8.1 Introduction

Retrial queues are characterized by the feature that a customer who finds the server busy upon arrival is obliged to leave the service area and repeat its demand after some time called "retrial time". Between trials, the blocked customer joins a group of unsatisfied customers called "orbit" or 'retrial group'. Retrial queuing systems with general service time and non-exponential retrial times have also received considerable attention during the last decade. The first investigation with general retrial time was done by Kapyrin (1977), but the methodology used was found to be incorrect by Falin (1986). Subsequently, Yang et al. (1994) have developed an approximation method to obtain the steady state performance for the model of Kapyrin. In recent years, several retrial models have been analyzed with general retrial times, details of which may be found in Gomez-Corral (1999), Atencia and Moreno (2005), Chang and Ke (2009), Krishnakumar et al. [2002(b)], Moreno (2004) and Wang and Li (2009).

The classical vacation scheme with Bernoulli service discipline was originated

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and significantly developed by Keilson and Servi (1986) and co-workers. Recently, there has been considerable attention paid to the study the $M/G/1$ type of queuing systems with two phases of service under Bernoulli vacation schedule and different vacation policies by Choudhury, et. al. (2007), Choudhury and Madan (2005) and Choudhury and Madan (2004). Further, Haridass and Arumuganathan (2011) investigated a bulk service queueing system with variant threshold policy for secondary jobs, which is related to our two phase queueing system.

A wide class of retrial policies for governing the vacation mechanism has also been discussed in the literature. However, a very few works on retrial models along with vacation mechanism for two phases of heterogeneous service system have been made. A number of papers by Ke and Chang [2009(b)], and Krishnakumer et. al. [2002(b)] has recently appeared in the queueing literature in which the concept of general retrial times has been introduced along with Bernoulli vacation schedule under the FCFS orbit retrial policy for batch arrival queueing system. Such type of queueing models occur in many real life situations where the server may be used for other secondary jobs, for instance to serve customers in other systems. Applications arise naturally in call centers with multi-task employees, customized manufacturing, telecommunication and computer networks, maintenance activities, production and quality control problems etc.

Again a number of researchers have made an extensive study on queueing models with unreliable server including Gaver (1962), Avi-Itzhak and Naor (1963), Thirurengadan (1963) and Mitrany and Avi-Itzhak (1968). On the other hand, retrial queues that take into account servers failures and repairs were introduced by Aissani (1988) and Kulkarni and Choi (1990). Wang et al. (2001) were the first to study a
repairable $M/G/1$ retrial queueing model from the viewpoint of reliability. They obtained both the queueing indices and the reliability characteristics. More recently, Ke and Chang [2009(b)] investigated similar type of $M^X/G/1$ retrial queueing model with two phases of service, Bernoulli vacation schedule, starting failure, and general retrial times. Although some aspects have been discussed separately on queueing systems with service interruptions, two phases of service, Bernoulli schedule vacation, repeated attempts with general retrial time, however, no work have been found that combines these features together for unreliable server queueing systems, even in the most recent studies. Moreover, another important characteristic for considering the retrial model with general retrial times is that we always obtain analytical solution in term of closed form expression. Hence to fill up to this gap, in this Chapter an attempt is made to study an $M^X/G/1$ retrial queue with two phases of service, general retrial times and Bernoulli vacation schedule for an unreliable server. To this end, the methodology is based on the embedded Markov chain and the inclusion of supplementary variables.

The following results have been obtained under the present study of this Chapter

(i) Joint distribution of the number in the orbit and state of the server

(ii) Stochastic decomposition

(iii) Distribution of number of customers in the system at a departure epoch

(iv) Particular cases

(v) System Performance measures

(vi) Optimal control policy

(vii) The numerical illustration.
8.2 The Model description

We consider an $M^x/G/1$ queueing system with two phases of service, where the primary customers arrive according to a compound Poisson process with mean arrival rate $\lambda$. The size of the successive arriving batches is $X_1, X_2, \ldots$; where $X_1, X_2, \ldots$ are independently identically distributed (i.i.d) random variables with probability mass function (p.m.f) $a_n = P_r\{X = n\}; n \geq 1$, probability generating function $a(z) = E[z^X]$ and the first two moments $a_1[1]$ and $a_2[2]$ respectively. The server provides to each primary customer two phases of heterogeneous service in succession, a first phase of service (FPS) followed by a second phase of service (SPS) (Busy periods). The service discipline is assumed to be FCFS. The service times random variables $B_i$, $i = 1, 2$ are assumed to follow general laws with probability distribution function (d.f) $B_i(x), i = 1, 2$, Laplace Stieltjes Transform (LST) $\beta_{i}^*(\theta) = E[e^{-\theta B_i}]$ and finite $k$-th moments $\beta_{i}^{(k)}$, $i = 1, 2$, where sub index $i = 1$ (respectively $i = 2$ ) denotes the FPS (respectively SPS). While the server is working with any phase of service, it may breakdown at any time and the service channel will fail for a short interval of time (Breakdown periods). The breakdowns i.e., server’s life times are generated by an exogenous Poisson process with rates $\alpha_1$ for FPS and $\alpha_2$ for SPS respectively. As soon as a breakdown occurs, the server is sent for repair during which time it stops providing service to the primary customers till the service channel is repaired. The customer which was being served just before server breakdown waits for the server to complete its remaining service. The repair time (denoted by $R_1$ for FPS and $R_2$ for SPS) distributions of the server for both phases of service are assumed to be arbitrarily
distributed with d.f $G_1(y)$ and $G_2(y)$, \( \text{LST} \ G'_1(\theta) = E[e^{-\theta t_1}] \) and \( \text{LST} \ G'_2(\theta) = E[e^{-\theta t_2}] \) and finite \( k \)-th moments \( g_1^{(k)} \) and \( g_2^{(k)} \), respectively. Immediately after the broken server is repaired, the server is ready to start its remaining service to customers in both phases of service. In this case, the service times are cumulative i.e., we consider preemptive-resume for service time, which we may refer to as \emph{generalized service times}. After each SPS service completion, the server may go for a vacation of random length \( V \) (\emph{vacation period}) with probability \( p (0 \leq p \leq 1) \) or may serve the next unit, if any, with probability \( q = 1 - p \), i.e., the server takes a Bernoulli vacation. Next, we assume that the vacation time random variable \( V \) follows a general law of probability with d.f \( V(y) \), \( \text{LST} \ \theta^* (\theta) = E[e^{-\theta V}] \) and finite \( k \)-th moments \( \theta^{(k)} \). This type of queueing model is known as Bernoulli vacation queue with two phases of service and unreliable server. The model of this nature, without a second phase of service, was first investigated by \textit{Li et al.} (1997). Now to further develop such a type of model, we may further introduce the concept of repeated attempts with \emph{FCFS orbit retrial policy}, where if a primary customer finds the server busy with any phase of service, on vacation, or down, then he/she leaves the service area and joins a group of unsatisfied customers, i.e. in a orbit in order to seek service again and again until it finds the server free. The time between successive repeated attempts i.e. retrial times \( \{R_n; n \geq 1\} \) of the customers are assumed to be a sequence of i.i.d random variables with d.f \( C(x) \) and \( \text{LST} \ C^*(\theta) = E[e^{-\theta R}] \). Further, we assume that input process, server’s life time, server’s delay time, server’s repair time, service time and retrial time random variables are mutually independent of each others. It should be noted here that this type of retrial model for reliable server, i.e. without service interruption, was also investigated by Ke
and Chang [2009(b)] and Kella (1990). However, our assumption on retrial policy varies from them, wherein they assume that the server always takes a vacation when the orbit is empty. Whereas here we assume that an arriving customer that finds the server unavailable joins the retrial box from which only the first customer retries for service.

8.3 Joint Distribution of Number in the Orbit and State of the Server

In this section first we investigate the necessary and sufficient condition for the system to be stable of our model.

Theorem 8.3.1

The inequality \( a_{[0]}(1 - C^*(\lambda)) + \rho_H < 1 \) is a necessary and sufficient condition for the system to be stable, where \( \rho_H = \lambda a_{[0]}(\beta_1^{(0)}{1 + \alpha_1 g_1^{(0)}} + \beta_2^{(0)}{1 + \alpha_2 g_2^{(0)}}) + p g^{(0)} \).

Proof: - It is easy to see that \( \{X_n; n \in \mathbb{N}^+\} \) is an irreducible and a periodic Markov chain. To prove the positive recurrence we may use the Foster's criterion [e.g. see section 1.5 of Chapter 1], which states that an irreducible and aperiodic Markov chain is positive recurrent if there exists a non-negative function \( f(s), s \in \mathbb{N}^+ \) and \( \varepsilon > 0 \) such that the mean drift \( \varphi_s = E[f(X_{n+1}) - f(X_n)|X_n = s] \) is finite for all \( s \in \mathbb{N}^+ \) and \( \varphi_s \leq -\varepsilon \) for all \( s \in \mathbb{N}^+ \) except perhaps a finite number. In our case, we take \( f(s) = s \) to obtain

\[
\varphi_j = \begin{cases} 
\lambda a_{[0]}(\beta_1^{(0)}{1 + \alpha_1 g_1^{(0)}}) + \lambda a_{[0]}(\beta_2^{(0)}{1 + \alpha_2 g_2^{(0)}}) + \lambda a_{[0]}p g^{(0)} - 1, & j = 0 \\
\lambda a_{[0]}(\beta_1^{(0)}{1 + \alpha_1 g_1^{(0)}}) + \lambda a_{[0]}(\beta_2^{(0)}{1 + \alpha_2 g_2^{(0)}}) + \lambda a_{[0]}p g^{(0)} + a_{[0]}(1 - C^*(\lambda)), & j = 1, 2, \ldots,
\end{cases}
\]

where \( C^*(\lambda) = \int_0^\infty e^{-\lambda x} dC(x) \) is the probability that no units arrive during the retrial times.

Obviously, \( \lambda a_{[0]}(\beta_1^{(0)}{1 + \alpha_1 g_1^{(0)}}) + \lambda a_{[0]}(\beta_2^{(0)}{1 + \alpha_2 g_2^{(0)}}) + p \lambda a_{[0]}(g^{(0)}) + a_{[0]}(1 - C^*(\lambda)) \)
\[ a_{i|j}(1 - C^*(\lambda)) + \rho_H < 1 \] is a sufficient condition for ergodicity.

The necessary condition follows readily from Kaplan's condition as noted in Sennott et al. (1983), namely \( \varphi_j < \infty \) for all \( j \geq 0 \) and there exists \( j_0 \in \mathbb{Z}^+ \) such that \( \varphi_j \geq 0 \) for \( j \geq j_0 \). It should be noted here that in our case, Kaplin's condition is satisfied because there is a such 'k' such that for \( \rho_{i,j} = 0 \) for \( j < i - k \) and \( i > 0 \), where \( P = (p_{i,j}) \) is the one step transition matrix of \( \{X_n; n \in \mathbb{Z}^+\} \). Then \( a_{i|j}(1 - C^*(\lambda)) + \rho_H \geq 1 \) implies the nonergodicity of the Markov chain. Hence if \( a_{i|j}(1 - C^*(\lambda)) + \rho_H < 1 \) and \( G_i(\ ), B_i(\ ), V(\ ) \) and \( C(\ ) \) for \( i = 1, 2 \) satisfy the regularity conditions, then the system is stable.

Since the arrival process is a Poisson process, it can be shown from Burke's theorem [e.g. see Cooper (1981), pp. 187 - 188 or section 1.5 of Chapter I] that the steady state probabilities of our bivariate Markov process \( \{N(t), X(t)\} \) exist and are positive under the same condition as the steady state probabilities of \( \{X_n, n \in \mathbb{Z}^+\} \) i.e., if and only if \( a_{i|j}(1 - C^*(\lambda)) + \rho_H < 1 \).

Next, attempts are made to obtain the PGF of the joint distribution of the state of the server and the number in the orbit by treating elapsed retrial time, elapsed service time of the customer and elapsed vacation time, elapsed delay time and elapsed repair time of the server as supplementary variables. Assuming the system is in steady state conditions, let \( N(t) \) be the orbit size (i.e., number of customers in the retrial group) at time \( t \), \( R^0(t), B^0_i(t) \) be the elapsed retrial time and service times for \( i^{th} \) phase of service, where sub index \( i = 1 \) (respectively \( i = 2 \)) denotes the FPS (respectively SPS) of the customers at time \( t \). In addition, let \( R^0_i(t) \) for \( i = 1, 2 \) and \( V^0(t) \), be the elapsed repair time of the server during which breakdown occurs in the system at \( i^{th} \) phase of
service time $t$ and elapsed vacation time of the server at time $t$ respectively. Further, we introduced the following random variable:

$$Y(t) = \begin{cases} 
0, & \text{if the server is idle with no customer in the system at time } t \\
1, & \text{if the server is idle during the retrial time at time } t \\
2, & \text{if the server is busy with FPS at time } t \\
3, & \text{if the server is busy with SPS at time } t \\
4, & \text{if the server is on vacation at time } t \\
5, & \text{if the server is under repair during FPS at time } t \\
6, & \text{if the server is under repair during SPS at time } t 
\end{cases}$$

So that the supplementary variables $R^0(t), B^0_i(t), V^0(t)$ and $R^0_i(t)$ for $i = 1, 2$ are introduced in order to obtain a bivariate Markov process $\{N(t), X(t)\}$, where $X(t) = 0$ if $Y(t) = 0$, $X(t) = R^0(t)$ if $Y(t) = 1$, $X(t) = B^0_i(t)$ if $Y(t) = 2$, $X(t) = B^0_i(t)$ if $Y(t) = 3$, $X(t) = V^0(t)$ if $Y(t) = 4$, $X(t) = R^0_i(t)$ if $Y(t) = 5$ and $X(t) = R^0_i(t)$ if $Y(t) = 6$.

Now we define following limiting probabilities:

$$P_0 = \lim_{t \to \infty} P_r \{N(t) = 0, X(t) = 0\}$$

$$\psi_n(x)dx = \lim_{t \to \infty} P_r \{N(t) = n, X(t) = R^0(t); x < R^0(t) \leq x + dx\}; x > 0, n \geq 1$$

$$Q_n(y)dy = \lim_{t \to \infty} P_r \{N(t) = n, X(t) = V^0(t); y < V^0(t) \leq y + dy\}; y > 0, n \geq 0$$

For $i = 1, 2$

$$P_{i,n}(x)dx = \lim_{t \to \infty} P_r \{N(t) = n, X(t) = B^0_i(t); x < B^0_i(t) \leq x + dx\}; x > 0, n \geq 0$$

and for fixed values of $x$ and $n \geq 0$

$$R_{i,n}(x, y)dy = \lim_{t \to \infty} P_r \{N(t) = n, X(t) = R^0_i(t); y < R^0_i(t) \leq y + dy\}; (x, y) > 0$$

Further, it is assumed that $C(0) = 0, C(\infty) = 1, V(0) = 0, V(\infty) = 1, B_i(0) = 0, B_i(\infty) = 1$,
$G_r(0) = 0$, $G_r(\infty) = 1$, that $C(x), B_i(x)$ are continuous at $x = 0$, and that $V(y)$ and $G_i(y)$ are continuous at $y = 0$ respectively, so that

$$\eta(x)dx = \frac{dC(x)}{1 - C(x)}; \quad \gamma(y)dy = \frac{dV(y)}{1 - V(y)};$$

and for $i = 1, 2$; $\mu_i(x)dx = \frac{dB_i(x)}{1 - B_i(x)}$ and $\xi_i(y)dy = \frac{dG_i(y)}{1 - G_i(y)}$ are the first order differential (hazard rate) functions of $C(\ ), V(\ ), B_i(\ )$ and $G_i(\ )$, respectively.

### 8.3.1 The steady state equations

The Kolmogorov forward equations to govern the system under steady state conditions [e.g. see Cox [1955(a)] or section 1.5 of Chapter I] can be written as follows:

\begin{align}
\frac{d}{dx} \psi_n(x) + [\lambda + \eta(x)] \psi_n(x) &= 0; \quad n \geq 1 \quad (8.3.1.1) \\
\frac{d}{dy} Q_n(y) + [\lambda + \gamma(y)] Q_n(y) &= \lambda(1 - \delta_{n,0}) \sum_{k=1}^{n} a_k Q_{n-k}(y); \quad n \geq 0 \quad (8.3.1.2)
\end{align}

and for $i = 1, 2$

\begin{align}
\frac{d}{dx} P_{i,n}(x) + [\lambda + \alpha_i + \mu_i(x)] P_{i,n}(x) &= \lambda \sum_{k=1}^{n} a_k P_{i,n-k}(x) + \int_{0}^{\infty} \xi_i(y) R_{i,n}(x, y) dy; \quad n \geq 1 \quad (8.3.1.3) \\
\frac{d}{dy} R_{i,n}(x, y) + [\lambda + \xi_i(y)] R_{i,n}(x, y) &= \lambda \sum_{k=1}^{n} a_k R_{i,n-k}(x; y); \quad n \geq 1 \quad (8.3.1.4)
\end{align}

\[ \lambda P_0 = \int_{0}^{\infty} \gamma(y) Q_0(y) dy + q \int_{0}^{\infty} \mu_2(x) P_{2,1}(x) dx \quad (8.3.1.5) \]
where \( R_{i,0}(x,y) = 0 \) and \( P_{i,0}(x) = 0 \) for \( i = 1,2 \) occurring in equations (8.3.1.3) and (8.3.1.4) respectively.

These set of equations are to be solved under the boundary conditions at \( x = 0 \):

\[
\psi_n(0) = \int_0^\infty \gamma(y)Q_n(y)dy + q \int_0^\infty \mu_2(x)P_{2,n+1}(x)dx; n \geq 1
\]  
(8.3.1.6)

\[
P_{1,n}(0) = \lambda \sum_{k=1}^n a_k \int_0^\infty \psi_{n-k}(x)dx + \lambda a_n P_0 + \int_0^\infty \eta(x)\psi_n(x)dx; n \geq 1
\]  
(8.3.1.7)

\[
P_{2,n}(0) = \int_0^\infty \mu_1(x)P_{1,n}(x)dx; n \geq 1
\]  
(8.3.1.8)

and at \( y = 0 \):

\[
Q_n(0) = p \int_0^\infty \mu_2(x)P_{2,n+1}(x)dx; n \geq 0
\]  
(8.3.1.9)

Also at \( y = 0 \) and fixed values of \( x \) and \( i = 1,2 \)

\[
R_{i,n}(x,0) = \alpha_i P_{i,n}(x); x > 0, n \geq 1
\]  
(8.3.1.10)

with normalizing condition

\[
P_0 + \sum_{n=1}^\infty \int_0^\infty \psi_n(x)dx + \sum_{n=0}^\infty \int_0^\infty Q_n(y)dy + \sum_{i=1}^2 \sum_{n=1}^\infty \left\{ \int_0^\infty P_{i,n}(x)dx + \int_0^\infty R_{i,n}(x,y)dxdy \right\} = 1
\]  
(8.3.1.11)

8.3.2 The model solution

To solve the system of equations (8.3.1) - (8.3.10), let us define following PGFs for \( |z| < 1 \):
\[ Q(y; z) = \sum_{n=0}^{\infty} z^n Q_n(y) ; \quad Q(0; z) = \sum_{n=0}^{\infty} z^n Q_n(0) \]

\[ \psi(x, z) = \sum_{n=1}^{\infty} z^n \psi_n(x) ; \quad \psi(0, z) = \sum_{n=1}^{\infty} z^n \psi_n(0) \]

and for \( i = 1, 2 \)

\[ P_i(x, z) = \sum_{n=0}^{\infty} z^n P_{i,n}(x) ; \quad P_i(0, z) = \sum_{n=0}^{\infty} z^n P_{i,n}(0) \]

\[ R_i(x, y; z) = \sum_{n=0}^{\infty} z^n R_{i,n}(x; y) ; \quad R_i(x, 0; z) = \sum_{n=0}^{\infty} z^n R_{i,n}(x; 0). \]

Let \( b(z) = \lambda(1 - \alpha(z)) \); It should be noted here that if \( a_n = \delta_{n1}; n \geq 1 \), then \( \alpha(z) = z \), then proceeding in the usual manner with equations (8.3.1.1) - (8.3.1.5), we get a set of differential equations of Lagrangian type whose solutions are given by:

\[ \psi(x; z) = \psi(0; z)[1 - C(x)] \exp\{-b(0)x\} \quad (8.3.2.1) \]

\[ Q(y; z) = Q(0; z)[1 - V(y)] \exp\{-b(z)y\} \quad (8.3.2.2) \]

\[ P_i(x; z) = P_i(0; z)[1 - B_i(x)] \exp\{-\lambda_i(z)x\}; \quad \text{for} \ i = 1, 2 \quad (8.3.2.3) \]

and \( R_i(x, y; z) = R_i(x, 0; z)[1 - G_i(y)] \exp\{-b(z)y\}; \quad \text{for} \ i = 1, 2 \quad (8.3.2.4) \)

where \( \lambda_i(z) = b(z) + \alpha_i \left(1 - G_i'(b(z))\right) \) for \( i = 1, 2 \).

Again multiplying both sides of equation (8.3.1.10) by \( z^n \) and then taking the summation over all possible values of \( n \geq 1 \) and utilizing (8.3.2.3), for \( i = 1, 2 \) we get on simplification

\[ R_i(x; 0; z) = \alpha_i P_i(0; z)[1 - B_i(x)] \exp\{-\lambda_i(z)x\} \quad (8.3.2.5) \]

Now multiplying equation (8.3.1.6) by \( z^n \) and then taking the summation over all possible values of \( n \geq 1 \), we get on simplification
\[ \psi(0,z) = qz^{-1}P_2(0,z)\beta_1^*(\lambda_2(z)) + Q(0,z)\theta^*(b(z)) - \lambda P_0 \]  
\tag{8.3.2.6}

Similarly from equations (8.3.1.7) - (8.3.1.9), we get
\[ zP_1(0,z) = \lambda b(z)P_0 + \psi(0,z)[C^*(\lambda) + b(z)\{1 - C^*(\lambda)\}] \]  
\tag{8.3.2.7}
\[ P_2(0,z) = P_1(0,z)\beta_1^*(\lambda_1(z)) \]  
\tag{8.3.2.8}

and \[ Q(0,z) = pz^{-1}P_2(0,z)\beta_2^*(\lambda_2(z)) \]  
\tag{8.3.2.9}

Utilizing (8.3.2.8) and (8.3.2.9) in (8.3.2.6) and then substituting it in (8.3.2.7) yield
\[ P_1(0,z) = \frac{P_0 zb(z)C^*(\lambda)}{L(z)[C^*(\lambda) + a(z)\{1 - C^*(\lambda)\}] - z} \]  
\tag{8.3.2.10}

where \[ L(z) = \{q + p\theta^*(b(z))\} \beta_1^*(\lambda_1(z))\beta_2^*(\lambda_2(z)) \]

Now utilizing (8.3.2.10) in (8.3.2.7) - (8.3.2.9), we get
\[ \psi(0,z) = \frac{\lambda P_0[z - b(z)L(z)]}{L(z)[C^*(\lambda) + a(z)\{1 - C^*(\lambda)\}] - z} \]  
\tag{8.3.2.11}
\[ P_2(0,z) = \frac{P_0 zb(z)C^*(\lambda)\beta_1^*(\lambda_1(z))}{L(z)[C^*(\lambda) + a(z)\{1 - C^*(\lambda)\}] - z} \]  
\tag{8.3.2.12}

and \[ Q(0,z) = \frac{P_0 zb(z)\beta_1^*(\lambda_1(z))\beta_2^*(\lambda_2(z))C^*(\lambda)}{L(z)[C^*(\lambda) + a(z)\{1 - C^*(\lambda)\}] - z} \]  
\tag{8.3.2.13}

Letting \( z \to 1 \) in (8.3.2.10), we obtain by the L’ Hospital’s rule
\[ P_1(0,1) = \frac{\lambda a_{[i]}C^*(\lambda)P_0}{[1 - a_{[i]}(1 - C^*(\lambda)) - \rho_H]} \]  
\tag{8.3.2.14}

This gives
\[ Q(y;1) = \frac{\lambda a_{[i]}PC^*(\lambda)P_0[1 - V(y)]}{[1 - a_{[i]}(1 - C^*(\lambda)) - \rho_H]} \]  
\tag{8.3.2.15}

and for \( i = 1,2 \)
\[ P_1(x, l) = \frac{\lambda a_{[0]} C^*(\lambda) P_0 \left[ 1 - B_1(x) \right]}{\left[ 1 - a_{[0]}(1 - C^*(\lambda)) - \rho_H \right]} \]  
\"(8.3.2.16)\"

\[ R_1(x, y, l) = \frac{\lambda a_{[0]} C^*(\lambda) P_0 \left[ 1 - B_1(x) \right][1 - G_1(y)]}{\left[ 1 - a_{[0]}(1 - C^*(\lambda)) - \rho_H \right]} \]  
\"(8.3.2.17)\"

Hence from the normalizing condition (8.3.1.11), we get

\[ P_0 = \frac{\left[ 1 - a_{[0]}(1 - C^*(\lambda)) - \rho_H \right]}{C^*(\lambda)} \]  
\"(8.3.2.18)\"

Note that the equation (8.3.2.18) represents steady state probability that the server is idle but available in the system. Thus we summarize our results in the following Theorem 8.3.1.

**Theorem 8.3.1**

Under the stability condition \( a_{[0]}(1 - C^*(\lambda)) + \rho_H < C^*(\lambda) \), the joint distribution of the state of the server and the size of the orbit has the following partial PGFs as

\[ \psi(x; z) = \frac{\lambda \left[ 1 - a_{[0]}(1 - C^*(\lambda)) - \rho_H \right] [z - b(z) L(x)][1 - C(x)] \exp\{-\lambda x\}}{C^*(\lambda) L(x)[C^*(\lambda) + a(z)[1 - C^*(\lambda)]] - z} \]  
\"(8.3.2.19)\"

\[ P_1(x; z) = \frac{z \left[ 1 - a_{[0]}(1 - C^*(\lambda)) - \rho_H \right] b(z)[1 - B_1(x)] \exp\{-\lambda_1(z) x\}}{L(x)[C^*(\lambda) + a(z)[1 - C^*(\lambda)]] - z} \]  
\"(8.3.2.20)\"

\[ P_2(x; z) = \frac{z \left[ 1 - a_{[0]}(1 - C^*(\lambda)) - \rho_H \right] \beta_1^*(\lambda_1(z)) b(z)[1 - B_2(x)] \exp\{-\lambda_2(z) x\}}{L(x)[C^*(\lambda) + a(z)[1 - C^*(\lambda)]] - z} \]  
\"(8.3.2.21)\"

\[ Q(y; z) = \frac{p \left[ 1 - a_{[0]}(1 - C^*(\lambda)) - \rho_H \right] \beta_1^*(\lambda_1(z)) \beta_2^*(\lambda_2(z)) b(z)[1 - P(y)] \exp\{-b(z) y\}}{L(x)[C^*(\lambda) + a(z)[1 - C^*(\lambda)]] - z} \]  
\"(8.3.2.22)\"

\[ R_1(x, y; z) = \frac{z \left[ 1 - a_{[0]}(1 - C^*(\lambda)) - \rho_H \right] b(z)[1 - B_1(x)] \exp\{-\lambda_1(z) x\} \times [1 - G_1(y)] \exp\{-b(z) y\}}{L(x)[C^*(\lambda) + a(z)[1 - C^*(\lambda)]] - z} \]
Next we are interested to investigate the marginal orbit size distribution due to system state of the server.

Theorem 8.3.2

Under the stability condition \( \rho_H < C^*(\lambda) \), the marginal PGFs of the server’s state orbit size distribution are given by

\[
\varphi(z) = \frac{[1-a_0][1-C^*(\lambda)] - \rho_H}{[1-G(z)][1-C^*(\lambda)]} \frac{z-b(z)L(z)[1-C^*(\lambda)]}{L(z)[1-C^*(\lambda)]} \]  

(8.3.2.25)

\[
P_1(z) = \frac{z[1-a_0](1-C^*(\lambda)) - \rho_H}{\lambda_1(z)[L(z)[1-C^*(\lambda)] - z]}
\]

(8.3.2.26)

\[
P_2(z) = \frac{z[1-a_0](1-C^*(\lambda)) - \rho_H}{\lambda_2(z)[L(z)[1-C^*(\lambda)] - z]}
\]

(8.3.2.27)

\[
Q(z) = \frac{[1-a_0](1-C^*(\lambda)) - \rho_H}{L(z)[1-C^*(\lambda)]} \frac{1-G(z)(b(z))}{[1-G(z)(b(z))][1-C^*(\lambda)] - z}
\]

(8.3.2.28)

\[
R_i(z) = \frac{za[1-a_0](1-C^*(\lambda)) - \rho_H}{\lambda_i(z)[L(z)[1-C^*(\lambda)] - z]}
\]

(8.3.2.29)
and \( R_2(z) = \frac{z \alpha \beta_1(\lambda_1(z))\left[1 - a_0(1 - C^*(\lambda)) - \rho_{xz} \left[1 - \beta_2^*(\lambda_2(z))\left(1 - G^*_2(b(z))\right)\right]\right]}{\lambda_2(z)\left[C^*(\lambda) + a(z)\left[1 - C^*(\lambda)\right]\right] - z} \) \hspace{1cm} (8.3.2.30)

**Proof:** Integrating expressions (8.3.2.19) - (8.3.2.21) with respect to \( x \) and using the well-known result of renewal theory

\[
\int_0^\infty e^{-sx}(1 - F(x))dx = \frac{1 - f^*(s)}{s} \hspace{1cm} (8.3.2.31)
\]

where \( F(\cdot) \) is the distribution function of a random variable and \( f^*(\cdot) \) represents LST of it, we get formulae (8.3.2.25) - (8.3.2.27), respectively. Similarly, integrating (8.3.2.22) with respect to \( y \), we get formula (8.3.2.28). Again, integrating equation (8.3.2.23) and (8.3.2.24) with respect to \( y \), we get

\[
R_i(x; z) = \int_0^\infty R_i(x, y; z)dy = \frac{\alpha_i P_i(x; z)(1 - G^*_i(b(z)))}{b(z)} \quad \text{for } i = 1, 2. \hspace{1cm} (8.3.2.32)
\]

Further, integrating (8.3.2.32) with respect to \( x \) for \( i = 1, 2 \), we claimed in formula (8.3.2.29) and (8.3.2.30), respectively.

Our next objective is to investigate the distribution of the number of customers in the system and orbit.

**Corollary 8.3.1**

Let \( \phi_i \) and \( \chi_i \) be the stationary distribution of the number of customers in the system and orbit respectively under the steady state condition, then its corresponding PGF \( \Phi(z) = \sum_{j=0}^{\infty} z^j \phi_j \) and \( \chi(z) = \sum_{j=0}^{\infty} z^j \chi_j \) are given by
\[ \Phi(z) = \frac{[1 - a_{ij}(1 - C^*(\lambda)) - \rho_n]}{[C^*(\lambda) + a(z)(1 - C^*(\lambda))][q + pr^*(b(z))]} \beta_1^*(\lambda_1(z)) \beta_2^*(\lambda_2(z)) - z \] (8.3.2.33)

and

\[ \chi(z) = \frac{[1 - a_{ij}(1 - C^*(\lambda)) - \rho_n]}{[C^*(\lambda) + a(z)(1 - C^*(\lambda))][q + pr^*(b(z))]} \beta_1^*(\lambda_1(z)) \beta_2^*(\lambda_2(z)) - z \] (8.3.2.34)

respectively.

**Proof:** - The result follows directly from Theorem 8.3.2. With the help of the PGFs \( \psi(z), Q(z), P_i(z), R_i(z) \) for \( i = 1,2 \) and \( P_0 \), we get the distribution of the number of customers in the orbit having PGF

\[ \Phi(z) = P_0 + \psi(z) + P_1(z) + P_2(z) + zQ(z) + R_1(z) + R_2(z). \]

By direct calculation we can obtain (8.3.2.33). Further utilizing the relationship between orbit size and system size distribution for an \( M^x/G/1 \) type retrial queue, we may write

\[ \chi(z)L(z) = \Phi(z) \] (8.3.2.35)

Now utilizing (8.3.2.33) in (8.3.2.35), we get (8.3.2.34)

### 8.4 Stochastic Decomposition

In this section we study the stochastic decomposition property of the system size distribution of our model. The literature on vacation models recognizes this property as one of the most interesting features in this mater, e.g., see Doshi (1986), and Fuhrmann and Cooper (1985). Stochastic decomposition for retrial models is also found in Yang et al. (1994) and Yang and Templeton (1987). The existence of the stochastic decomposition property for our model can be demonstrated easily by showing that
\( \Phi(z) = \Phi_0(z)M(z) \) \hspace{1cm} (8.4.1)

where \( \Phi_0(z) \) is the PGF of the system size distribution of an \( M^X/G/1 \) queue with unreliable server and delaying repair under Bernoulli vacation schedule given by formula (8.4.1) by putting \( C^*(\lambda) = 1 \) in the formula (8.3.2.33). Thus we have

\[
\Phi_0(z) = \frac{[1-\rho_H](1-z)[q + p y^*(b(z))]\beta_1^*(\lambda_1(z))\beta_2^*(\lambda_2(z))}{[q + p y^*(b(z))]\beta_1^*(\lambda_1(z))\beta_2^*(\lambda_2(z))-z}
\]

Note that for \( \alpha_i = 0 \) (i.e. for reliable service systems) the above result is consistent with the result obtained by Choudhury and Madan (2004) and \( M(z) \) is the PGF of the conditional distribution of the number of customers in the orbit given that the system is idle = \( \frac{P_0 + \psi(z)}{P_0 + \psi(1)} \) which is equal to

\[
M(z) = \frac{[1-\alpha_0][1-C^*(\lambda)]-\rho_H}{(1-\rho_H)[[1-C^*(\lambda)]a(z)+C^*(\lambda)[q + p y^*(b(z))]\beta_1^*(\lambda_1(z))\beta_2^*(\lambda_2(z))-z]} \hspace{1cm} (8.4.2)
\]

More specifically, we may call it additional system size distribution due to retrial time.

The expected number of customers in the orbit during the retrial time i.e. mean of additional system size due to retrials is found to be

\[
L_M = M'(1) = \frac{\rho_H(1-C^*(\lambda))a_0}{[1-a_0][(1-C^*(\lambda))-\rho_H]}
\]

\[
+ \left[ \lambda a_0 \right]^2 \left[ \beta_1(1+\alpha_1 g_1)^2 + \beta_2(1+\alpha_2 g_2)^2 + \alpha_1 \beta_1 g_1(1+\alpha_2 g_2)^2 + \alpha_2 \beta_2 g_2(1+\alpha_1 g_1)^2 \right] \frac{[1-C^*(\lambda)]a_0}{2(1-\rho_H)[1-a_0][(1-C^*(\lambda))-\rho_H]}
\]

\[
+ \left[ \lambda a_0 \right]^2 \left[ p y^*(1+\alpha_1 g_1)^2 + 2y^*(1+\alpha_2 g_2)^2 \right] \left[ \beta_1(1+\alpha_1 g_1)^2 + \beta_2(1+\alpha_2 g_2)^2 \right] \frac{[1-C^*(\lambda)]a_0}{2(1-\rho_H)[1-a_0][(1-C^*(\lambda))-\rho_H]}
\]

\[
+ \left[ \lambda a_0 \right]^2 \left[ \beta_1(1+\alpha_1 g_1)^2 + \beta_2(1+\alpha_2 g_2)^2 \right] \frac{[1-C^*(\lambda)]a_0}{(1-\rho_H)[1-a_0][(1-C^*(\lambda))-\rho_H]}
\]
Remarks 8.4.1

From expression (8.4.1), we observe that the system size distribution at a departure epoch of the unreliable $M^X/G/1$ retrial queue with delaying repair and generalized repeated attempts under Bernoulli vacation schedule can be decomposed into distributions of two independent random variables viz.-

1. The system size distribution of an $M^X/G/1$ queue with unreliable server and delaying repair under Bernoulli vacation schedule [represented by first term of right hand side of expression (8.4.1)] and
2. The additional system size distribution due to retrials [represented by the second term of right hand side of expression (8.4.1)].

8.5 Distribution of number of customers in the system at departure epoch

In this section first we derive the PGF of the stationary system size distribution at a departure epoch. The key result of this section is now stated in Theorem 8.5.1.

Theorem 8.5.1

Under the steady state condition $a_{11}(1-C^*(\lambda))+\rho_H < 1$, the PGF of the steady state system size distribution at a departure epoch for this type of model is given by

$$
\pi(z) = \left[1 - a_{11}(1-C^*(\lambda)) - \rho_H \left[1 - a(z)\right]q + p\gamma^*(b(z))\right] \beta_1^*(\lambda_1(z))\beta_2^*(\lambda_2(z)) - z
$$

(8.5.1)
Proof: - Let \( \tau_n \) be the time of \( n \)-th total service completion epoch i.e. we consider the epochs at which the required service time requested by a unit expires. Then the sequence \( N_n = N(\tau_n+) \) form a Markov chain which is the embedded Markov renewal process of the continuous time Markov process \( Z(t) \) as \( t \to \infty \). Next we define the following limiting probabilities

\[
\pi_j = \lim_{n \to \infty} P_r\{N_n = j\} ; j \geq 0.
\]

Now since the arrival process is compound Poisson process, it follows from Burke’s theorem [e.g. Cooper (1981), pages 187-188 or section 1.5 of Chapter I] that the departure point probabilities \[ \pi_j \] exist and positive under the same condition

\[
a_0[1-C^*(\lambda)]+\rho < 1.
\]

Then following argument of PASTA [e.g. see section 1.5 of Chapter I] we note that a departing customer will see ‘\( j \)’ units in the system just after a departure if and only if there were \( (j+1) \) units in the system just before the departure and therefore we have

\[
\pi_j = A_0 \int_0^\infty \mu(x)P_{2,j+1}(x)dx + A_0 \int_0^\infty \gamma(x)Q_j(x)dx ; j \geq 0,
\]

where \( A_0 \) is the normalizing constant.

Let \( \pi(z) = \sum_{j=0}^{\infty} z^j \pi_j \) be the PGF of \( \{\pi_j ; j \geq 0\} \), and then multiplying equation (8.5.2) by both sides of \( z^j \) and then taking summation over \( j \geq 0 \) and utilizing equations (8.3.2.2) and (8.3.2.3), we get on simplification

\[
\pi(z) = \frac{B_0[1-a_0(1-C^*(\lambda))-\rho \beta(z)[q + p\gamma^*(b(z))\beta^*(\lambda_1(z))\beta^*_2(\lambda_2(z))]}{(C^*(\lambda)+a(z)[1-C^*(\lambda)])[q + p\gamma^*(b(z))\beta^*_1(\lambda_1(z))\beta^*_2(\lambda_2(z))]} - z
\]

Utilizing normalizing condition \( \pi(1) = 1 \), we get
\[ B_0 = \left[\lambda a_{\lfloor l \rfloor}\right]^{-1}. \] \tag{8.5.4}

Now inserting (8.5.4) in (8.5.3), we get required expression (8.5.1).

**Remark 8.5.1**

If we compare expression (8.3.233) with expression (8.5.1), then we have

\[
\pi(z) = \frac{1 - a(z)}{a_{\lfloor l \rfloor}(1 - z)} \Phi(z) = H(z)\Phi(z); \text{ as expected,} \tag{8.5.5}
\]

where \( H(z) = \frac{1 - a(z)}{a'(1)(1 - z)} \) is the PGF of the number of customers placed before an arbitrary test customer (tagged customer) in an admission batch in which the tagged customer arrives. This number is given as backward recurrence time in the discrete time renewal process where renewal points are generated by the arrival size random variable and \( a_{\lfloor l \rfloor} = a'(1). \)

This is the relationship between the system size distributions at a random point of time and at departure point of time for \( M^X/G/1 \) type of retrial queue with unreliable server. Now form above expression (8.5.5) it is observed that the PGF of system size distribution at a departure epoch of this type of model is the convolution of distributions of three independent random variables, one of which is the number of customers placed before a tagged customer in a batch in which the tagged customer arrives. This is due to randomness property of the arriving size of the batches. The interpretation of the other two random variables is provided in Remark 8.4.1.
Next, we are interested about the distribution for the number of customers in the system immediately after those time points at which a customer departs or an idle period is ended.

**Theorem 8.5.2**

Let \( \Psi(z) \) be the PGF of the number of customers in the system at immediately after those time points at which a customer departs or an idle period is terminated of an \( M^x / G/1 \) retrial queue with Bernoulli vacation schedule and unreliable server, then under steady state condition \( a_{[1]}(1-C^*(\lambda)) + \rho_H < 1 \) we have

\[
\Psi(z) = \frac{1 - a_{[1]}(1-C^*(\lambda)) - \rho_H \left[ g + p \gamma^*(b(z)) \right] \beta_1^*(\lambda_1(z)) \beta_2^*(\lambda_2(z)) - za(z)}{1 + a_{[1]} - \rho_H \left[ g + p \gamma^*(b(z)) \right] \gamma^*(\lambda) + a(z) \left[ 1 - C^*(\lambda) \right] \beta_1^*(\lambda_1(z)) \beta_2^*(\lambda_2(z)) - z} \]

(8.5.6)

**Proof:** The result follows directly by utilizing stochastic decomposition property for an \( M^x / G/1 \) retrial queue with Bernoulli vacation schedule and unreliable server viz-

[ e.g. see section 8.4 ]

\[
\Psi(z) = \Psi_0(z)M(z) \quad (8.5.7)
\]

where \( \Psi_0(z) \) is the PGF of the stationary queue size distribution at which a customer departs or at termination epoch of an idle period for an \( M^x / G/1 \) type of retrial queue with Bernoulli vacation schedule and unreliable server. This can be obtained easily by replacing the original service time distribution by our generalized service time
distribution or equivalently $B(z) = \left\{ q + py^* (b(z)) \right\} \beta_1^* (\lambda_1(z)) \beta_2^* (\lambda_2(z))$ i.e. the PGF of a batch of customers who arrived during our generalized service time, in the well-known result of Gross and Harris (1985) which is of the form [see section (5.1.9), page 237]

$$\Psi_0(z) = C_0 \frac{[B(z) - za(z)]}{[B(z) - z]} \quad (8.5.8)$$

where $C_0$ is a constant to be evaluated.

Now utilizing $B(z)$ in (8.5.8), we may write

$$\Psi_0(z) = C_0 \left[ q + py^* (b(z)) \right] \beta_1^* (\lambda_1(z)) \beta_2^* (\lambda_2(z) - za(z)) \quad (8.5.9)$$

Since $\Psi_0(1) = 1$, we get

$$C_0 = \frac{(1 - \rho_H)}{(1 + a_1 - \rho_H)} \quad (8.5.10)$$

Hence formula (8.5.6) follows by inserting (8.5.9), (8.5.10) and (8.4.2) in (8.5.7).

### 8.6 Particular Cases

In this section we discuss briefly some particular cases of our model, which are consistent with the existing literature. Suppose the retrial time distribution is exponential with d.f $C(x) = 1 - \exp{-\lambda x}; x > 0$, then $C^*(\lambda) = (\lambda + \nu)^{-1}$ and therefore equations (8.3.2.33) and (8.3.2.34) yield

$$\Phi(z) = \frac{[(\lambda + \nu)(1 - \rho_H) - \lambda a_1](1 - z)(p + qy^* (b(z))) \beta_1^* (\lambda_1(z)) \beta_2^* (\lambda_2(z))}{[\nu + \lambda a(z)](q + py^* (a(z))) \beta_1^* (\lambda_1(z)) \beta_2^* (\lambda_2(z)) - z(\lambda + \nu)} \quad (8.6.1)$$

$$\chi(z) = \frac{[(\lambda + \nu)(1 - \rho_H) - \lambda a_1](1 - z)}{[\nu + \lambda a(z)](q + py^* (a(z))) \beta_1^* (\lambda_1(z)) \beta_2^* (\lambda_2(z)) - z(\lambda + \nu)} \quad (8.6.2)$$
and \( \pi(z) = \frac{[\lambda + \nu(1 - \rho_H) - \lambda a(z)][1 - a(z)](p + q \gamma^* (b(z))) \beta_2^*(\lambda_1(z)) \beta_2^*(\lambda_2(z))}{a(z)[q + p \gamma^* (a(z))] \beta_1^*(\lambda_1(z)) \beta_2^*(\lambda_2(z)) - z(\lambda + \nu)} \)  

(8.6.3)

These are the PGFs of the orbit size and system size distributions of the unreliable \( M^X / G / 1 \) retrial queue with two phases of service and constant repeated attempts under Bernoulli vacation schedule. Note that for \( p = 0 \) (i.e., there is no Bernoulli vacation in the system), \( \alpha_i = 0 \) (i.e., there is no breakdown in the system) for \( i = 1, 2 \) and \( a(z) = z \) with \( \beta_2^*(\lambda_2(z)) = 1 \), the above expression (8.6.1) is consistent with expression (2.13) of Choi et al. [1993(b)]

Suppose that there is no retrial time in the system, then with \( C^*(\lambda) = 1 \), then equation (8.5.1) yields

\[
\pi(z) = \frac{(1 - \rho_H)(1 - a(z))[q + p \gamma^* (b(z))] \beta_1^*(A_1(z)) \beta_2^*(A_2(z))}{a(z)[q + p \gamma^* (b(z))] \beta_1^*(A_1(z)) \beta_2^*(A_2(z)) - z}
\]

which is the PGF of the system size distribution of an \( M^X / G / 1 \) unreliable queue with two phases of service and is consistent with expression (3.4.4) of our previous Chapter-III.

Again if we take \( z = \gamma^*(b(z)) \) in the above expressions (8.6.1) - (8.6.3), then we get on simplifications

\[
\Phi(z) = \frac{[q - a(z)(1 - C^*(\lambda)) - \rho_H][1 - z](p + qz) \beta_1^*(\lambda_1(z)) \beta_2^*(\lambda_2(z))}{[C^*(\lambda) + a(z)(1 - C^*(\lambda))][q + pz] \beta_1^*(\lambda_1(z)) \beta_2^*(\lambda_2(z)) - z}
\]

and

\[
\chi(z) = \frac{[q - a(z)(1 - C^*(\lambda)) - \rho_H][1 - z]}{[C^*(\lambda) + a(z)(1 - C^*(\lambda))][q + pz] \beta_1^*(\lambda_1(z)) \beta_2^*(\lambda_2(z)) - z}
\]
and \[ \pi(z) = \frac{[q - a(z)(1 - C^*(\lambda))] - \rho_H \{1 - a(z)(p + qz)\beta^*_{1}(\lambda_1(z))\beta^*_{2}(\lambda_2(z))}{a(z)[C^*(\lambda) + a(z)(1 - C^*(\lambda))]\beta^*_{1}(\lambda_1(z))\beta^*_{2}(\lambda_2(z)) - z} \]

These are the PGFs of the number of customers in the system at a service completion epoch, the number of customers in the orbit and number of customers in the system at a departure epoch respectively for an \(M^X/G/1\) unreliable queue with two phases of service and Bernoulli feedback mechanism under generalized repeated attempts.

Further, if we take \(p = 0\) (i.e. there is no Bernoulli feedback mechanism in the system), then \(\rho_H = \lambda a(z)\left\{\beta^*(1 + \alpha_1 g_{1}^{(1)}) + \beta^*_2(1 + \alpha_2 g_{2}^{(1)})\right\} < 1\) and the expression (8.3.2.33), (8.3.2.34) and (8.5.1) yields

\[ \Phi(z) = \frac{[q - a(z)(1 - C^*(\lambda))] - \rho_H \{1 - z\beta^*_{1}(\lambda_1(z))\beta^*_{2}(\lambda_2(z))}{[C^*(\lambda) + a(z)(1 - C^*(\lambda))]\beta^*_{1}(\lambda_1(z))\beta^*_{2}(\lambda_2(z)) - z} \]

\[ \chi(z) = \frac{[q - a(z)(1 - C^*(\lambda))] - \rho_H \{1 - z\}}{[C^*(\lambda) + a(z)(1 - C^*(\lambda))]\beta^*_{1}(\lambda_1(z))\beta^*_{2}(\lambda_2(z)) - z} \]

and \[ \pi(z) = \frac{[q - a(z)(1 - C^*(\lambda))] - \rho_H \{1 - a(z)\beta^*_{1}(\lambda_1(z))\beta^*_{2}(\lambda_2(z))}{a(z)[C^*(\lambda) + a(z)(1 - C^*(\lambda))]\beta^*_{1}(\lambda_1(z))\beta^*_{2}(\lambda_2(z)) - z} \]

These are the corresponding PGFs of the number of customers in the system at a random epoch, the number of customers in the orbit and number of customers in the system at a departure epoch respectively for an \(M^X/G/1\) unreliable queue with two phases of service and generalized repeated attempts. Note that for \(\beta^*_2(\lambda_2(z)) = 1\, , \alpha_1 = 0\) and \(a(z) = z\) the above expressions are consistent with the result obtained by Gomez-Corral (1999).

**8.7 System Performance Measures**
Our next object is to provide explicit expressions for the system state probabilities and some important performance measures of the system. First of all, we derive the system state probabilities and results are summarized in the following theorem.

**Theorem 8.7.1**

If the system is in steady state conditions, then we have

(i) the probability that the server is idle and system is empty is

\[ P_E = \frac{[1 - a_{[i]} (1 - C^* (\lambda)) - \rho_H]}{C^* (\lambda)} \]

(ii) the probability that the server is idle but system is nonempty is

\[ P_{NE} = \frac{[1 - a_{[i]} (1 - C^* (\lambda)) - \rho_H]}{\rho_H} \]

(iii) the probability that the server is busy with FPS is

\[ P_{B_1} = \lambda a_{[i]} \beta_1^{(i)} \]

(iv) the probability that the server is busy with SPS is

\[ P_{B_2} = \lambda a_{[i]} \beta_2^{(i)} \]

(v) the probability that the server is under repair during FPS is

\[ P_{R_1} = \alpha_1 \lambda a_{[i]} \beta_1^{(i)} g_1^{(i)} \]

(vi) the probability that the server is under repair during SPS is

\[ P_{R_2} = \alpha_2 \lambda a_{[i]} \beta_2^{(i)} g_2^{(i)} \]

(vii) the probability that the server is on vacation is

\[ P_v = p \lambda a_{[i]} \gamma^{(i)} \]

**Proof:** Note that \( P_{NE} = \lim_{z \to 1} \psi (z) \), \( P_v = \lim_{z \to 1} Q (z) \) and for \( i = 1, 2 \) \( P_{B_i} = \lim_{z \to 1} P_{z} (z), \)

\[ P_{R_i} = \lim_{z \to 1} R_{i} (z) \] and \( P_E = 1 - \left\{ P_{NE} + P_v + P_{B_1} + P_{B_2} + P_{R_1} + P_{R_2} \right\} \). The stated formula follow by direct calculation.
Secondly, we derive the mean orbit size and the mean system size of this model under stability conditions.

**Theorem 8.7.2**

Let $L_0$, $L_s$ and $L_D$ be the expected number of units in the orbit, system and at departure point of time respectively, then under the steady state condition we have

$$L_0 = \frac{\rho_H \left(1 - C^* (\lambda) \right) a[i]}{1 - a[i] \left(1 - C^* (\lambda) \right) - \rho_H} + \frac{p \left[ \lambda a[i] \beta^{(2)} + 2 \gamma^{(2)} \beta^{(2)} (1 + \alpha_i g^{(1)}) + \beta^{(2)} (1 + \alpha_i g^{(2)} + \alpha_2 g^{(2)}) \right] - C^* (\lambda)}{2 \left[1 - a[i] \left(1 - C^* (\lambda) \right) - \rho_H \right]}$$

$$+ \frac{\lambda a[i] \beta^{(2)} (1 + \alpha_i g^{(2)}) + \beta^{(2)} (1 + \alpha_i g^{(2)}) \beta^{(2)} (1 + \alpha_i g^{(2)}) + \alpha_i \beta^{(2)} (1 + \alpha_i g^{(2)}) + \alpha_2 \beta^{(2)} g^{(2)} \beta^{(2)} (1 - C^* (\lambda))}{2 \left[1 - a[i] \left(1 - C^* (\lambda) \right) - \rho_H \right]}$$

$$+ \frac{a[i] \left(1 - C^* (\lambda) \right) + \rho_H}{1 - a[i] \left(1 - C^* (\lambda) \right) - \rho_H} a[i]$$

(8.7.1)

$$L_s = L_0 + \rho_H$$

and $L_D = L_s + a[i]$;

where $a[i] = \frac{a[i]}{2a[i]}$ is the mean residual batch size.

**Proof:** The results follow directly by differentiating (8.3.2.34), (8.3.2.33) and (8.5.1) with respect to $z$ and then taking limit $z \to 1$ by using the L' Hospital's rule.

**Remark 8.7.1:** Once we have obtained the expected number of units in the orbit, it is trivial to obtain the other related performance measure viz.- mean waiting time. The
mean waiting time $W$ in the steady state can be easily obtained with the help of little’s formula viz. $L_0 = \lambda a_{[1]} W$.

Next we derive the mean busy period and expected length of a busy cycle under the steady state condition.

**Theorem 8.7.3**

Let $T_b$ and $T_c$ be the length of a busy period and length of a busy cycle, respectively, then under the steady state conditions, we have

$$E(T_b) = \frac{\beta_1 (1 + \alpha_1 g_1^{(0)}) + \beta_2 (1 + \alpha_2 g_2^{(0)})}{1 - a_{[1]} (1 - C^*(\lambda)) - \rho_H} + \frac{p_H^{(i)}}{1 - a_{[1]} (1 - C^*(\lambda)) - \rho_H}$$

$$+ \frac{(1 - C^*(\lambda)) (a_{[1]} - 1)}{\lambda a_{[1]} [1 - a_{[1]} (1 - C^*(\lambda)) - \rho_H]}$$

(8.7.2)

and $E(T_c) = \frac{C^*(\lambda)}{\lambda a_{[1]} [1 - a_{[1]} (1 - C^*(\lambda)) - \rho_H]}$.  

(8.7.3)

**Proof:** The results follow directly by applying the argument of alternating renewal process, which lead to the well known result

$$E(T_b) = \frac{1}{\lambda a_{[1]} \left( \frac{1}{P_0} - 1 \right)}$$

(8.7.4)

$$E(T_c) = \left[ \lambda a_{[1]} P_0 \right]^{-1}$$

(8.7.5)

Inserting (8.3.2.18) in (8.7.4) and (8.7.5), we get (8.7.2) and (8.7.3), respectively.

Finally, we will consider two reliability indices of the system viz.- the system availability and failure frequency under the steady state conditions. Let $A_v(t)$ be the system availability at time $t$, that is, the probability that the server is either working for
Theorem 8.7.4

The availability of the server and failure frequency of the server under the steady state condition are respectively given by

\[
A_y = \frac{1 - a_0 \{1 - C^*(\lambda)(1 + \lambda \beta_1^{(i)} + \lambda \beta_2^{(i)}) + \lambda (\alpha g_1^{(i)} \beta_1^{(i)} + \alpha \beta_2^{(i)} \beta_2^{(i)} + P_{f(i)}) \}}{C^*(\lambda)}, \tag{8.7.6}
\]

and

\[
M_f = \alpha_1 \lambda a_0 \beta_1^{(i)} + \alpha_2 \lambda a_0 \beta_2^{(i)} \tag{8.7.7}
\]

Proof:- The result follows directly by considering the following equations

\[
A_y = P_0 + \int_0^\infty P_1(x;1)dx + \int_0^\infty P_2(x,1)dx
\]

\[
M_f = \alpha_1 \int_0^\infty P_1(x;1)dx + \alpha_2 \int_0^\infty P_2(x,1)dx
\]

Now since \( \int_0^\infty [1 - B_i(x)]dx = \int_0^\infty xdB_i(x) = \beta_i^{(i)} \) for \( i = 1,2 \); therefore from equations (8.3.2.16), we get (8.7.6) and (8.7.7) respectively.

8.8 A real world application with the optimum control policy

In this section, we present a possible application and some numerical examples in some situations to explain that our model can represent the possible application reasonably well. In wireless network, access point is used to connect between wireless
and wired networks. Wireless access point provides services essentially in two phases viz. FPS (first phase of service) followed by SPS (second phase of service) parallel to a WLAN. It serves as an interface point and bridges between wireless devices and wired network, so the wireless devices can access the resources of the wired network. It further provides wireless linkage between wireless devices that may be out of range of each other. Typically, wireless device's connection requests arrive at the access point following the Poisson stream. When requests arrive at the access point, one request is selected to be served and the rest of the requests will join the buffer and retry again after a random period. In the buffer, each message waits a certain amount of time and requires the service again. There is a daemon program implemented at access point to manage the connection requests from buffer. Each time it tries but fails, it will wait another amount of time and then try again. At the completion of a service, access point may perform one of maintenance activities such as virus scan to keep the access point functioning well or serve the next connection request. In addition, in practice the access point may fail and can be repaired. The repair time depends on the degree of failure of access point. Because the system performance may be heavily affected by access point breakdown, it is well worth to investigate such system from the queueing theory viewpoint to develop a proper management policy. In this scenario, access point, buffer in the access point, retransmission policy, and maintenance activities correspond to the server, the orbit, the retrial discipline, and the vacation policy, respectively, in the queueing terminology.

To determine a proper management policy of such a system, we now consider the problem of determining an optimal control policy for the retrial system with an unreliable server. $C^*(\lambda)$, being the probability that no units arrive during the retrial
times, will be taken as our decision variable. Denoting the total expected cost per unit of time by $TC(C^*(\lambda))$, we have,

$$TC(C^*(\lambda)) = c_n L_n + c_0 \frac{E(T_b)}{E(T_{bc})} + c_s \frac{1}{E(T_c)} + c_a \frac{E(T_0)}{E(T_c)}$$

where $c_n$ be the holding cost per unit time for each customer present in the system, $c_0$ be the cost per unit time for keeping the server on and in operation, $c_s$ be the setup cost per busy cycle, and $c_a$ be the startup cost per unit time for the preparatory work of the server before staring the service. Utilizing (8.7.2) and (8.7.3), we have

$$TC(C^*(\lambda)) = c_n \left[ \rho + c \frac{[1 - C^*(\lambda)]}{[1 - a_{11} (1 - C^*(\lambda)) - \rho_H]} \right] + c_0 +$$

$$\left[ c_n + \lambda a_{11} c_s - c_0 \right] \frac{[1 - a_{11} (1 - C^*(\lambda)) - \rho_H]}{C^*(\lambda)}$$

where $a = a_{11} \rho_H + \frac{a_{11}^2}{2} \left[ \beta_1^{(2)} (1 + \alpha_1 g_1^{(0)})^2 + \beta_2^{(2)} (1 + \alpha_2 g_2^{(0)})^2 + \alpha_1 \beta_1^{(0)} g_1^{(0)} + \alpha_2 \beta_2^{(0)} g_2^{(0)} \right]$

$$+ \frac{a_{11}^2}{2} \left[ \beta_1^{(2)} (1 + \alpha_1 g_1^{(0)})^2 + \beta_2^{(2)} (1 + \alpha_2 g_2^{(0)})^2 + \alpha_1 \beta_1^{(0)} g_1^{(0)} + \alpha_2 \beta_2^{(0)} g_2^{(0)} \right] +$$

$$+ \left[ \alpha_{ii} (1 + \alpha_1 g_1^{(0)}) (1 + \alpha_2 g_2^{(0)}) \right]$$

The optimal value of $C^*(\lambda)$, denoted by $C^*(\lambda)'$, is obtained by solving

$$\frac{dTC(C^*(\lambda))}{dC^*(\lambda)} = 0$$

which is given by

$$C^*(\lambda)' = -\frac{B + \sqrt{B^2 - 4AK}}{2A} = d^* \text{(Say)};$$
Where \( B = c_2 \{2a_{[2]}^2 - 2a_{[2]}(1 - \rho_H)\}, \quad A = c_1 - c_2 a_{[2]}^2, \quad c_1 = c_h \{a_{[2]} + 2c(1 - \rho_H)\} \)

\[c_2 = 2(c_s + \lambda a_{[2]} c_s - c_0)\left(\rho_H + a_{[1]} - 1\right), \quad K = c_2 \{2a_{[2]}(1 - \rho_H) - a_{[2]}^2 - (1 - 2\rho_H + \rho_H^2)\}\]

It can be verified that

\[
\frac{d^2 TC(C^* (\lambda))}{dC^* (\lambda)^2} = \frac{a_{[1]} \{a_{[2]} + 2c(1 - \rho_H)\}}{[1 - a_{[2]}(1 - C^*(\lambda)) - \rho_H]^5} + \frac{2(1 - a_{[1]} - \rho_H)(c_s + \lambda a_{[2]} c_s - c_0)}{(C^*(\lambda))^5} < 0
\]

8.9. Numerical Example

We now present some illustrative examples in this section. Note that the total expected cost per unit of time only requires first and second moments of the different distributions involved. For example let us assume that the service time, repair time and vacation time follow the exponential distribution the distribution of the sizes of successive arriving batches is assumed to be geometric with parameter \(a\), so that

\[a_{[1]} = \frac{1}{a} \quad \text{and} \quad a_{[2]} = \frac{(2 - a)}{a^2}.\]

To show how any parameter can affect the system performance, let us consider the following arbitrary values for the non-monetary system parameters: \(\lambda = 0.01, \quad \alpha_1 = 0.5, \quad \alpha_2 = 0.5, \quad \sigma = 0.5, \quad \beta^{(0)}_1 = 0.5, \quad \beta^{(0)}_2 = 0.4, \quad g^{(0)}_1 = \beta^{(0)}_1 / 5, \quad g^{(0)}_2 = \beta^{(0)}_2 / 5, \quad g^{(0)} = (\beta^{(0)}_1 + \beta^{(0)}_2) / 4,\) and the following values for the monetary system parameters:

\[c_h = 5, c_0 = 100, c_s = 1000, c_a = 100.\] The variations of the optimal cost per unit of time \((OC)\) are depicted in Figure 1 below
The optimal cost is \( OC = 50.53 \) and it is reached at the optimal value of \( d^* = .356648 \)

Next we conduct a sensitivity analysis on the system parameters involved to investigate how sensitive is the optimal policy to the system parameters. We first investigate the monetary system parameters. Keeping all parameters fixed, we vary each unit cost, one at a time, in Tables 1-4 below

Table 1: Effect of \( C_h \) on optimal policy

<table>
<thead>
<tr>
<th>( C_h )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( OC )</td>
<td>76.30</td>
<td>69.86</td>
<td>63.42</td>
<td>56.98</td>
<td>50.53</td>
<td>44.09</td>
<td>37.65</td>
<td>31.21</td>
<td>24.77</td>
<td>18.33</td>
</tr>
</tbody>
</table>

Table 2: Effect of \( C_0 \) on optimal policy

<table>
<thead>
<tr>
<th>( C_0 )</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>130</th>
<th>140</th>
<th>150</th>
<th>160</th>
<th>170</th>
</tr>
</thead>
<tbody>
<tr>
<td>( OC )</td>
<td>13.27</td>
<td>31.99</td>
<td>50.53</td>
<td>69.16</td>
<td>87.79</td>
<td>106.42</td>
<td>125.05</td>
<td>143.68</td>
<td>162.31</td>
<td>180.94</td>
</tr>
</tbody>
</table>

Table 3: Effect of \( C_a \) on optimal policy

<table>
<thead>
<tr>
<th>( C_a )</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>130</th>
<th>140</th>
</tr>
</thead>
<tbody>
<tr>
<td>( OC )</td>
<td>93.68</td>
<td>85.05</td>
<td>76.42</td>
<td>67.79</td>
<td>59.16</td>
<td>50.53</td>
<td>41.9</td>
<td>33.27</td>
<td>24.64</td>
<td>16</td>
</tr>
</tbody>
</table>

Table 4: Effect of \( C_s \) on optimal policy

<table>
<thead>
<tr>
<th>( C_s )</th>
<th>600</th>
<th>700</th>
<th>800</th>
<th>900</th>
<th>1000</th>
<th>1100</th>
<th>1200</th>
<th>1300</th>
<th>1400</th>
<th>1500</th>
</tr>
</thead>
<tbody>
<tr>
<td>( OC )</td>
<td>57.44</td>
<td>55.7</td>
<td>53.786</td>
<td>52.26</td>
<td>50.53</td>
<td>48.81</td>
<td>47.08</td>
<td>45.36</td>
<td>43.63</td>
<td>41.9</td>
</tr>
</tbody>
</table>
We observe that only the increase of $C_0$ causes the optimal cost to increase in Table 2. On the other hand in Table 1, Table 3 and Table 4 we see that $OC$ decreases as any of the other three unit costs increases.

Next, we investigate the effect of nonmonetary system parameters viz., Bernoulli vacation schedule $p$, breakdown rates $\alpha_i, (i = 1,2)$, arrival rate $\lambda$ and arriving batch size $a$ on the optimal cost ($OC$) in Tables 5-9. All the other parameters are kept unchanged. In Table 5 we vary the probability of a vacation from 0 to 1 and record the corresponding values of system optimal cost. We see that system optimum cost increases as $p$ increases.

The effects of breakdown rates $\alpha_i, (i = 1,2)$ are examined in Tables 6-7. We observe that as breakdown rates increase then optimum total cost increases.

Next we investigate the effect of arrival rate on the optimal cost. Table 8 below shows that higher the value of the arrival rate, the lower is the value of the optimal cost.

Again the effect of batch size on the optimal cost has been examined in Table 9. The Table reports that $OC$ first increases ($a \leq 0.48$) and then decreases with increasing batch size.

Table 5: Effect of the Bernoulli vacation on the optimal cost

<table>
<thead>
<tr>
<th>$p$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$OC$</td>
<td>50.51</td>
<td>50.52</td>
<td>50.52</td>
<td>50.53</td>
<td>50.53</td>
<td>50.53</td>
<td>50.54</td>
<td>50.54</td>
<td>50.54</td>
<td>50.54</td>
<td>50.55</td>
</tr>
</tbody>
</table>

Table 6: Effect of the FPS breakdown rate $\alpha_1$ on the optimal cost

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$OC$</td>
<td>50.53</td>
<td>50.53</td>
<td>50.53</td>
<td>50.53</td>
<td>50.53</td>
<td>50.53</td>
<td>50.54</td>
<td>50.54</td>
<td>50.54</td>
</tr>
</tbody>
</table>

Table 7: Effect of the SPS breakdown rate $\alpha_2$ on the optimal cost

<table>
<thead>
<tr>
<th>$\alpha_2$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$OC$</td>
<td>50.53</td>
<td>50.53</td>
<td>50.53</td>
<td>50.53</td>
<td>50.53</td>
<td>50.53</td>
<td>50.53</td>
<td>50.54</td>
<td>50.54</td>
</tr>
</tbody>
</table>
Table 8: Effect of the arrival rate on the optimal cost

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.006</th>
<th>0.007</th>
<th>0.008</th>
<th>0.009</th>
<th>0.01</th>
<th>0.011</th>
<th>0.012</th>
<th>0.013</th>
<th>0.014</th>
</tr>
</thead>
<tbody>
<tr>
<td>OC</td>
<td>57.16</td>
<td>55.54</td>
<td>53.90</td>
<td>52.23</td>
<td>50.53</td>
<td>48.81</td>
<td>47.07</td>
<td>45.29</td>
<td>43.50</td>
</tr>
</tbody>
</table>

Table 9: Effect of arriving batch sizes on the optimal cost

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0.4</th>
<th>0.42</th>
<th>0.44</th>
<th>0.46</th>
<th>0.48</th>
<th>0.5</th>
<th>0.52</th>
<th>0.54</th>
<th>0.56</th>
<th>0.58</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>OC</td>
<td>29.55</td>
<td>36.31</td>
<td>41.68</td>
<td>45.81</td>
<td>48.77</td>
<td>50.53</td>
<td>50.98</td>
<td>49.78</td>
<td>46.28</td>
<td>38.97</td>
<td>24.05</td>
</tr>
</tbody>
</table>

Next, a sensitivity analysis of some of the parameters on the system can be performed with the help of graphs. For the effect of parameters $p, \alpha_1, \alpha_2, \lambda$ and $\alpha$ on the optimal policy, two dimensional graphs are drawn in Figures 2-6 such that the stability condition is satisfied.

Keeping fixed the base values given above, one parameter can be varied at a time and the corresponding objective function value can be computed. The graphs below show the effect of some of the system parameters on the optimal expected cost. Figure 2 shows the variations of the optimal expected cost. To investigate the effect of Bernoulli vacation schedule on the system performance we vary the probability of a vacation from 0 to 1 and record the corresponding value of the optimal expected cost ($OC$). We observe that the higher the probability of a vacation, the higher the system total cost.

Figure 3 and Figure 4 below show that $OC$ increases as mean failure rates increase.
Next, we want to investigate the effect of the arrival rate on the system performance. Keeping the values of the system parameters as above, we vary the rate of arrival from 0.006 to 0.014 and again record the corresponding values of the system total expected cost. Figure 5 reports that $OC$ decrease as $\lambda$ increases.

Again to study the effect of batch size on the system performance, we give different values to the batch size and record the corresponding value of the system optimal cost ($OC$). Figure 6 below shows the variations of the optimal cost. $OC$ first increases ($a \leq 0.5$) and then decreases with increasing batch size $a$. 
8.10 Concluding Remark

We have studied $M/G/1$ type of unreliable queuing model as a batch generalization of the previous chapter VII with the following features: each customer requires two successive phases of service, a customer who finds the server busy joins an orbit and retries for service after a random time generally distributed. We have obtained the following results: the condition under which steady state is reached, the system size distribution at a departure epoch, the probability generating function of the joint distributions of the server state and orbit size, the reliability indices of the system.

This study can be complemented in various ways. For example, it may be worth generalizing the arrival process to the case of a Markov Arrival Process (MAP), batch Markov Arrival Process (BMAP).