma(y)dy = \frac{dV(y)}{1-V(y)} \quad \mu_i(x)dx = \frac{dB_i(x)}{1-B_i(x)} \quad \text{and} \quad \xi_i(y)dy = \frac{dG_i(y)}{1-G_i(y)}$$

are the first order differential (hazard rate) functions of $V, B_i$ and $G_i$ respectively for $i = 1,2$.

2.3.1 The steady state equations

The Kolmogorov forward equations to govern the system under steady state conditions [e.g. See Cox [1955(a)] or section 1.5 of Chapter I] for $i = 1,2$; where sub index $i = 1$ (respectively $i = 2$) denotes the FPS (respectively SPS) can be written as follows:

$$\frac{d}{dx} P_{i,n}(x) + \left[ \lambda + \alpha_i + \mu_i(x) \right] P_{i,n}(x) = \lambda \left( 1 - \delta_{n,0} \right) P_{i,n-1}(x) + \int_0^\infty \xi_i(y) R_{i,n}(x,y)dy; n \geq 0$$

(2.3.1.1)

$$\frac{d}{dy} Q_{n}(y) + \left[ \lambda + \gamma(y) \right] Q_{n}(y) = \lambda \left( 1 - \delta_{n,0} \right) Q_{n-1}(y); n \geq 0$$

(2.3.1.2)

$$\frac{d}{dy} R_{i,n}(x,y) + \left[ \lambda + \xi_i(y) \right] R_{i,n}(x,y) = \lambda \left( 1 - \delta_{n,0} \right) R_{i,n-1}(x,y); n \geq 0$$

(2.3.1.3)
\[ \lambda U_0 = \int_0^\infty \gamma(y)Q_0(y)dy + q \int_0^\infty \mu_2(x)P_{2,0}(x)dx \quad ; \quad (2.3.1.4) \]

where \( \delta_{n,m} \) denotes Kronecker’s delta function.

These set of equations are to be solved under the following boundary condition at \( x = 0 \):

\[ P_{1,n}(0) = \lambda \delta_{n,0}U_0 + \int_0^\infty \mu_2(x)P_{2,n}(x)dx + q \int_0^\infty \gamma(y)Q_n(y)dy \quad ; \quad n \geq 0 \quad (2.3.1.5) \]

\[ P_{2,n}(0) = \int_0^\infty \mu_1(x)P_{1,n}(x)dx; \quad n \geq 0 \quad (2.3.1.6) \]

\[ Q_{n}(0) = p \int_0^\infty \mu_2(x)P_{2,n}(x)dx; \quad n \geq 0 \quad , \quad at \, \, y = 0 \quad (2.3.1.7) \]

and at \( y = 0 \) for \( i = 1,2 \) and fixed values of \( x \):

\[ R_{i,n}(x;0) = \alpha_i P_{i,n}(x); n \geq 0 \quad (2.3.1.8) \]

With normalizing condition

\[ U_0 + \sum_{n=0}^\infty \left[ \int_0^\infty Q_n(y)dy + \sum_{i=1}^2 \int_0^\infty P_{i,n}(x)dx + \int_0^\infty R_{i,n}(x,y)dy \right] = 1 \quad (2.3.1.9) \]

2.3.2 The model solution

To solve the system of equations (2.3.1.1) - (2.3.1.8), let us introduce the following PGFs for \( i = 1,2 \) and \(|z| < 1\):

\[ R_i(x,y;z) = \sum_{n=0}^\infty z^n R_{i,n}(x;y) \quad ; \quad R_i(x,0;z) = \sum_{n=0}^\infty z^n R_{i,n}(x;0) \]

\[ Q(y;z) = \sum_{n=0}^\infty z^n Q_n(y) \quad ; \quad Q(0;z) = \sum_{n=0}^\infty z^n Q_n(0) \]
Let $b(z) = \lambda (1 - z)$, and then proceeding in usual manner with equation (2.3.1.2) and (2.3.1.3), we get a set of differential equations of Lagrangian type whose solutions are given by:

$$Q(y; z) = Q(0; z)[1 - V(y)] \exp\{-b(z)y\}; y > 0$$  \hspace{1cm} (2.3.2.1)

$$R_i(x, y; z) = R_i(x, 0; z)[1 - G_i(y)] \exp\{-b(z)y\}; y > 0 \quad \text{for } i = 1, 2$$  \hspace{1cm} (2.3.2.2)

where $R_i(x, 0; z)$ can be obtained from equations (2.3.1.8), which after simplification yields

$$R_i(x, 0; z) = \alpha_i P_i(x; z) \quad \text{for } i = 1, 2$$  \hspace{1cm} (2.3.2.3)

Now solving the differential equations (2.3.1.1), we get

$$P_i(x; z) = P_i(0; z)[1 - B_i(x)] \exp\{-A_i(x)\}; x > 0 \quad \text{for } i = 1, 2;$$  \hspace{1cm} (2.3.2.4)

where $A_i(x) = b(z) + \alpha_i(1 - G_i^*(b(z)))$ for $i = 1, 2$.

Utilizing (2.3.2.4) and (2.3.2.3) in (2.3.2.2), we get for $i = 1, 2$

$$R_i(x, y; z) = \alpha_i P_i(x; z) \beta_i^*(A_2(z)) \exp\{-b(z)y\}$$  \hspace{1cm} (2.3.2.5)

Multiplying equation (2.3.1.5) by $z^n$ and then taking summation over all possible values of $n \geq 0$, we get on simplification

$$zP_1(0; z) = qP_2(0; z)\beta_2^*(A_2(z)) + Q(0; z)\theta^*(\lambda(z)) - \lambda(z)U_0$$  \hspace{1cm} (2.3.2.6)

Similarly from equations (2.3.1.6) and (2.3.1.7), we have

$$P_2(0; z) = pP_1(0; z)\beta_1^*(A_1(z))$$  \hspace{1cm} (2.3.2.7)

and $Q(0; z) = pP_2(0; z)\beta_2^*(A_2(z))$, respectively.  \hspace{1cm} (2.3.2.8)

Now utilizing (2.3.2.7) and (2.3.2.8) in (2.3.1.6) and then simplifying, we get
\[ P_1(0;z) = \frac{b(z)U_0}{[q + p g^*(b(z))]\beta_1^*(A_1(z))\beta_2^*(A_2(z)) - z} \]  
(2.3.2.9)

Let \( z \to 1 \) in (2.3.2.9), we obtain by the L'Hospital's rule

\[ P_1(0;1) = \frac{\lambda U_0}{(1 - \rho_H)}; \]

where \( \rho_H = \rho_i (1 + \alpha_1 g_1^{(i)}) + \rho_2 (1 + \alpha_2 g_2^{(i)}) + p \rho_v \) is the utilizing factor of the system,

\( \rho_i = \lambda \beta_i^{(i)} \) for \( i = 1,2 \) and \( \rho_v = \lambda g^{(i)} \).

This gives for \( i = 1,2 \).

\[ P_i(x;1) = \frac{\lambda U_0 [1 - B_i(x)]}{(1 - \rho_H)} \]  
(2.3.2.10)

\[ R_i(x,y;1) = \frac{\alpha_i \lambda U_0 [1 - B_i(x)][1 - G_i(y)]}{(1 - \rho_H)} \]

and

\[ Q(y;1) = \frac{p \lambda U_0 [1 - V(y)]}{(1 - \rho_H)}. \]

Now utilizing the normalizing condition (2.3.1.9), we get

\[ U_0 = (1 - \rho_H) \]  
(2.3.2.11)

Note that equation (2.3.2.10) represents steady-state probability that the server is idle but available in the system. Also, from equation (2.3.2.10), we have \( \rho_H < 1 \), which is the necessary and sufficient condition under which steady-state solution exists.

Thus we summarize our results in the following Theorem 2.3.1.

\textbf{Theorem 2.3.1}

Under the stability condition \( \rho_H < 1 \), the joint distribution of the state of the server and the queue size has the following partial \( PGF \)s
\[ P_1(x, z) = \frac{(1 - \rho_H)b(z)[1 - B_1(x)]\exp\{-A_1(z)x\}}{[q + p g^*(b(z))]B_1^*(A_1(z))B_2^*(A_2(z)) - z} \quad (2.3.2.12) \]

\[ P_2(x, z) = \frac{(1 - \rho_H)b(z)\beta_1^*(A_1(z))[1 - B_2(x)]\exp\{-A_2(z)x\}}{[q + p g^*(b(z))]B_1^*(A_1(z))B_2^*(A_2(z)) - z} \quad (2.3.2.13) \]

\[ Q(y, z) = \frac{p(1 - \rho_H)\lambda(z)\beta_1^*(A_1(z))\beta_2^*(A_2(z))[1 - V(y)]\exp\{-b(z)y\}}{[q + p g^*(b(z))]B_1^*(A_1(z))B_2^*(A_2(z)) - z} \quad (2.3.2.14) \]

\[ R_1(x, y; z) = \frac{\alpha_1(1 - \rho_H)b(z)[1 - B_1(x)]\exp\{-A_1(z)x\}\times [1 - G_1(y)]\exp\{-b(z)y\}}{[q + p g^*(b(z))]B_1^*(A_1(z))B_2^*(A_2(z)) - z} \quad (2.3.2.15) \]

and

\[ R_2(x, y; z) = \frac{\alpha_2(1 - \rho_H)b(z)\beta_1^*(A_1(z))[1 - B_2(x)]\exp\{-A_2(z)x\}\times [1 - G_2(y)]\exp\{-b(z)y\}}{[q + p g^*(b(z))]B_1^*(A_1(z))B_2^*(A_2(z)) - z} \quad (2.3.2.16) \]

where \( b(z) = \lambda(1 - z) \) and \( A_i(z) = b(z) + \alpha_i(1 - G_i^*(b(z))) \); respectively for \( i = 1, 2 \).

**Remark 2.3.1**

It is important to note here that such types of joint distributions are important to obtain the distribution of each state of the server in more comprehensive manner, which helps us to obtain marginal distributions of the server’s states as well as stationary queue size distribution at a departure epoch.

**Theorem 2.3.2**

Under the stability condition \( \rho_H < 1 \), the marginal PGFs of the server’s state queue size distributions are given by

\[ P_1(z) = \frac{(1 - \rho_H)\lambda(z)[1 - \beta_1^*(A_1(z))]}{A_1(z)[q + p g^*(\lambda(z))]B_1^*(A_1(z))B_2^*(A_2(z)) - z} \quad (2.3.2.17) \]

\[ P_2(z) = \frac{(1 - \rho_H)\lambda(z)\beta_1^*(A_1(z))[1 - \beta_2^*(A_2(z))]}{A_2(z)[q + p g^*(\lambda(z))]B_1^*(A_1(z))B_2^*(A_2(z)) - z} \quad (2.3.2.18) \]
\[ Q(z) = \frac{p(1 - \rho_H)\beta_1^*(A_1(z))\beta_2^*(A_2(z))[1 - g^*(\lambda(z))]}{[q + p g^*(\lambda(z))]B_1^*(A_1(z))B_2^*(A_2(z)) - z} \]  
\hspace{1cm} (2.3.2.19)

\[ R_1(z) = \frac{\alpha_1(1 - \rho_H)[1 - G_1(\lambda(z))][1 - \beta_1^*(A_1(z))]}{A_1(z)[q + p g^*(\lambda(z))]B_1^*(A_1(z))B_2^*(A_2(z)) - z} \]  
\hspace{1cm} (2.3.2.20)

and \[ R_2(z) = \frac{\alpha_2(1 - \rho_H)[1 - G_2(\lambda(z))][1 - \beta_2^*(A_2(z))]}{A_2(z)[q + p g^*(\lambda(z))]B_1^*(A_1(z))B_2^*(A_2(z)) - z} \]  
\hspace{1cm} (2.3.2.21)

**Proof:** Integrating (2.3.2.12), (2.3.2.13) and (2.3.2.14) with respect to \( x \) and \( y \) respectively and then using the well known result of renewal theory

\[
\int_0^\infty e^{-\alpha}(1 - B_i(x))dx = \frac{[1 - \beta_i^*(\theta)]}{\theta} \text{ for } i = 1,2
\]

and

\[
\int_0^\infty e^{-\alpha}(1 - V(y))dy = \frac{[1 - g^*(\theta)]}{\theta},
\]

we get formulae (2.3.2.17), (2.3.2.18) and (2.3.2.19).

Similarly, integrating equation (2.3.2.5) with respect to \( y \), we get for \( i = 1,2 \)

\[
R_i(x,z) = \int_0^\infty R_i(x,y;z)dy = \alpha_i [p(z)]^{-1} [1 - G_i(b(z))]P_i(0;z)[1 - B_i(x)] \exp(-A_i(z)x)
\]  
\hspace{1cm} (2.3.2.22)

Further integrating (2.3.2.22) with respect to \( x \) and utilizing (2.3.2.9) and (2.3.2.11), we claimed in formulae (2.3.2.20) and (2.3.2.21).

Next the system state probabilities are given in corollary 2.3.1

**Corollary 2.3.1**

If the system is in steady-state conditions, then
(i) the probability that the system is idle is
\[ P_i = 1 - \rho_1 \left( 1 + \alpha_1 g_1^{(i)} \right) - \rho_2 \left( 1 + \alpha_2 g_2^{(i)} \right) + \lambda p g^{(i)} \]

(ii) the probability that the server is busy with FPS is \( P_{B_i} = \rho_1 \)

(iii) the probability that the server is busy with SPS is \( P_{B_2} = \rho_2 \)

(iv) the probability that the server is on vacation, \( P_v = p \rho_v \)

(v) the probability that the server is under repair during FPS is,
\[ P_{R_1} = \alpha_1 \rho_1 g_1^{(i)} \]

(vi) the probability that the server is under repair during SPS is,
\[ P_{R_2} = \alpha_2 \rho_2 g_2^{(i)} \]

Proof: - Note that
\[ P_v = \lim_{z \to 1} Q(z), \quad P_{B_i} = \lim_{z \to 1} P_i(z), \quad P_{R_i} = \lim_{z \to 1} R_i(z) \quad \text{for } i = 1, 2 \]

and \( P_i = 1 - \sum_{n=1}^2 \left( P_{B_n} + P_{R_n} \right) - P_v \)

The stated formulae follow by direct calculation.

2.4 Queue size distribution at Departure epoch

To obtain the PGF of the queue size distribution at departure epoch, we follow the argument of PASTA [see Wolff (1982) or section 1.5 of Chapter 1], we state that a departing customer will see \( j \) customer in the queue just after a departure if and only if there were \( j \) customer in the queue SPS or a vacation just before the departure. Now denoting \( \{ \pi_j : j \geq 0 \} \) as the probability that there are \( j \) units in the queue at a departure epoch, then for \( j \geq 0 \) we may write
\[
\pi_j = K_0q_0 \int_0^\infty \mu_2(x)P_{2,j}(x)dx + K_0 \int_0^\infty \gamma(y)Q_j(y)dy; \tag{2.4.1}
\]

where \(K_0\) is the normalizing constant.

Now multiplying both sides of Eq. (2.4.1) by \(z^j\) and then taking summation over \(j \geq 0\) and utilizing equations (2.3.2.1) and (2.3.2.4), we get on simplification

\[
\pi(z) = \frac{K_0U_0 \lambda(z) [q + p g^*(\lambda(z))] \beta_1^*(A_1(z)) \beta_2^*(A_2(z))}{[q + p g^*(\lambda(z))] \beta_1^*(A_1(z)) \beta_2^*(A_2(z)) - z} \tag{2.4.2}
\]

Utilizing normalizing condition \(\pi(1) = 1\), we get

\[
K_0 = \frac{(1 - \rho_H)}{\lambda U_0} \tag{2.4.3}
\]

Hence from equations (2.4.2) and (2.4.3) we have,

\[
\pi(z) = \frac{(1 - \rho_H)(1 - z) [q + p g^*(\lambda(z))] \beta_1^*(A_1(z)) \beta_2^*(A_2(z))}{[q + p g^*(\lambda(z))] \beta_1^*(A_1(z)) \beta_2^*(A_2(z)) - z} \tag{2.4.4}
\]

Which is the PGF of the stationary queue size at a departure epoch of this \(M/G/1\) queue with two phases of service and Bernoulli vacation schedule.

Next the mean queue size of this model is given in the corollary 2.4.1.

**Corollary 2.4.1**

Under the stability conditions, the mean number of customers in the system (i.e. mean queue length) \(E[N_Q(t)]\) is given by

\[
E[N_Q(t)] = \rho_H + \rho_1 \rho_2 \frac{1 + \alpha_1 g_1(0)}{1 - \rho_H} \frac{1 + \alpha_2 g_2(0)}{}
\]

+ \frac{\lambda \left[ \rho_1 \alpha_1 g_1^{(2)} + 2(1 + \alpha_1 g_1^{(0)})^2 \beta_R^{(1)} \right] + \rho_2 \left[ \alpha_2 g_2^{(2)} + 2(1 + \alpha_2 g_2^{(0)})^2 \beta_R^{(2)} \right]}{2(1 - \rho_H)}
\[ \frac{\lambda P(2g_1^{(1)}P_1(1 + \alpha_1 g_1^{(1)}) + 2g_2^{(1)}P_2(1 + \alpha_2 g_2^{(1)}) + \lambda g^{(2)}}{2(1 - \rho_H)} \]

where \( \beta_{R}^{(i)} = \frac{\beta_{(i)}^{(2)}}{2\beta_{(i)}^{(i)}} \) is the residual service time of \( i \)-th phase of service for \( i = 1, 2 \).

**Proof:** The result follows directly by differentiating (2.4.4) with respect to \( z \) and then taking limit \( z \to 1 \) by using the L'Hospital's rule.

### 2.5 Particular cases

The results obtained in this section are consistent with the existing literature. For example, if we suppose \( \alpha_1 = \alpha_2 = 0 \) (i.e. there is no breakdown in the system) in the above expressions (2.4.4) and (2.4.5), then we have

\[
\pi(z) = \frac{(1 - \rho)(1 - z)}{[q + p g^*(\lambda - \lambda z)]\beta_1^*(\lambda - \lambda z)\beta_2^*(\lambda - \lambda z) - z}
\]

and

\[
E[N_0(t)] = \rho + \frac{\lambda^2 [\beta_2^{(2)} + \beta_2^{(3)} + 2\beta_1^{(1)} \beta_2^{(2)}]}{2(1 - \rho)} + \frac{p [g^{(3)} + 2g^{(4)}(\beta_1^{(1)} + \beta_2^{(1)})]}{2(1 - \rho)};
\]

where \( \rho = \lambda [\beta_1^{(1)} + \beta_2^{(2)} + p g^{(1)}] \) is the utilization factor of such system.

Note that the expression (2.5.1) is the well known Pollaczek-Khinchin formula [e.g. see section 1.5 of Chapter I] for an \( M/G/1 \) queue with two phases of service under Bernoulli vacation schedule and these two results are consistent with the results obtained by Choudhury and Madan (2004) for single unit arrival case.

Similarly, if we take \( z = g^*(\lambda(z)) \) in expression (2.4.4), then we get on simplification
\[ \pi(z) = \frac{(q - \rho)(1 - z)[q + pz]\beta_1'(A_1(z))\beta_2'(A_2(z))}{[q + pz]\beta_1'(A_1(z))\beta_2'(A_2(z)) - z} \] (2.5.2)

which is the expression for the PGF of queue size distribution at the service completion epoch of an $M/G/l$ unreliable server queue with two phases of service under Bernoulli feedback mechanism. Further, we observe that for $\alpha_1 = \alpha_2 = 0$ the above formula (2.5.2) reduces to Pollaczek-Khinchin formula of the classical $M/G/l$ queue with two phases of service under Bernoulli feedback mechanism and this verifies the results of Takagi (1991) (see Eq.(4.29), Page.- 51) for $\beta_2'(A_2(z)) = 1$ in our expression (2.5.2). From above discussion, we may conclude that the above formula (2.4.4) represents the classical generalization of the Pollaczek-Khinchin formula for $M/G/l$ queue with two phases of service under Bernoulli vacation schedule for unreliable server.

### 2.6 Busy period distribution

We define busy period as a length of time interval that makes the server busy and it continues to the instant when the system becomes empty and denote

\[ T_b = \text{length of the busy period.} \]
\[ T_c = \text{length of the busy cycle.} \]
\[ T_0 = \text{length of the idle period.} \]

Let $T_b^*(\theta) = E[e^{-\theta T_b}]$ be the LST of $T_b$, and then Taka'č's functional equation under the steady state condition is given by

\[ T_b^*(\theta) = H^*(\theta + \lambda - \lambda T_b^*(\theta)) ; \]
where \( H^*(\theta) = \frac{q + p g^*(\theta) + \beta_1^*(\theta + \alpha_1(1-G_1^*(\theta))) + \beta_2^*(\theta + \alpha_2(1-G_2^*(\theta)))}{\lambda} \) is the LST of \( \mathcal{B} \)
i.e., our modified service time distribution.

The mean busy period is found to be

\[
E(T_b) = \frac{\frac{d}{d\theta} T_b^*(\theta)}{d\theta} \bigg|_{\theta=0}
\]

\[
= \frac{\beta_1^{(0)}(1 + \alpha_1 g_1^{(0)}(\lambda)) + \beta_2^{(0)}(1 + \alpha_2 g_2^{(0)}(\lambda)) + \rho g^{(0)}}{(1 - \rho_H)}
\]

\[
= \frac{\rho_H}{\lambda(1 - \rho_H)}
\]

Now, since \( E(T_0) = \frac{1}{\lambda} \), therefore utilizing the relationship \( E(T_c) = E(T_b) + E(T_0) \), we get

\[
E(T_c) = \frac{1}{\lambda(1 - \rho_H)}
\]

Similarly, the waiting time distribution of a test customer for our model has the following LST.

2.7 Waiting time distribution

Let \( W_Q^*(\theta) \) be the LST of the waiting time distribution of a test customer for this model under steady state condition, then

\[
W_Q^*(\theta) = \frac{\theta(1 - \rho_H)(q + p g^*(\theta)) \beta_1^*(\theta + \alpha_1(1-G_1^*(\theta))) \beta_2^*(\theta + \alpha_2(1-G_2^*(\theta)))}{\theta - \frac{\lambda}{1 - \{q + p g^*(\theta)\beta_1^*(\theta + \alpha_1(1-G_1^*(\theta))) \beta_2^*(\theta + \alpha_2(1-G_2^*(\theta)))\}}
\]

(2.7.1)

and \( E[W_Q] = \frac{E[N_Q(t)]}{\lambda} \)

(2.7.2)
Proof: - The results follow directly from formula (2.4.4) by utilizing distributional form of Little’s Law [e.g. see Keilson and Servi (1990) or section 1.5 of Chapter I]:

\[ W^*_\varphi(\lambda - \lambda z) = \pi(z) \]  

(2.7.3)

Now setting \( \lambda - \lambda z = \theta \) in equation (2.7.3) and utilizing (2.4.4), we get (2.7.1). Similarly formula (2.7.2) follows directly by routine differentiation in (2.7.1) with respect to \( \theta \) and then taking limit \( \theta \to 0 \) by using the L’Hospital’s rule.

2.8 Reliability Analysis

Our final goal is to derive some reliability indices of this model. First of all we will discuss two reliability indices of the system viz. - the system availability and failure frequency under the steady state conditions. Suppose that the system is initially empty. Let \( A_v(t) \) be the point wise availability of the server at time \( t \) that is, the probability that the server is either serving a customer or the server is available if the server is free and up during an idle period, such that the steady state availability of the server will be \( A_v = \lim_{t \to \infty} A_v(t) \)

Theorem 2.8.1

The steady state availability of the server is given by,

\[ A_v = 1 - \rho_1 \alpha_1 \beta_1^{(i)} - \rho_2 \alpha_2 \beta_2^{(i)} - \lambda \theta \beta^{(i)} \]  

(2.8.1)

Proof: - The result follows directly from theorem (2.3.2) by considering the following equation

\[ A = U_0 + \sum_{n=1}^{2} \int_0^{\infty} P_n(x,1)dx = U_0 + \lim_{z \to 1} [P_1(z) + P_2(z)] \]
By using (2.3.2.11), (2.3.2.17) and (2.3.2.18) we get (2.8.1).

**Theorem 2.8.2**

The steady state failure frequency of the server is given by,

\[ M_f = \alpha_1 \rho_1 + \alpha_2 \rho_2 \]  

(2.8.2)

**Proof:** - The result follows directly from equation (2.3.2.10) by utilizing the argument of Li et al. [1997(a)].

\[ M_f = \alpha_1 \int_0^\infty P_1(x,1)dx + \alpha_2 \int_0^\infty P_2(x,1)dx \]

Now since \( \int_0^\infty [1 - B_i(x)]dx = \int_0^\infty xdB_i(x) = \beta_i^{(1)} \) for \( i=1, 2 \); therefore from equation (2.3.2.10) we have (2.8.2).

Next, we derive Laplace transform of reliability function and denote \( \tau \) by the time to the first failure of the server, and then the reliability function of the server is

\[ R(t) = P(\tau > t) \]

**Theorem 2.8.3**

The Laplace transform of \( R(t) \) is given by

\[
R^*(\theta) = U_0^* (\theta) + \frac{[(\theta + \alpha_2)(1 - \beta_2^* (\theta + \alpha_1)) + (\theta + \alpha_1)\beta_2^* (\theta + \alpha_1)(1 - \beta_1^* (\theta + \alpha_2))][1 - \theta U_0^* (\theta)]}{(\theta + \alpha_1)(\theta + \alpha_2)[1 - (q + p \vartheta^* (\theta + \lambda(z)))]\beta_1^* (\theta + \alpha_1 + \lambda(z))\beta_2^* (\theta + \alpha_2 + \lambda(z))}
\]

\[ + \frac{p \beta_1^* (\theta + \alpha_1)\beta_2^* (\theta + \alpha_2)(1 - \vartheta^* (\theta)) [1 - \theta U_0^* (\theta)]}{\theta[1 - (q + p \vartheta^* (\theta + \lambda(z)))]\beta_1^* (\theta + \alpha_1 + \lambda(z))\beta_2^* (\theta + \alpha_2 + \lambda(z))}; \]

(2.8.3)

where \( U_0^* (\theta) = \frac{1}{\lambda + \theta - \lambda \omega_0 (\theta)} \)
and \( \omega_0(\theta) \) is the unique root of the equation

\[
    z = \left\{ q + p \delta^* (\theta + \lambda(z)) \right\} \beta_1^* (\theta + \alpha_1 + \lambda(z)) \beta_2^* (\theta + \alpha_2 + \lambda(z)) / \gamma(0 + \alpha_1 \lambda(z) + A(z)) / \gamma_2(0 + \alpha_2 + \lambda(z))
\]

inside \(|z| = 1, \text{Re}(\theta) > 0\) and \( \lambda(z) = \lambda(1 - z) \).

**Proof:** - In order to find the reliability function of the server, we assume that the failure state of the server be the absorbing states, and then we obtain a new system. In this new system, we use the same notations as in the previous sections, and then we can write the following set of Kolmogorov forward equations:

\[
    \frac{\partial U_i(t)}{\partial t} + \lambda U_i(t) = \int_0^\infty \gamma(y) Q_i(y, t) dy + q \int_0^\infty \mu_2(x) P_{2,0}(x, t) dx \tag{2.8.4}
\]

\[
    \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial y} \right) Q_n(y, t) + [\lambda + \gamma(y)] Q_n(y, t) = \lambda(1 - \delta_{n,0}) Q_{n-1}(y, t); n \geq 0 \tag{2.8.5}
\]

and for \( i = 1,2 \) and \( n \geq 0 \)

\[
    \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) P_{i,n}(x, t) + [\lambda + \alpha_i + \mu_i(x)] P_{i,n}(x, t) = \lambda(1 - \delta_{n,0}) P_{i,n-1}(x, t) \tag{2.8.6}
\]

These equations are to be solved with initial condition \( U_i(0) = 1 \) and subject to the boundary conditions at \( x = 0 \):

\[
    P_{1,n}(0; t) = \lambda \delta_{n,0} U_0(t) + q \int_0^\infty \mu_2(x) P_{2,n}(x, t) dx + \int_0^\infty \gamma(y) Q_n(y, t) dy; n \geq 0 \tag{2.8.7}
\]

\[
    P_{2,n}(0; t) = \int_0^\infty \mu_1(x) P_{1,n}(x, t) dx; n \geq 0 \tag{2.8.8}
\]

and at \( y = 0 \):

\[
    Q_n(0; t) = p \int_0^\infty \mu_2(x) P_{2,n}(x, t) dx; n \geq 0 \tag{2.8.9}
\]
We now introduce the following Laplace transform of generating functions for $|z| < 1$:

$$P^*_i(x, \theta; z) = \sum_{n=0}^{\infty} z^n P_{i,n}^*(x; \theta) \quad P^*_i(0, \theta; z) = \sum_{n=0}^{\infty} z^n P_{i,n}^*(0; \theta) \quad \text{for } i = 1, 2$$

$$Q^*_i(y, \theta; z) = \sum_{n=0}^{\infty} z^n Q_{i,n}^*(y; s) \quad Q^*_i(0, \theta; z) = \sum_{n=0}^{\infty} z^n Q_{i,n}^*(0; \theta) \quad \text{for } i = 1, 2$$

and $U^*_0(\theta) = \int_0^\infty e^{-\theta} dU_0(t)$

Now performing Laplace transform with respect to these equations (2.8.4) and (2.8.5) we get

$$\begin{align*}
(\theta + \lambda)U^*_0(\theta) - 1 &= \int_0^\infty \mathcal{G}(y)Q^*_0(y; \theta)dy + q\int_0^\infty \mu_2(x)P^*_{2,0}(x; \theta)dx \\
(\theta + \lambda + \mathcal{G}(y))Q^*_n(y; \theta) + \frac{\partial}{\partial y}Q^*_n(y, \theta) &= \lambda \left(1 - \delta_{n,0}\right)Q_{n-1}^*(y; \theta) \quad n \geq 0
\end{align*}$$

Similarly from Eqs. (2.8.6) – (2.8.9) we have for $n \geq 0$ and $i = 1, 2$

$$\begin{align*}
\left(\theta + \lambda + \alpha_i + \mu_i(x)\right)P^*_{i,n}(x; \theta) + \frac{\partial}{\partial x}P^*_{i,n}(x; \theta) &= \lambda \left(1 - \delta_{n,0}\right)P_{i,n-1}^*(x; \theta) \\
P^*_{1,n}(0; \theta) &= \delta_{n,0} \lambda U^*_0(\theta) + q\int_0^\infty \mu_2(x)P^*_{2,n+1}(x; \theta)dx + \int_0^\infty \mathcal{G}(y)Q^*_{n+1}(y, \theta)dy \\
P^*_{2,n}(0; \theta) &= \int_0^\infty \mu_1(x)P^*_{1,n}(x; \theta)dx
\end{align*}$$

and $Q^*_n(0; \theta) = p\int_0^\infty \mu_2(x)P^*_{2,n}(x; \theta)dx$

Now multiplying equation (2.8.11) by $z^n$ and then taking summation over all possible values of $n \geq 0$, we get a set of differential equation of Lagrangian type whose solution is given by
\[ Q'(y,e-,z) = Q'(0,0,z) \exp[-(\theta + b(z))y][1-V(y)] \] (2.8.16)

where \( b(z) = \lambda(1-z) \) is as defined in section 2.3.

Again from equation (2.8.15) we have
\[ Q^*(0,0;z) = P_2^*(0,0;z) = \beta_2^*(\theta + \alpha_2 + \lambda(z)) \] (2.8.17)

Now similarly multiplying equation (2.8.12) by \( z^n \) and then taking summation over all possible values of \( n \geq 0 \), we get a set of similar type of differential equation of Lagrangian type whose solution is given by,
\[ P_i^*(x,0;z) = P_i^*(0,0;z) \exp[-\{\theta + \alpha_i + \lambda(z)x\}[1-B_i(x)] \text{ for } i = 1,2 \] (2.8.18)

Again from equations (2.8.13) and (2.8.14) we have
\[ zP_i^*(0,0,z) + (\lambda(z) + \theta U_0^*(\theta)) = 1 + qP_i^*(0,0,z)\beta_2^*(\theta + \alpha_2 + \lambda(z)) + Q^*(0,0,z)\theta^*(\theta + \lambda(z)) \] (2.8.19)

and \( P_2^*(0,0,z) = P_1^*(0,0,z)\beta_1^*(\theta + \alpha_1 + \lambda(z)) \) (2.8.20)

Similarly from equation (2.8.19) after utilizing (2.8.17) and (2.8.20), we have
\[ P_i^*(0,0,z) = \frac{[U_0^*(\theta)(s + \lambda(z)) - 1]}{[q + p \theta^*(\theta + \lambda(z))]\beta_1^*(\theta + \alpha_1 + \lambda(z))\beta_2^*(\theta + \alpha_2 + \lambda(z)) - z} \] (2.8.21)

Further from equation (2.8.18) for \( i = 1 \) we can obtain
\[ P_1^*(\theta,z) = \int_0^\infty P_1^*(x,s;z)dx \]

\[ = \frac{[1 - \beta_1^*(\theta + \alpha_1 + \lambda(z))][U_0^*(\theta)(s + \lambda(z)) - 1]}{(\theta + \alpha_1 + \lambda(z))\{[q + p \theta^*(\lambda(z))]\beta_1^*(\theta + \alpha_1 + \lambda(z))\beta_2^*(\theta + \alpha_2 + \lambda(z)) - z\} \] (2.8.22)

Similarly from equation (2.8.18) for \( i = 2 \) we have
\[ P_2^*(\theta, z) = \frac{\beta_1^*(\theta + \alpha_1 + \lambda(z))(1 - \beta_2^*(\theta + \alpha_2 + \lambda(z))) U_0^*(\theta)(\theta + \lambda(z)) - 1}{(\theta + \alpha_2 + \lambda(z))(q + p \theta^*(\theta + \lambda(z)))\beta_1^*(\theta + \alpha_1 + \lambda(z))\beta_2^*(\theta + \alpha_2 + \lambda(z)) - z} \]  

(2.8.23)

Finally from equation (2.8.16)

\[ Q^*(\theta; z) = \frac{p \beta_1^*(\theta + \alpha_1 + \lambda(z)) \beta_2^*(\theta + \alpha_2 + \lambda(z))(1 - \theta^*(\theta + \lambda(z))) U_0^*(\theta)(\theta + \lambda(z)) - 1}{(\theta + \lambda(z))(q + p \theta^*(\theta + \lambda(z)))\beta_1^*(\theta + \alpha_1 + \lambda(z))\beta_2^*(\theta + \alpha_2 + \lambda(z)) - z} \]  

(2.8.24)

Now consider the coefficient

\[ f(z) = \left\{ q + p \theta^*(\theta + \lambda(z))\right\} \left(\beta_1^*(\theta + \alpha_1 + \lambda(z))\beta_2^*(\theta + \alpha_2 + \lambda(z)) - z \right\} \]

from which it can be shown that the function \( f(z) \) is convex. Hence by Rouche’s theorem \( f(z) \) has only exactly one root \( \omega_0(\theta) \) inside the unit circle \( |z| = 1 \) for Re(\( z \))

Therefore we have

\[ U_0^*(\theta) = \frac{1}{\theta + \lambda - \lambda \omega_0(\theta)} \]

where \( \omega_0(\theta) \) is the unique root of the equation

\[ z = \left\{ q + p \theta^*(\theta + \lambda(z))\right\} \left(\beta_1^*(\theta + \alpha_1 + \chi(z))\beta_2^*(\theta + \alpha_2 + \chi(z)) \right\} \]

Hence from equations (2.8.22), (2.8.23) and (2.8.24) we have

\[ R^*(\theta) = U_0^*(\theta) + P_1^*(\theta) + P_2^*(\theta) + Q^*(\theta) \]

\[ = U_0^*(\theta) + P_1^*(\theta, 1) + P_2^*(\theta, 1) + Q^*(\theta, 1) \]  

(2.8.25)

where
\[ P_1^*(\theta,1) = \frac{[1 - \beta_1^*(\theta + \alpha_1)][\theta U_0^*(\theta) - 1]}{[\theta + \alpha_1][(q + p)\vartheta^*(\theta)]\beta_1^*(\theta + \alpha_1)\beta_2^*(\theta + \alpha_2) - 1]} \]

\[ P_2^*(\theta,1) = \frac{\beta_1^*(\theta + \alpha_1)[1 - \beta_2^*(\theta + \alpha_2)][\theta U_0^*(\theta) - 1]}{[\theta + \alpha_2][(q + p)\vartheta^*(\theta)]\beta_1^*(\theta + \alpha_1)\beta_2^*(\theta + \alpha_2) - 1]} \]

\[ Q^*(\theta,1) = \frac{\beta_1^*(\theta + \alpha_1)\beta_2^*(\theta + \alpha_2)[1 - \vartheta^*(\theta)][\theta U_0^*(\theta) - 1]}{\theta[(q + p)\vartheta^*(\theta)]\beta_1^*(\theta + \alpha_1)\beta_2^*(\theta + \alpha_2) - 1]} \]

By substitution, we obtain formula (2.8.3).

Finally, we discuss mean time to the first failure in corollary 2.8.1.

**Corollary 2.8.1.**

The mean time to the first failure (MTFF) of the server is given by

\[ MTFF = U_0^*(0) + \frac{(\rho_1 + \rho_2 + p\rho_\lambda)[\alpha_1(1 - \beta_1^*(\alpha_1)) + \alpha_2\beta_1^*(\alpha_1)(1 - \beta_2^*(\alpha_2))]}{\alpha_1\alpha_2[1 - \beta_1^*(\alpha_1)\beta_2^*(\alpha_2)]} \] \hspace{1cm} (2.8.26)

**Proof:** From (2.8.3) and the following equations:

\[ MTFF = \int_0^\infty R(\theta)dt = R^*(\theta) \] \hspace{1cm} (2.8.27)

Since by Tauberian theorem of L.T. we have \( \lim_{\theta \to 0} \theta U_0^*(\theta) = \lim_{t \to \infty} U_0(t) = U_0 \) and therefore we get \( U_0 = 1 - (\rho_1 + \rho_2 + p\rho_\lambda) \). Hence by substituting \( U_0 \) in (2.8.27) we obtain (2.8.26).

### 2.9 Numerical Illustration

Some numerical results along with its related graphs based on the mean queue size and the system total expected cost per unit of time are shown in this section. Our
main intention is to investigate the effect of the system parameters namely, the breakdown rates $\alpha_1$ and $\alpha_2$, the Bernoulli vacation probability $p$ and the arrival rate $\lambda$ on the mean queue size $L_s$ and on the optimal cost $TC$. Note that for the sake of computational convenience, let us assume that service times follow the exponential distributions with $\beta_i^{(1)} = \frac{1}{\mu_i}$, $\beta_i^{(2)} = \frac{2}{\mu_i^2}$ for $i = 1, 2$, the repair times follow the exponential distributions with $\beta_i^{(1)} = \frac{1}{\xi_i}$, $\beta_i^{(2)} = \frac{2}{\xi_i^2}$ and vacation time follows an exponential distribution with $\beta^{(1)} = \frac{1}{\gamma}$, $\beta^{(2)} = \frac{2}{\gamma^2}$.

2.9.1. Optimal design

The optimal design of a queueing system is to determine the optimal system parameters using some cost functions. To illustrate, let $c_h$ be the holding cost per unit time for each customer present in the system, $c_0$ be the cost per unit time for keeping the server on and in operation, $c_s$ be the setup cost per busy cycle, and $c_a$ be the startup cost per unit time for the preparatory work of the server before starting the service. These unit costs can be combined with the performance measures obtained above to write the system total expected cost per unit of time as

$$TC = c_h L_s + c_0 \frac{E(T_h)}{E(T_c)} + c_s \frac{1}{E(T_c)} + c_a \frac{E(T_0)}{E(T_c)}$$

$$= c_h L_s + c_0 \rho_H + c_s \lambda (1 - \rho_H) + c_a (1 - \rho_H)$$
Here $E(T_0) = \frac{1}{\lambda a_{01}}$ is the expected length of an idle period, while $E(T_b)$ and $E(T_C)$ are the expected length of a busy period and a busy cycle, respectively, and are given in section 2.5 of this Chapter.

For the following values of the parameters: $p = 0.5$, $\lambda = 0.68$, $c_s = 5$, $c_0 = 100$, $c_s = 1000$, $c_a = 100$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$, $\beta_1(t) = 0.5$, $\beta_2(t) = 0.4$, $g_1(t) = \beta_1(t) / 5$, $g_2(t) = \beta_2(t) / 5$, $\vartheta(t) = 0.45$, we find a total expected cost per unit of time is $TC = 255.788$.

The effect of the system parameters on the optimal policy can be easily undertaken numerically. For example, Tables 1-5 show the effect on the mean queue size $L_s$ and on the optimal cost $TC$ of the parameters of interest to us in this paper, namely, the breakdown rates $\alpha_1$ and $\alpha_2$, the probability of the Bernoulli vacation $p$, and the arrival rate $\lambda$, respectively. All the other parameters are kept unchanged.

Again in Table 6 we observe that for higher values of $\lambda$, the rate of increase in $L_s$ is faster than the lower values of $\lambda$ for various values of $p$ for both the cases, as it should be.

Finally in Table 7 and in Table 8, the numerical results are summarized in which the steady-state server availability $A_s$ and failure frequency $M_f$ are calculated with the given data. Clearly, high value of $\alpha$, (i=1, 2) results in low server availability and high failure frequency.
Table 1: Effect of the FPS breakdown rate $\alpha_1$ on the optimal cost

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
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<tr>
<td>$L_s$</td>
<td>2.76618</td>
<td>2.81934</td>
<td>2.8742</td>
<td>2.93083</td>
<td>2.98932</td>
<td>3.04976</td>
<td>3.11225</td>
<td>3.1769</td>
<td>3.24382</td>
</tr>
<tr>
<td>$TC$</td>
<td>263.92</td>
<td>261.874</td>
<td>259.837</td>
<td>257.808</td>
<td>255.788</td>
<td>253.778</td>
<td>251.779</td>
<td>249.79</td>
<td>247.813</td>
</tr>
</tbody>
</table>

Table 2: Effect of the SPS breakdown rate $\alpha_2$ on the optimal cost

<table>
<thead>
<tr>
<th>$\alpha_2$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_s$</td>
<td>2.8468</td>
<td>2.88134</td>
<td>2.91659</td>
<td>2.95258</td>
<td>2.98932</td>
<td>3.02684</td>
<td>3.06516</td>
<td>3.10432</td>
<td>3.14433</td>
</tr>
<tr>
<td>$TC$</td>
<td>260.994</td>
<td>259.687</td>
<td>258.384</td>
<td>257.084</td>
<td>255.788</td>
<td>254.496</td>
<td>253.208</td>
<td>251.924</td>
<td>250.645</td>
</tr>
</tbody>
</table>

Table 3: Effect of the Bernoulli vacation on the $L_s$ and optimal cost

<table>
<thead>
<tr>
<th>$p$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_s$</td>
<td>1.6997</td>
<td>1.93249</td>
<td>2.2114</td>
<td>2.55423</td>
<td>2.98932</td>
<td>3.56464</td>
<td>4.36843</td>
<td>5.58254</td>
<td>7.65161</td>
</tr>
<tr>
<td>$TC$</td>
<td>332.572</td>
<td>312.928</td>
<td>293.515</td>
<td>274.421</td>
<td>255.788</td>
<td>237.857</td>
<td>221.068</td>
<td>206.33</td>
<td>195.931</td>
</tr>
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</table>

Table 4: Effect of the arrival rate on the $L_s$ and optimal cost

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.58</th>
<th>0.605</th>
<th>0.63</th>
<th>0.655</th>
<th>0.68</th>
<th>0.705</th>
<th>0.73</th>
<th>0.755</th>
<th>0.78</th>
<th>0.805</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_s$</td>
<td>1.69865</td>
<td>1.9279</td>
<td>2.20577</td>
<td>2.5502</td>
<td>2.9893</td>
<td>3.56964</td>
<td>4.37414</td>
<td>5.56664</td>
<td>7.52201</td>
<td>11.3273</td>
</tr>
</tbody>
</table>

Table 5: Effect of the arrival rate on the $L_s$ and optimal cost

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.325</th>
<th>0.35</th>
<th>0.375</th>
<th>0.4</th>
<th>0.425</th>
<th>0.45</th>
<th>0.475</th>
<th>0.5</th>
<th>0.525</th>
<th>0.55</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_s$</td>
<td>0.546276</td>
<td>0.6117</td>
<td>0.6831</td>
<td>0.7614</td>
<td>0.8478</td>
<td>0.94386</td>
<td>1.05139</td>
<td>1.17283</td>
<td>1.3113</td>
<td>1.47099</td>
</tr>
<tr>
<td>$TC$</td>
<td>304.573</td>
<td>310.224</td>
<td>314.447</td>
<td>317.247</td>
<td>318.604</td>
<td>319.604</td>
<td>317.604</td>
<td>314.604</td>
<td>310.604</td>
<td>304.64</td>
</tr>
</tbody>
</table>

Table 6: Effect of the arrival rate and the Bernoulli vacation on the mean queue size

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.53</th>
<th>0.555</th>
<th>0.58</th>
<th>0.605</th>
<th>0.63</th>
<th>0.655</th>
<th>0.68</th>
<th>0.705</th>
<th>0.73</th>
<th>0.755</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_s$</td>
<td>0.95412</td>
<td>1.03943</td>
<td>1.13156</td>
<td>1.23171</td>
<td>1.3414</td>
<td>1.46251</td>
<td>1.5975</td>
<td>1.74956</td>
<td>1.92297</td>
<td></td>
</tr>
<tr>
<td>$TC$</td>
<td>1.04621</td>
<td>1.14539</td>
<td>1.25376</td>
<td>1.37316</td>
<td>1.50593</td>
<td>1.65512</td>
<td>1.82484</td>
<td>2.02047</td>
<td>2.24985</td>
<td></td>
</tr>
<tr>
<td>$p$</td>
<td>1.1482</td>
<td>1.2641</td>
<td>1.39246</td>
<td>1.53608</td>
<td>1.69865</td>
<td>1.88516</td>
<td>2.10251</td>
<td>2.36053</td>
<td>2.67376</td>
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</tr>
<tr>
<td>$TC$</td>
<td>1.26192</td>
<td>1.39818</td>
<td>1.55146</td>
<td>1.72607</td>
<td>1.92792</td>
<td>2.16534</td>
<td>2.45041</td>
<td>2.80144</td>
<td>3.24743</td>
<td></td>
</tr>
<tr>
<td>$L_s$</td>
<td>1.38964</td>
<td>1.55104</td>
<td>1.73586</td>
<td>1.95088</td>
<td>2.20577</td>
<td>2.51485</td>
<td>2.99019</td>
<td>3.39775</td>
<td>4.07032</td>
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</tr>
<tr>
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<td>1.72714</td>
<td>1.9526</td>
<td>2.22153</td>
<td>2.5502</td>
<td>2.96417</td>
<td>3.50596</td>
<td>4.25203</td>
<td>5.35497</td>
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</tr>
<tr>
<td>$p$</td>
<td>1.6997</td>
<td>1.93249</td>
<td>2.2114</td>
<td>2.55423</td>
<td>2.98932</td>
<td>3.56464</td>
<td>4.36843</td>
<td>5.58254</td>
<td>7.65161</td>
<td></td>
</tr>
<tr>
<td>$TC$</td>
<td>1.89087</td>
<td>2.17538</td>
<td>2.52632</td>
<td>2.97386</td>
<td>3.56963</td>
<td>4.41013</td>
<td>5.69873</td>
<td>7.95054</td>
<td>12.9606</td>
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</tr>
<tr>
<td>$p$</td>
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<td>2.46753</td>
<td>2.91849</td>
<td>3.52064</td>
<td>4.37414</td>
<td>5.69254</td>
<td>8.02708</td>
<td>13.3698</td>
<td>38.7388</td>
<td></td>
</tr>
</tbody>
</table>
Next, fixing the base values given above a sensitivity analysis of the same parameters on the system can be performed with help of graphs.

The graphs below show the effect of some of the system parameters on the total expected cost per unit of time and on mean queue size in the system. To investigate the effect of server failures on the total expected cost per unit of time, we give different values to mean failure rates and record the corresponding value of the system total expected cost and mean queue size. Figure 1 and Figure 3 below show that $TC$ decreases as mean failure rates increases. From Figure 2 and Figure 4, we observe that $L_s$ increases as $\alpha_i$ $(i = 1,2)$ increases. When breakdown rates increase, the server is unable to provide service for the customers, which leads to the expected number of customers in the system becoming larger and the completion period longer.

![Figure 1](image1.png)

![Figure 2](image2.png)
Suppose now we are interested in the effect of the Bernoulli vacation schedule on the system performance. Keeping the values of the system parameters unchanged, we vary the probability of a vacation from 0 to 0.9 and again record the corresponding values of the system total expected cost and mean queue size in the system. Figures 5 and 6 below depict the variations of the cost and mean queue size respectively with the probability of a vacation. We see that $TC$ first decreases as $p$ increases and then becomes stably as $p$ becomes large. Figure 6 reports that mean queue size increases as $p$ increases. As expected, a larger $p$ implies that the number of customers and the completion period becomes larger, due to ongoing preventative maintenance having a higher probability.
Next, we are interested to investigate the cases that the effect of arrival rate when all the data are kept unchanged. Figure 7 and Figure 8 show the variations of the system total cost and mean queue size in the system respectively when the arrival rate varies from 0.58 to 0.805. We see that $TC$ first decreases as $\lambda$ increases and then becomes stably as $\lambda$ becomes large. Figure 8 reports that mean queue size increases as $\lambda$ increases.

Finally, we want to see the effect of arrival rate when the arrival rate varies from 0.125 to 0.35. Figure 9 and Figure 10 show the variations of the system total cost and mean queue size in the system respectively. Figure 9 reveals that $TC$ first increases ($\lambda \leq 0.425$) and then decreases ($\lambda > 0.425$). We find that the maximum cost is $TC = 318.63$, and it is obtained when $\lambda = 0.425$ and in Figure 10 it appears that $L_s$ increases as $\lambda$ increases.
2.10 Concluding Remarks

We have studied in this chapter an $M/G/1$ queue with the following features: each customer requires two successive phases of service, the server is unreliable and may breakdown during any phase of service, the server is subject to a Bernoulli vacation schedule. We have obtained the following results: the probability generating function of the joint distributions of the server state and queue size, the queue size distribution at the departure epoch, waiting time distribution, busy period distribution of the system reliability function.

This study can be complemented in various ways by introducing concepts of new vacation policies like modified vacation policy, work vacation policy etc [for details we refer to see Ke et. al[2010(a)]. Further present model can be generalized for the arrival process to the case of a compound Poisson process.