3.1 Introduction

Recently, there have been several contributions considering queueing models with service breakdowns or some other kind of interruptions. Among some early papers in this area, we refer the readers to the papers by White and Christie (1958), Heathcote (1959), Keilson (1962), Gaver (1962), Avi-Itzhak and Naor (1963), Thirurengadan (1963) and Mitrany and Avi-Itzhak (1968) for some fundamental works. In most of the previous studies, it is assumed that once service channel fails, it instantaneously undergoes repairs. However, in many real-life situations, it may not be feasible to start the repairs immediately due to the non-availability of the repairman or of the apparatus needed for the repair, in which case the system may also be turned off and there is delay in repair during which time the server stops providing service to the customers. As related works, we should mention the paper of Chao (1995) which dealt with queueing systems in which a catastrophe or disaster that can be viewed as general breakdown removes all the work present in the system. A disaster can be viewed as a general breakdown which causes all the jobs in the system to be lost. A typical example

* Some parts of this chapter have been published in “International Journal of Operations Research”, (2013), 10(3), 134-152. [See Ref. 46 of Bibliography]
is the distributed database system with site failure considered by Towsley and Tripathi (1991).

The classical vacation scheme with Bernoulli service discipline was initiated and developed significantly by Keilson and Servi (1986) and co-workers, where they incorporated the concept of modified service time. Recently, there have been several contributions considering $M/G/1$ type queueing systems with two phases of service under Bernoulli vacation schedule and different vacation policies by Choudhury and Madan [2004, 2005], Choudhury and Paul (2006) and Choudhury et.al (2007) among others. In most of the previous studies, the server is assumed to be available in the service station on a permanent basis and the service station never fails. But in practice, we often experience cases where service stations fail and are repaired. Similar phenomena always occur in the area of computer communication networks, flexible manufacturing systems, etc. Because the performance of such systems may be heavily affected by the service station breakdown and delay in repair, these systems with a repairable service station are well worth investigating from the queueing theory viewpoint, as well as from the reliability point of view. Hence, Li et al. [1997(a)] considered the reliability analysis for an unreliable server under Bernoulli vacation schedule.

A wide class of vacation policies for governing the vacation mechanism has also been discussed in the most recent survey by Ke et. al [2010(a)]. In this context, recently, Choudhury and Deka (2012) investigate an $M/G/1$ unreliable server Bernoulli vacation queue with two phases of service system. However, in this present Chapter, our purpose is to generalized such a type of $M/G/1$ unreliable server queue with batch arrivals for two phases of service system under Bernoulli vacation schedule,
where concepts of delay time is also introduced. These types of phenomena usually occur in the area of computer communication networks and flexible manufacturing systems. To this end, the methodology used will be based on the inclusion of supplementary variables.

The following results have been obtained under the present study of this chapter-

(i) The stationary queue size distribution
(ii) The queue size distribution at departure epoch
(iii) Some particular cases
(iv) The busy period distribution
(v) The waiting time distribution
(vi) The Reliability analysis
(vii) The numerical illustration

3.2 The model description:

We consider an $M^x/G/1$ queueing system, where the number of individual primary customers arrive to the system according to a compound Poisson process with arrival rate $\lambda$. The sizes of successive arriving batches are i.i.d random variables $X_1, X_2, \ldots$, distributed with probability mass function $a_n = \text{Prob} \{X = n\}; \ n \geq 1$, probability generating function (PGF) $a(z) = \mathbb{E}[z^X]$, and finite factorial moments $a_{[k]} = \mathbb{E}[X(X-1)\ldots (X-k+1)]$.

The server provides to each unit two phases of heterogeneous service in succession, the first phase service (FPS) followed by the second phase service (SPS). The service discipline is assumed to be FCFS. Further, it is assumed that the service time $B_i$ of the $i^{th}$ phase service follows a general probability law with distribution
function \( d.f \) \( B_i(x), i = 1,2 \), Laplace Stieltjes Transform (LST) \( \beta_i^*(\theta) = E[e^{-\theta x}], \) and finite \( k \)-th moments \( \beta_i^{(k)}, i = 1,2 \), where sub-index \( i = 1 \) (respectively \( i = 2 \)) denotes the FPS (respectively SPS). As soon as the SPS of a unit is completed the server may go for a vacation of random length \( V \) with probability \( p (0 \leq p \leq 1) \) or may continue to serve the next unit, if any, with probability \( q = (1 - p) \). Otherwise, it remains in the system.

Next, we assume that the vacation time \( V \) follows a general probability law with d.f. \( V(y), LST \gamma^*(\theta) \) and finite moments \( \gamma^{(k)} \), which is independent of the service time random variables and the arrival process. Further, it is also assumed that if, after returning from a vacation, the server does not find any units in the system, even then it joins the system without taking any further vacations and this policy is termed as single vacation with Bernoulli schedule (BS). While the server is working with any phase of service, it may breakdown at any time and the service channel will fail for a short interval of time (Breakdown periods). The breakdowns i.e., server’s life times, are generated by an exogenous Poisson process with rates \( \alpha_1 \) for FPS and \( \alpha_2 \) for SPS, which we may call some sort of disaster during FPS and SPS periods, respectively. As soon as a breakdown occurs, the server is sent for repair during which time it stops providing service to the customers waiting in the queue till the service channel is repaired. The customer being served just before server breakdown waits for repair to start, which we may refer to as waiting period of the server. We define this waiting time as delay time. The delay time \( D_i \) of the server for the \( i \)-th phase of service follows a general law of probability with d.f \( D_i(y), LST \gamma_i^*(\theta) = E[e^{-\theta x}], \) and \( k \)-th finite moments \( \gamma_i^{(k)} \); for \( i = 1,2 \). The repair time (denoted by \( R_i \) for FPS and \( R_2 \) for SPS) distributions of the server for both phases of service are assumed to be arbitrarily
distributed with d.f \( G_1(y) \) and \( G_2(y) \), LST \( G'_1(\theta) = E[e^{-\theta G_1}] \) and \( G'_2(\theta) = E[e^{-\theta G_2}] \), and finite \( k \)-th moments \( g_1^{(k)} \) and \( g_2^{(k)} \), respectively. Immediately after the broken server is repaired, it is ready to start its remaining service to customers in either phase of service. In this case, the service times are cumulative, i.e., we consider a preemptive-resume policy for service time, which may be referred to as generalized service times. Further we assume that input process, server's lifetime, server's repair time, server's delay time, service time and vacation time random variables are mutually independent of each others.

Thus the time required by a unit to complete the service cycle, which may be called as modified service time is given by,

\[
B = \begin{cases} 
B_1 + B_2 + V, & \text{with probability } p \\
B_1 + B_2, & \text{with probability } q = (1 - p) 
\end{cases}
\]

By \( H_i \) we denote the generalized service time for \( i \)-th phase of service and \( H'_i(\theta) = E[e^{-\theta H_i}] \) as its LST then

\[
H'_i(\theta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-\theta x} e^{-\alpha_i x} \left[ \frac{\alpha_i x^n}{n!} \right] \left[ \gamma'_i(\theta) G'_i(\theta) \right]^n dB_i(x) \\
= B_i \left[ \frac{\theta + \alpha_i (1 - \gamma'_i(\theta) G'_i(\theta))}{(1 - \gamma'_i(\theta) G'_i(\theta))} \right] \text{ for } i = 1, 2
\]

Hence the first two moments are found to be

\[
h_i^{(0)} = \frac{dH'_i(\theta)}{d\theta} \bigg|_{\theta=0} = \beta_i^{(0)} \left[ 1 + \alpha_i \left( \gamma_i^{(1)} + g_i^{(1)} \right) \right]
\]

and \( h_i^{(2)} = (-1)^2 \frac{d^2H'_i(\theta)}{d\theta^2} \bigg|_{\theta=0} = \beta_i^{(2)} \left[ 1 + \alpha_i \left( \gamma_i^{(1)} + g_i^{(1)} \right) \right]^2 + \alpha_i \beta_i^{(1)} \left( \gamma_i^{(2)} + g_i^{(2)} + 2\gamma_i^{(1)} g_i^{(1)} \right) \]

Where \( h_i^{(k)} \) is the \( k \)-th moment of the \( i \)-th phase of generalized service time distribution.
3.3 Stationary queue size distribution:

In this section, we first setup the system state equations for its stationary queue size, by treating elapsed vacation time, elapsed service time, elapsed repair time, and elapsed delay time of the server, for both phases of service, as supplementary variables. Then we solve the equations and derive the PGFs of the stationary queue size distribution. Assume that the system is in steady-state conditions. Let $N(t)$ be the queue size (including one being served, if any) at time $t$, $V^0(t)$ be the elapsed vacation time of the server, and $B_i^0(t)$ be the elapsed service time of the customer for the $i$-th phase of service at time $t$, with $i = 1, 2$ denoting FPS and SPS respectively. In addition, let $R_i^0(t)$ and $D_i^0(t)$ be the elapsed repair time and elapsed delay time of the server for $i$-th phase of service during which breakdown occurs in the system at time $t$, where sub-index $i = 1$ (respectively $i = 2$) denotes FPS (respectively SPS). Further, we introduce the following random variable:

$$Y(t) = \begin{cases} 
0, & \text{if the server is idle at time } t. \\
1, & \text{if the server is busy with FPS at time } t. \\
2, & \text{if the server is busy with SPS at time } t. \\
3, & \text{if the server is on vacation at time } t. \\
4, & \text{if the server is waiting for repair during FPS at time } t. \\
5, & \text{if the server is waiting for repair during SPS at time } t. \\
6, & \text{if the server is under repair during FPS at time } t. \\
7, & \text{if the server is under repair during SPS at time } t.
\end{cases}$$

The supplementary variables $V^0(t)$, $B_i^0(t)$, $D_i^0(t)$ and $R_i^0(t)$ for $i = 1, 2$ are introduced in order to obtain a bivariate Markov process $\{N(t), X(t)\}$, where $X(t) = 0$ if $Y(t) = 0$, $X(t) = B_i^0(t)$ if $Y(t) = 1$, $X(t) = B_j^0(t)$ if $Y(t) = 2$, $X(t) = V^0(t)$ if
$Y(t) = 3$, $X(t) = D_1^0(t)$ if $Y(t) = 4$, $X(t) = D_2^0(t)$ if $Y(t) = 5$, $X(t) = R_1^0(t)$ if $Y(t) = 6$, and $X(t) = R_2^0(t)$ if $Y(t) = 7$. Next we define following limiting probabilities

\[ U_0(t) = P\{N(t) = 0, X(t) = 0\} \]

\[ Q_n(y,t) = P\{N(t) = n, X(t) = V^0(t); y < V^0(t) \leq y + dy\}; y > 0, n \geq 0 \]

and for $i = 1,2$ and $n \geq 0$

\[ P_{i,n}(x,t)dx = P_r\{N(t) = n, X(t) = B_i^0(t); x < B_i^0(t) \leq x + dx\}; x > 0 \]

\[ S_{i,n}(x,y)dy = P_r\{N(t) = n, X(t) = D_i^0(t); y < D_i^0(t) \leq y + dy; B_i^0(t) = x\}; (x,y) > 0 \]

\[ R_{i,n}(x,y,t)dy = P_r\{N(t) = n, X(t) = R_i^0(t); y < R_i^0(t) \leq y + dy; B_i^0(t) = x\}; (x,y) > 0 \]

Now, the analysis of the limiting behaviour of this queueing process at a random epoch can be performed with the help of Kolmogorov forward equations provided limiting probabilities

\[ U_0 = \lim_{t \to \infty} U_0(t), \quad Q_n(y)dy = \lim_{t \to \infty} Q_n(y,t), n \geq 0 \]

for $i = 1,2$ and $n \geq 0$

\[ P_{i,n}(x)dx = \lim_{t \to \infty} P_{i,n}(x), \quad S_{i,n}(x,y)dy = \lim_{t \to \infty} S_{i,n}(x,y,t), \quad R_{i,n}(x,y)dy = \lim_{t \to \infty} R_{i,n}(x,y,t) \]

exist and positive under the condition that they are independent of the initial state. Further, it is assumed that

\[ V(0) = 0, V(\infty) = 1, B_i(0) = 0, B_i(\infty) = 1, D_i(0) = 0, D_i(\infty) = 1, G_i(0) = 0, G_i(\infty) = 1 \text{ for } i = 1,2, \text{ and } V(y) \text{ is continuous at } y = 0 \text{ for } i = 1,2; B_i(x) \text{ is continuous at } x = 0, \text{ and } D_i(y) \text{ and } G_i(y) \text{ are continuous at } y = 0 \text{ for } i = 1,2 \]

respectively, so that
\[ \gamma(y)dy = \frac{dP(y)}{1-V(y)} \]

\[ \mu_i(x)dx = \frac{dB_i(x)}{1-B_i(x)}; \quad \eta_i(y)dy = \frac{dD_i(y)}{1-D_i(y)}; \quad \xi_i(y)dy = \frac{dG_i(y)}{1-G_i(y)} \quad \text{for } i=1,2 \]

are the first order differential (hazard rate) functions of \( V, B_i, D, \) and \( G_i \), respectively for \( i=1,2 \).

### 3.3.1 The steady-state equations

The Kolmogorov forward equations to govern the system under steady-state conditions [e.g. see Cox [1955(a)] or section 1.5 of Chapter I]

\[ \frac{d}{dx}P_{i,n}(x) + \left[ \lambda + \alpha_i + \mu_i(x) \right]P_{i,n}(x) = \lambda \sum_{k=1}^{n} a_k P_{i,n-k}(x) + \int_{0}^{\infty} \xi_i(y)R_{i,n}(x,y)dy; \quad n \geq 1 \]

(3.3.1.1)

\[ \frac{d}{dy}Q_{n}(y) + \left[ \lambda + \gamma(y) \right]Q_{n}(y) = \lambda \left( 1 - \delta_{0,n} \right) \sum_{k=1}^{n} a_k Q_{n-k}(y); \quad x > 0, n \geq 0 \]

(3.3.1.2)

\[ \frac{d}{dy}S_{i,n}(x,y) + \left[ \lambda + \eta_i(y) \right]S_{i,n}(x,y) = \lambda \sum_{k=1}^{n} a_k S_{i,n-k}(x,y); \quad n \geq 1 \]

(3.3.1.3)

\[ \frac{d}{dy}R_{i,n}(x,y) + \left[ \lambda + \xi_i(y) \right]R_{i,n}(x,y) = \lambda \sum_{k=1}^{n} a_k R_{i,n-k}(x,y); \quad n \geq 1 \]

(3.3.1.4)

\[ \lambda U_0 = \int_{0}^{\infty} \gamma(y)Q_{0}(y)dy + q \int_{0}^{\infty} \mu_2(x)P_{2,1}(x)dx \]

(3.3.1.5)

where \( \delta_{m,n} \) denotes Kronecker's function and \( P_{i,0}(x) = 0, \quad Q_{i,0}(x,y) = 0 \), and \( R_{i,0}(x,y) = 0 \) for \( i=1,2 \) occurring in equations (3.3.1.1) - (3.3.1.4). These sets of equations are to be solved under the boundary conditions at \( x=0 \):

\[ P_{i,n}(0) = \lambda a_{n}U_0 + q \int_{0}^{\infty} \mu_2(x)P_{2,n+1}(x)dx + \int_{0}^{\infty} \gamma(y)Q_{n}(y)dy; \quad n \geq 1 \]

(3.3.1.6)
\[ P_{2,n}(0) = \int_0^\infty \mu_1(x)P_{1,n}(x) \, dx \quad ; \quad n \geq 1 \quad (3.3.1.7) \]

and at \( y = 0 \):

\[ Q_n(0) = p \int_0^\infty \mu_2(x)P_{2,n+1}(x) \, dx \quad ; \quad n \geq 1 \quad (3.3.1.8) \]

also at \( y = 0 \) for \( i = 1,2 \) and fixed values of \( x \)

\[ S_{i,n}(x,0) = \alpha_i P_{i,n}(x) \quad ; \quad x > 0, n \geq 1 \quad (3.3.1.9) \]

\[ R_{i,n}(x,0) = \int_0^\infty \eta_i(y)S_{i,n}(x,y) \, dy \quad ; \quad x > 0, n \geq 0 \quad (3.3.1.10) \]

These equations are to be solved under the following normalizing condition

\[
U_0 + \sum_{n=0}^\infty \left[ \int_0^\infty Q_n(y) \, dy + 2 \left\{ \int_0^\infty P_{i,n}(x) \, dx + \int_0^\infty S_{i,n}(x,y) \, dy + \int_0^\infty R_{i,n}(x,y) \, dy \right\} \right] = 1
\]

(3.3.1.11)

3.3.2 The model solution

To solve the system of equations (3.3.1.1) - (3.3.1.10), let us introduce the following PGFs for \( |z| < 1 \) and \( i = 1,2 \):

\[ S_i(x,y;z) = \sum_{n=1}^\infty z^n S_{i,n}(x,y) \quad ; \quad S_i(x,0;z) = \sum_{n=1}^\infty z^n S_{i,n}(x;0) \]

\[ R_i(x,y;z) = \sum_{n=1}^\infty z^n R_{i,n}(x,y) \quad ; \quad R_i(x,0;z) = \sum_{n=1}^\infty z^n R_{i,n}(x;0) \]

\[ Q(y,z) = \sum_{n=1}^\infty z^n Q_n(y) \quad ; \quad Q(0,z) = \sum_{n=1}^\infty z^n Q_n(0) \]

\[ P_i(x,z) = \sum_{n=1}^\infty z^n P_{i,n}(x) \quad ; \quad P_i(0,z) = \sum_{n=1}^\infty z^n P_{i,n}(0) \]
Let \( b(z) = \lambda (1 - a(z)) \). It should be noted here that if \( a_n = \delta_{n,1}; n \geq 1 \), then \( a(z) = z \). Now proceeding in the usual manner with equations (3.3.1.2) - (3.3.1.4), we get a set of differential equations of Lagrangian type whose solutions are given by:

\[
Q(y; z) = Q(0; z)[1 - V(y)]\exp\{-b(z)y\}; \quad y > 0
\]  

(3.3.2.1)

\[
S_i(x, y; z) = S_i(x, 0; z)[1 - D_i(y)]\exp\{-b(z)y\}; \quad \text{for } i = 1, 2 \text{ and } (x, y) > 0
\]  

(3.3.2.2)

\[
R_i(x, y; z) = R_i(x, 0; z)[1 - G_i(y)]\exp\{-b(z)y\}; \quad \text{for } i = 1, 2 \text{ and } (x, y) > 0
\]  

(3.3.2.3)

where \( S_i(x, 0; z) \) and \( R_i(x, 0; z) \) for \( i = 1, 2 \) can be obtained from equations (3.3.1.9) and (3.3.1.10), which after simplification yield

\[
S_i(x, 0; z) = \alpha_i P_i(x; z)
\]  

(3.3.2.4)

\[
R_i(x, 0; z) = \alpha_i P_i(x, z)\gamma^* (b(z))
\]  

(3.3.2.5)

Now solving the differential equation (3.3.1.1), we get

\[
P_i(x; z) = P_i(0; z)[1 - B_i(x)]\exp\{- A_i(z)x\}; \quad x > 0 \quad \text{for } i = 1, 2
\]  

(3.3.2.6)

where \( A_i(z) = b(z) + \alpha_i [1 - \gamma^* (b(z))G^* (b(z))] \) for \( i = 1, 2 \).

Utilizing (3.3.2.4) - (3.3.2.6) in (3.3.2.2) we get for \( i = 1, 2 \)

\[
S_i(x, y; z) = \alpha_i P_i(0; z)[1 - B_i(x)]\exp\{- A_i(z)x\} [1 - D_i(y)]\exp\{-b(z)y\}
\]  

(3.3.2.7)

Utilizing (3.3.2.5) and (3.3.2.6) in (3.3.2.3) we get for \( i = 1, 2 \)

\[
R_i(x, y; z) = \alpha_i P_i(0; z)[1 - B_i(x)]\exp\{- A_i(z)x\} [1 - G_i(y)]\exp\{-b(z)y\}\gamma^* (b(z))
\]  

(3.3.2.8)

Multiplying equation (3.3.1.6) by \( z^n \) and then taking summation over all possible values of \( n \geq 1 \), we get on simplification

\[
zP_i(0, z) = qP_2(0; z)\beta'_2 (A_2(z)) + zQ(0; z)\beta^* (b(z)) - zU_0 b(z)
\]  

(3.3.2.9)

Similarly from equations (3.3.1.7) and (3.3.1.8), we have

\[
P_1(0, z) = P_1(0; z)\beta^* (A_1(z))
\]  

(3.3.2.10)
\[ zQ(0, z) = pP_z(0; z)\beta'_2(A_2(z)) \]  

(3.3.2.11)

Utilizing (3.3.2.10) and (3.3.2.11) in (3.3.2.9) and simplifying we get

\[ P_1(0, z) = \frac{zU_0b(z)}{g + p\hat{a}^*(b(z))\beta_1'(A_1(z))\beta'_2(A_2(z)) - z} \]  

(3.3.2.12)

Letting \( z \to 1 \) in (3.3.2.12), we obtain by L’Hospital’s rule

\[ P_1(0, 1) = \frac{\lambda a_{[1]}U_0}{(1 - \rho_H)} \]  

(3.3.2.13)

Where \( \rho_H = \rho_1\left[1 + \alpha_1(s_1^{(1)} + y_1^{(1)})\right] + \rho_2\left[1 + \alpha_2(s_2^{(1)} + y_2^{(1)})\right] + p\lambda a_{[1]}g^{(1)} \) is the utilizing factor of the system and \( \rho_i = \lambda a_{[1]}\beta_i^{(0)} \) for \( i = 1, 2 \).

This gives for \( i = 1, 2 \)

\[ P_i(x, 1) = \frac{\lambda a_{[1]}U_0[1 - B_i(x)]}{(1 - \rho_H)} \]  

(3.3.2.14)

\[ S_i(x, y, 1) = \frac{\alpha_i\lambda a_{[1]}U_0[1 - B_i(x)][1 - D_i(y)]}{(1 - \rho_H)} \]  

(3.3.2.15)

\[ R_i(x, y, 1) = \frac{\alpha_i\lambda a_{[1]}U_0[1 - B_i(x)][1 - G_i(y)]}{(1 - \rho_H)} \]  

(3.3.2.16)

And

\[ Q(y, 1) = \frac{p\lambda U_0a_{[1]}[1 - V(y)]}{(1 - \rho_H)} \]  

(3.3.2.17)

Now utilizing normalizing condition (3.3.1.11), we get

\[ U_0 = (1 - \rho_H) \]  

(3.3.2.18)

Note that equation (3.3.2.14) represents steady-state probability that the server is idle but available in the system. Also, from equation (3.3.2.14), we have \( \rho_H < 1 \), which is the necessary and sufficient condition under which steady-state solution exists. Thus, we summarize our results in the following Theorem 3.3.1.
Theorem 3.3.1

Under the stability condition $\rho_H < 1$, the joint distribution of the state of the server and the queue size has the following partial PGFs

$$P_1(x;z) = \frac{zb(z)(1 - \rho_H)(1 - B_1(x))\exp\{-(A_1(z))x\}}{\langle g + p\vartheta'(b(z))\rangle\beta_1^*(A_1(z))\beta_1^*(A_2(z)) - z} \quad (3.3.2.19)$$

$$P_2(x;z) = \frac{zb(z)(1 - \rho_H)\beta_1^*(A_1(z))(1 - B_2(x))\exp\{-(A_2(z))x\}}{\langle g + p\vartheta'(b(z))\rangle\beta_1^*(A_1(z))\beta_2^*(A_2(z)) - z} \quad (3.3.2.20)$$

$$Q(y;z) = \frac{pb(z)(1 - \rho_H)\beta_1^*(A_1(z))\beta_2^*(A_2(z))(1 - V(y))\exp\{-(b(z))y\}}{\langle g + p\vartheta'(b(z))\rangle\beta_1^*(A_1(z))\beta_2^*(A_2(z)) - z} \quad (3.3.2.21)$$

$$R_1(x,y;z) = \frac{\alpha_1 zb(z)(1 - \rho_H)(1 - B_1(x))\exp\{-(A_1(z))x\}\gamma_1^*(b(z))(1 - G_1(y))\exp\{-(b(z))y\}}{\langle g + p\vartheta'(b(z))\rangle\beta_1^*(A_1(z))\beta_2^*(A_2(z)) - z} \quad (3.3.2.22)$$

$$R_2(x,y;z) = \frac{\alpha_2 zb(z)(1 - \rho_H)(1 - B_2(x))\beta_1^*(A_1(z))\exp\{-(A_2(z))x\}\times[1 - G_2(y)]\gamma_2^*(b(z))\exp\{-(b(z))y\}}{\langle g + p\vartheta'(b(z))\rangle\beta_1^*(A_1(z))\beta_2^*(A_2(z)) - z} \quad (3.3.2.23)$$

$$S_1(x,y;z) = \frac{\alpha_1 zb(z)(1 - \rho_H)(1 - B_1(x))\exp\{-(A_1(z))x\}[1 - G_1(y)]\exp\{-(b(z))y\}}{\langle g + p\vartheta'(b(z))\rangle\beta_1^*(A_1(z))\beta_2^*(A_2(z)) - z} \quad (3.3.2.24)$$

$$S_2(x,y;z) = \frac{\alpha_2 zb(z)(1 - \rho_H)\beta_1^*(A_1(z))(1 - B_2(x))\exp\{-(A_2(z))x\}[1 - D_2(y)]\exp\{-(b(z))y\}}{\langle g + p\vartheta'(b(z))\rangle\beta_1^*(A_1(z))\beta_2^*(A_2(z)) - z} \quad (3.3.2.25)$$

where $A_i(z) = b(z) + \alpha_i(1 - G_i^*(b(z))\gamma_i^*(b(z)))$ for $i = 1,2$ and $b(z) = \lambda(1 - a(z))$. 
Remark 3.3.1

It is important to note here that such types of joint distributions are important to obtain the distribution of each state of the server in more comprehensive manner, which helps us to obtain marginal distributions of the server’s states as well as stationary queue size distribution at a departure epoch.

Theorem 3.3.2

Under the stability condition $\rho_H < 1$, the marginal PGFs of the server’s state queue size distribution are given by

$$P_1(z) = \frac{(1 - \rho) z b(z) [1 - \beta^*_1(A_1(z))]}{A_1(z) [g + p \theta^*(b(z))] \beta^*_1(A_1(z)) \beta^*_2(A_2(z)) - z}$$  (3.3.2.26)

$$P_2(z) = \frac{(1 - \rho) z b(z) \beta^*_1(A_1(z)) [1 - \beta^*_2(A_2(z))]}{A_2(z) [g + p \theta^*(b(z))] \beta^*_1(A_1(z)) \beta^*_2(A_2(z)) - z}$$  (3.3.2.27)

$$Q(z) = \frac{p (1 - \rho) \beta^*_1(A_1(z)) \beta^*_2(A_2(z)) [1 - \theta^*(b(z))]}{[g + p \theta^*(b(z))] \beta^*_1(A_1(z)) \beta^*_2(A_2(z)) - z}$$  (3.3.2.28)

$$R_1(z) = \frac{\alpha_1 (1 - \rho) z [1 - G_1^*(b(z))] [1 - \beta^*_1(A_1(z))]}{A_1(z) [g + p \theta^*(b(z))] \beta^*_1(A_1(z)) \beta^*_2(A_2(z)) - z}$$  (3.3.2.29)

$$R_2(z) = \frac{\alpha_2 (1 - \rho) z \beta^*_1(A_1(z)) \gamma_2^*(b(z)) [1 - \beta^*_2(A_2(z))] [1 - G_2^*(b(z))]}{A_2(z) [g + p \theta^*(b(z))] \beta^*_1(A_1(z)) \beta^*_2(A_2(z)) - z}$$  (3.3.2.30)

$$S_1(z) = \frac{\alpha_1 (1 - \rho) z [1 - \gamma^*_1(b(z))] [1 - \beta^*_1(A_1(z))]}{A_1(z) [g + p \theta^*(b(z))] \beta^*_1(A_1(z)) \beta^*_2(A_2(z)) - z}$$  (3.3.2.31)
\[
S_i(z) = \frac{\alpha_i (1 - \rho_i) z \beta_i' (A_i(z)) [1 - \beta_i' (A_i(z)) [1 - \lambda_i' (b(z))] A_i(z) [g + p \theta' (b(z)) \beta_i' (A_i(z)) \beta_i' (A_i(z)) - z]}
\]
(3.3.2.32)

**Proof:** Integrating (3.3.2.19) and (3.3.2.20) with respect to \(x\) and (3.3.2.21) with respect to \(y\) and then using the well known result of renewal theory

\[
\int_0^\infty e^{-sx} (1 - B_i(x)) dx = \frac{[1 - \beta_i' (s)]}{s} \quad \text{for} \quad i = 1, 2
\]

and

\[
\int_0^\infty e^{-sy} (1 - V(y)) dy = \frac{[1 - \theta' (s)]}{s};
\]

We get formulae (3.3.2.26) - (3.3.2.28). Similarly, integrating equations (3.3.2.22) - (3.3.2.25) with respect to \(y\), we get for \(i = 1, 2\)

\[
R_i(x, z) = \int_0^\infty R_i(x, y; z) dy = \alpha_i [b(z)]^{-1} \gamma_i' (b(z)) [1 - G_i' (b(z))] P_i (0; z) [1 - B_i(x)] \exp \{- A_i(z)x\}
\]
(3.3.2.33)

\[
S_i(x, z) = \int_0^\infty S_i(x, y; z) dy = \alpha_i [b(z)]^{-1} [1 - \gamma_i' (b(z))] P_i (0; z) [1 - B_i(x)] \exp \{- (A_i(z)x)\}
\]
(3.3.2.34)

Further integrating (3.3.2.33) and (3.3.2.34) with respect to \(x\), and utilizing (3.3.2.12) and (3.3.2.18), we claimed in formulae (3.3.2.29) - (3.3.2.32).

Next the system state probabilities are given in corollary 3.3.1

**Corollary 3.3.1**

If the system is in steady state conditions, then

(i) The probability that the system is idle is,

\[
P_1 = 1 - \rho_1 \left[ 1 + \alpha_1 \left( g_1^{(i)} + \gamma_1^{(i)} \right) \right] - \rho_2 \left[ 1 + \alpha_2 \left( g_2^{(i)} + \gamma_2^{(i)} \right) \right] - p \lambda a_0 (g^{(i)})
\]
(ii) The probability that the server is busy with FPS is $P_{bi} = \rho_1$

(iii) The probability that the server is busy with SPS is $P_{b2} = \rho_2$

(iv) The probability that the server is on vacation, $P_{b} = p\lambda a_t g^{(i)}$

(v) The probability that the server is waiting for repair during FPS is $P_{w1} = \alpha_1 \rho_1 \gamma_1^{(i)}$

(vi) The probability that the server is waiting for repair during SPS is $P_{w2} = \alpha_2 \rho_2 \gamma_2^{(i)}$

(vii) The probability that the server is under repair during FPS is $P_{r1} = \alpha_1 \rho_1 \gamma_1^{(i)}$

(viii) The probability that the server is under repair during SPS is $P_{r2} = \alpha_2 \rho_2 \gamma_2^{(i)}$

Proof:- Noting that $P_{\nu} = \lim_{z \to 1} Q(z)$, $P_{bi} = \lim_{z \to 1} P_t(z)$, $P_{b2} = \lim_{z \to 1} R_t(z)$, $P_{b} = \lim_{z \to 1} S_t(z)$

for $i = 1, 2$, and $P_I = 1 - \sum_{i=1}^{2} \{P_{bi} + P_{b2} + P_{b}\}$, the stated formulae follow by direct calculation.

Theorem 3.3.3

Let $\psi_j$ be the stationary distribution of the number of customers in the queue at a random epoch, then its corresponding PGF, i.e. $\psi(z) = \sum_{j=0}^{\infty} z^j \psi_j$ is given by

$$\psi(z) = \frac{(1 - \rho_H)(1 - z)}{[q + p g^*(b(z))]\beta_1^*(A_1(z))\beta_2^*(A_2(z))}.$$  (3.3.2.35)

Proof: - The result follows directly from Theorem 3.3.2 with the help of PGFs $Q(z)$, $P_t(z)$, $R_t(z)$ and $S_t(z)$ for $i = 1, 2$, since the distribution of the number of customers in the queue has the PGF

$$\psi(z) = \sum_{i=1}^{2} \{P_i(z) + R_i(z) + S_i(z)\} + zQ(z).$$
By direct calculation we obtain (3.3.2.35).

### 3.4 Queue size distribution at Departure epoch

To obtain the PGF of the queue size distribution at departure epoch, we follow the argument of PASTA [see Wolff (1982) or section 1.5 of Chapter I], we state that a departing customer will see 'j' customer in the queue just after a departure if and only if there were 'j' customer in the queue SPS or a vacation just before the departure. Now denoting \( \{ \pi_j : j \geq 0 \} \) as the probability that there are \( j \) units in the queue at a departure epoch, then for \( j \geq 0 \) we may write

\[
\pi_j = K_0 \left[ q \int_0^\infty \mu_x(x)P_{2,j+1}(x)dx + \int_0^\infty g(y)Q_j(y)dy \right] \quad (3.4.1)
\]

Where \( K_0 \) is the normalizing constant.

Now multiplying both sides of equation (3.3.2.37) by \( z^j \) and then taking summation over \( j \in \mathbb{Z}^+ \) and utilizing equations (3.3.2.10) and (3.3.2.12), we get on simplification

\[
\pi(z) = \frac{K_0 U_0 b(z) \{ q + p g^*(b(z)) \} \beta_1^*(A_1(z)) \beta_2^*(A_2(z))}{\{ q + p g^*(b(z)) \} \beta_1^*(A_1(z)) \beta_2^*(A_2(z)) - z} \quad (3.4.2)
\]

Utilizing normalizing condition \( \pi(1) = 1 \), we get

\[
K_0 = \frac{1 - \rho_H}{\lambda U_0 a_{[1]}} \quad (3.4.3)
\]

Hence from equations (3.4.2) and (3.4.3) we have,

\[
\pi(z) = \frac{(1 - \rho_H) (1 - a(z)) \{ q + p g^*(b(z)) \} \beta_1^*(A_1(z)) \beta_2^*(A_2(z))}{a_{[1]} \{ q + p g^*(b(z)) \} \beta_1^*(A_1(z)) \beta_2^*(A_2(z)) - z} \quad (3.4.4)
\]
Which is the PGF of the stationary queue size at a departure epoch of this $M^X/G/1$ queue with two phases of service and Bernoulli vacation schedule.

Next the mean queue size of this model is given in the corollary 3.4.1.

**Corollary 3.4.1**

Under the stability conditions, the mean number of customers in the system (i.e. mean queue length) $L_s$ is given by

$$L_s = \rho_H + \frac{\left(\lambda a_{[1]}\right)^2 \left[\beta_1^{(3)} \left[1 + \alpha_1 \left(g_1^{(1)} + \gamma_1^{(1)}\right)\right]^2 + \beta_2^{(3)} \left[1 + \alpha_2 \left(g_2^{(1)} + \gamma_2^{(1)}\right)\right]^2 + p g^{(2)}\right]}{2(1 - \rho_H)}$$

$$+ \frac{\left(\lambda a_{[1]}\right)^2 \left[\alpha_1 \beta_1^{(1)} \left(g_1^{(2)} + \gamma_1^{(2)} + 2g_1^{(1)} \gamma_1^{(1)}\right) + \alpha_2 \beta_2^{(1)} \left(g_2^{(2)} + \gamma_2^{(2)} + 2g_2^{(1)} \gamma_2^{(1)}\right)\right]}{2(1 - \rho_H)}$$

$$+ \frac{\left(\lambda a_{[1]}\right)^2 \left[p g^{(1)} \left[\beta_1^{(1)} \left[1 + \alpha_1 \left(g_1^{(1)} + \gamma_1^{(1)}\right)\right] + \beta_2^{(1)} \left[1 + \alpha_2 \left(g_2^{(1)} + \gamma_2^{(1)}\right)\right]\right]\right]}{(1 - \rho_H)}$$

$$+ \frac{\left(\lambda a_{[1]}\right)^2 \left[\beta_1^{(1)} \beta_2^{(1)} \left[1 + \alpha_1 \left(g_1^{(1)} + \gamma_1^{(1)}\right)\right] + \alpha_2 \left[g_2^{(1)} + \gamma_2^{(1)}\right]\right]}{(1 - \rho_H)} \frac{a_{[2]}}{2a_{[1]}(1 - \rho_H)}$$

(3.4.5)

**Proof:** The result follows directly by differentiating (3.4.4) with respect to $z$ and then taking limit $z \to 1$ by using the L'Hopital's rule.

Next, the relationship between the stationary queue size distributions at a random epoch and at a departure epoch is stated in Corollary 3.4.2.

**Corollary 3.4.2**

Under the steady-state condition, the relationship between the PGFs of the queue size distributions at a random epoch and at a departure epoch of an $M^X/G/1$ unreliable
The server queue with two phases of service and Bernoulli vacation schedule under N-policy is given by

\[ \pi(z) = \left[ \frac{1 - a(z)}{a_0(1 - z)} \right] \psi(z) = H_t(z)\psi(z); \quad (3.4.6) \]

where \( H_t(z) \) is the PGF of the number of customers placed before an arbitrary test customer (tagged customer) in a batch in which the tagged customer arrives. This number is given as the backward recurrence time in the discrete time renewal process where renewal points are generated by the arrival size random variable [e.g., see Takagi (1991), p. 46], i.e.,

\[ H_t(z) = \frac{[1 - a(z)]}{a'(1)(1 - z)} \quad \text{and} \quad a_0 = a'(1). \]

**Remark 3.4.1**

From Corollary 3.4.2, it is observed that queue size distribution at the departure epoch of an \( M^x/G/1 \) unreliable server queue with two phases of service and Bernoulli vacation schedule is the convolution of the distributions of two independent random variables. The first one is the number of units placed before a tagged customer in a batch in which the tagged customer arrives. This is due to the randomness property of the arriving batch size. The interpretation of the other random variables is the stationary distribution an \( M^x/G/1 \) unreliable server queue with two phases of service and Bernoulli vacation schedule.

**Remark 3.4.2**

Now setting \( z = 0 \) in equation \((3.4.6)\), we get

\[ \pi(0) = P_r \quad \{ \text{No customer is waiting in the system at the departure epoch} \} \]
Thus the relationship between \( U_0 \) and \( \pi_0 \) is given by

\[
\pi_0 a_{[1]} = U_0.
\]

This exhibits an interesting phenomenon. It states that a random observer is more likely to find the system empty than a departing customer leaving the system.

### 3.5 Particular cases

Now if we put \( D_i = 0; \ i = 1,2 \) (i.e. there is no delay in the system) or equivalently \( \gamma'_i(\theta) = 1 \) and \( a(z) = z \) (i.e. for single unit arrival case), then we have

\[
\rho_H = \rho_1 \{1 + \alpha_1 g_1\} + \rho_2 \{1 + \alpha_2 g_2\} + p\lambda g^{(i)}, \quad \rho_i = \lambda \beta_i^{(i)}
\]

and therefore our expression (3.4.4) is reduced to

\[
\pi(z) = \frac{(1 - \rho_H)(1 - z)}{[g + p\gamma^*(b(z))\beta_1^*(b(z) + \alpha_1(1 - G_1^*(b(z))))\beta_2^*(b(z) + \alpha_2(1 - G_2^*(b(z)))) - z]}
\]

which is consistent with expression (2.4.4) of our previous Chapter-II.

Also, we note that if suppose \( \alpha_i = 0 \) for \( i = 1,2 \) (i.e. there is no breakdown in the system) our expression (3.5.1) is consistent with the expression (3.20) of Choudhury and Madan (2004).

### 3.6 Busy Period Distribution

The \textit{LST} of the busy period distribution of this \( M^+ / G / 1 \) unreliable queue with two phases of service and delayed repair under Bernoulli vacation schedule can be
obtained as follows. We define the busy period as a length of time interval that makes
the server busy and it continues to the instant when the system becomes free i.e., the
system becomes empty again and denote

\[ T_b = \text{length of the busy period}. \]

Let \( T_b^*(\theta) = E[e^{-\theta T_b}] \) be the LST of \( T_b \), and then Taka'cs functional equation
under the steady state condition is given by

\[ T_b^*(\theta) = H^*(\theta + \lambda - \lambda X(T_b^*(\theta))) ; \]

where \( H^*(\theta) = \{q + p \theta^*(\theta)\} \beta_1^*(\theta + \alpha_1 (1 - G_1^*(\theta)) (b(z))) \beta_2^*(\theta + \alpha_2 (1 - G_2^*(\theta)) (b(z))) \}

is the LST of our modified service time distribution.

The mean busy period is found to be

\[ E(T_b) = \frac{dT_b^*(\theta)}{d\theta} \bigg|_{\theta=0} \]

\[ = \left( \frac{\rho_H}{1 - \rho_H} \right) \frac{1}{\lambda \alpha_{[1]}}, \quad (3.6.1) \]

Where \( \rho_H = \rho_1 \{1 + \alpha_1 (g_1^{(1)} + \gamma_1^{(1)})\} + \rho_2 \{1 + \alpha_2 (g_2^{(1)} + \gamma_2^{(1)})\} + p \lambda \alpha_{[1]}^{(1)} \) is the utilizing
factor of the system.

Next we define \( T_0 \) as the length of the corresponding idle period. Now since
arrival process is Poisson, therefore \( T_b \) and \( T_0 \) generate an alternating renewal Process
and hence we may write

\[ \frac{E(T_b)}{E(T_0)} = \frac{P_r(T_b)}{1 - P_r(T_b)} \]

(3.6.2)

where \( P_r(T_b) \) is the probability that long fraction of time the server remains busy and
this is equivalent to \( \rho_H \).
Now utilizing (3.6.1) in (3.6.2), we have

\[ E(T_0) = [\lambda a_{[1]}]^1 \]  

(3.6.3)

Further, if we define \( T_c \) as length of the busy cycle, then we get

\[ E(T_c) = E(T_0) + E(T) \]

\[ = \left( \frac{1}{\lambda a_{[1]} (1 - \rho_H)} \right) \]  

(3.6.4)

Similarly, the waiting time distribution of a test customer for our model has the following LST.

3.7 Waiting time distribution:

To obtain the waiting time distribution in the queue, we first derive the waiting time of the first customer in an arriving batch, \( W_1 \) (say) and use \( \gamma_1^*(\theta) \) to denote LST of \( W_1 \).

Now if we identify a batch with a single customer, then its service time is just the modified service time of customers constituting the batch. In this case, the batch will have as its batch size \( X(z) = z \). The mean arrival rate will \( \lambda \) and LST of the modified service time of the batch will replace

\[ U^*(\theta) = (q + p \theta^* (\theta))I^*_1 (\theta)I^*_2 (\theta), \quad \text{where} \quad I^*_i (\theta) = \lambda^*_i (A_i (\theta)) \]

by \( X(U^*(\theta)) \)

Using the transformation and the results by Chaudhry and Templeton (1983) (see Chapter 3), from equation (3.3.2.36) we have,
\[
\pi(z) = \frac{(1 - \rho) \lambda (1 - z) X \left[ U^*(b(z)) \right]}{\lambda X \left( U^*(b(z)) - z \right)}
\] (3.7.1)

If the waiting time of each batch is independent of the part of arrival process following
the arrival time of the batches left behind a departing batch are those that arrive during
the time it spends in the queue and in service, it follows that

\[
\pi(z) = W_1^* (\lambda - \lambda z) X \left[ U^*(\lambda - \lambda z) \right]
\] (3.7.2)

Now putting \( \theta = \lambda - \lambda z \) in (3.7.2) and utilizing (3.7.1) in (3.7.2), we get finally,

\[
W_1^*(\theta) = \frac{(1 - \rho) \theta}{\theta - \lambda + \lambda X \left[ U^*(\theta) \right]}
\] (3.7.3)

Now, let \( W \) be the waiting time of an arbitrary customer in a batch and denote by
\( W^*(\theta) \) the LST of \( W \). If \( j \geq 1 \) is the position of the customer within arrival batch, then

\[
W = W_1 + \sum_{i=1}^{j-1} U_i' \quad ; \quad j \geq 1
\] (3.7.4)

Where \( U_i' \) denotes the difference between modified service time and inter arrival
time of the i customer in the batch.

If \( \chi_j \) is the probability of an arbitrary customer being the \( j^{th} \) position of an
arriving batch, then applying the results of Chaudhry & Templeton (1983), we may
write,

\[
\Pr \left[ \sum_{i=1}^{j-1} G_i' \leq t \right] = \sum_{j=1}^{\infty} \chi_j G(t)^{U_{j-1}}
\]

where \( U(t) = \Pr[U_i' \leq t] \) and \( \chi_j = \left( 1 - \sum_{i=1}^{j-1} a_i \right) / E(X) \)
Consequently, taking LST of (3.7.4), we get on simplification

\[ W^*(\theta) = E[e^{-\theta W}] = E[e^{-\theta \sum_{n=1}^{\infty} W_n}] = \frac{W_1^*(\theta) \left[ 1 - \lambda X(U^*(\theta)) \right]}{a_{[1]} \left[ 1 - U^*(\theta) \right]} \]

and therefore LST of the waiting time distribution in the queue for this model is given by

\[ W^*(\theta) = \frac{\theta \left[ 1 - X(U^*(\theta)) \right]}{\theta - \lambda + \lambda \left[ U^*(\theta) \right] \left[ 1 - U^*(\theta) \right]} \]  

(3.7.5)

3.8 Reliability Analysis:

Our final goal is to derive some reliability indices of this model. First of all we will discuss two reliability indices of the system viz. - the system availability and failure frequency under the steady state conditions. Suppose that the system is initially empty. Let \( A_v(t) \) be the point wise availability of the server at time \( t \) that is, the probability that the server is either serving a customer or the server is available if the server is free and up during an idle period, such that the steady state availability of the server will be \( A_v = \lim_{t \to \infty} A_v(t) \)

**Theorem 3.8.1**

The steady state availability of the server is given by,
\[ A_v = 1 - \rho_1 \alpha_1 (g_1^{(i)} + y_1^{(i)}) - \rho_2 \alpha_2 (g_2^{(i)} + y_2^{(i)}) - \lambda p a_{11} \theta^{(i)} \]  

(3.8.1)

**Proof:** - The result follows directly from theorem (3.3.2) by considering the following equation

\[ A_v = U_0 + \sum_{i=1}^{2} \int_0^\infty P_i(x,1)dx = U_0 + \lim_{z \to 1} [P_1(z) + P_2(z)] \]

By using (3.29), (3.37) and (3.38), we get (3.8.1).

**Theorem 3.8.2**

The steady state failure frequency of the server is given by,

\[ M_f = \alpha_1 \rho_1 + \alpha_2 \rho_2 \]  

(3.8.2)

**Proof:** - The result follows directly from equation (3.3.2.14) by utilizing the argument of Li et al. [1997(b)].

\[ M_f = \alpha_1 \int_0^\infty P_1(x,1)dx + \alpha_2 \int_0^\infty P_2(x,1)dx \]

Now since \( \int_0^\infty [1 - B_i(x)]dx = \int_0^\infty x dB_i(x) = \beta_i^{(0)} \); for \( i = 1, 2 \); therefore from equation (3.3.2.14) we have (3.8.2)

Next we denote by \( \tau \) the time to the first failure of the server, and then the reliability function of the server is \( R(t) = P(\tau > t) \)

**Theorem 3.8.3**

The Laplace transform of \( R(t) \) is given by
\[ R^*(\theta) = U_0^*(\theta) + \frac{1}{\lambda + \theta - \lambda \omega_0(\theta)} \]

where \( U_0^*(\theta) = \frac{1}{\lambda + \theta - \lambda \omega_0(\theta)} \)

and \( \omega_0(\theta) \) is the unique root of the equation

\[ z = \left\{ q + p \delta^*(\theta + b(z)) \right\} \beta_1^*(\theta + \alpha_1 + b(z)) \beta_2^*(\theta + \alpha_2 + b(z)) \]

inside \(|z| = 1, \text{Re}(\theta) > 0\) and \( b(z) = \lambda(1 - a(z)) \).

**Proof:** - In order to find the reliability function of the server, we assume that the failure state of the server be the absorbing states, then we obtain a new system. In this new system, we use the same notations as in the previous sections, and then we can write the following set of Kolmogorov forward equations:

\[ \frac{d}{dt} U(t) + \lambda U(t) = \int_0^\infty \gamma(y) Q_0(y, t) dy + q \int_0^\infty \mu_2(x) P_{2,1}(x; t) dx \]  \hspace{1cm} (3.8.4)

\[ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial y} \right) Q_n(y, t) + [\lambda + \gamma(y)] Q_n(y, t) = \lambda \left( 1 - \delta_{n,0} \right) \sum_{k=1}^n a_k Q_{n-k}(y, t), \quad n \geq 0 \]  \hspace{1cm} (3.8.5)

And for \( i=1,2 \)

\[ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) P_{i,n}(x, t) + [\lambda + \alpha_i + \mu_i(x)] P_{i,n}(x, t) = \lambda \sum_{k=1}^n a_k P_{i,n-k}(x, t), \quad n \geq 1 \]  \hspace{1cm} (3.8.6)

These equations are to be solved with initial condition \( U_0(0) = 1 \) subject to the boundary conditions at \( x=0 \):

\[ P_{1,n}(0, t) = \lambda a_n U_0(t) + q \int_0^\infty \mu_2(x) P_{2,n+1}(x; t) dx + \int \mathcal{G}(y) Q_n(y, t) dy, \quad n \geq 1 \]  \hspace{1cm} (3.8.7)
\[ P_{2,n}(0,t) = \int_0^\infty \mu_1(x) P_{1,n}(x,t) \, dx, \quad n \geq 1 \quad (3.8.8) \]

and \[ Q_n(0,t) = p \int_0^\infty \mu_2(x) P_{2,n+1}(x,t) \, dx, \quad n \geq 0 \quad (3.8.9) \]

We now introduce following Laplace transform of generating functions for \(|z| \leq 1;\)

\[ P^*_i(x,\theta; z) = \sum_{n=1}^{\infty} z^n P^*_n(x; \theta), \quad P^*_i(0, \theta; z) = \sum_{n=1}^{\infty} z^n P^*_n(0; \theta) \quad \text{for } i=1,2 \]

and \[ U^*_0(\theta) = \int_0^\infty e^{-\theta} dU_0(t) \]

Now performing Laplace transform with respect to these equations (3.8.4) and (3.8.5) we get

\[ (\theta + \lambda) U^*_0(\theta) - 1 = \int_0^\infty \gamma(y) Q^*_0(y; \theta) \, dy + q \int_0^\infty \mu_2(x) P^*_2(x; \theta) \, dx \quad (3.8.10) \]

\[ (\theta + \lambda + \gamma(y)) Q^*_n(y; \theta) + \frac{\partial}{\partial y} Q^*_n(y; \theta) = \lambda \left( 1 - \delta_{0,n} \right) \sum_{k=1}^{n} a_k Q^*_{n-k}(y; \theta) \quad n \geq 0 \quad (3.8.11) \]

Similarly from equations (3.8.6)-(3.8.9) we have for \( i=1, 2 \)

\[ (\theta + \lambda + \alpha_i + \mu_i(x)) P^*_{i,n}(x; \theta) + \frac{\partial}{\partial \theta} P^*_{i,n}(x; \theta) = \lambda \sum_{k=1}^{n} a_k P^*_{i,n-k}(x; \theta) \quad n \geq 1 \quad (3.8.12) \]

\[ P^*_{1,n}(0; \theta) = \lambda a_n U^*_0(\theta) + q \int_0^\infty \mu_2(x) P^*_{2,n+1}(x; \theta) \, dx + \int_0^\infty \beta(y) Q^*_n(y; \theta) \, dy \quad n \geq 1 \quad (3.8.13) \]

\[ P^*_{2,n}(0; \theta) = \int_0^\infty \mu_1(x) P^*_{1,n}(x; \theta) \, dx \quad n \geq 1 \quad (3.8.14) \]

and \[ Q^*_n(0; \theta) = p \int_0^\infty \mu_2(x) P^*_{2,n+1}(x; \theta) \, dx \quad n \geq 0 \quad (3.8.15) \]
Now multiplying equation (3.8.11) by $z^n$ and then taking summation over all possible values of $n \geq 0$, we get a set of differential equation of Lagrangian type whose solution is given by,

$$Q^*(y, \theta; z) = Q^*(0, \theta; z) \exp\left\{ - (b(z) + \theta) y \right\} \left[ 1 - V(y) \right]$$

(3.8.16)

Where $b(z) = \lambda(1 - a(z))$ is as defined in section 3.3

Again from equation (3.8.15) we have

$$zQ^*(0, \theta; z) = pP_1^*(0, \theta; z) \beta_1^*(\theta + \alpha_1 + b(z))$$

(3.8.17)

Now similarly multiplying equation (3.8.12) by $z^n$ and then taking summation over all positive values of $n \geq 0$, we get a set of similar type of differential equation of type whose solution is given by,

$$P_i^*(x, \theta; z) = P_i^*(0, \theta; z) \exp\left\{ - (b(z) + \alpha_i + \theta) x \right\} \left[ 1 - \phi_i(x) \right]$$

for $i = 1, 2$ (3.8.18)

Again from equations (3.8.13) and (3.8.14) we have

$$zP_1^*(0, \theta; z) + z[b(z) + \theta]U_0^*(\theta) = z + qP_1^*(0, \theta; z) \beta_1^*[b(z) + \theta + \alpha_1^2] + zQ^*(0, \theta; z) \beta^*(\theta + b(z))$$

(3.8.19)

$$P_2^*(0, \theta; z) = P_1^*(0, \theta; z) \beta_1^*(\theta + \alpha_1 + b(z))$$

(3.8.20)

Similarly from equation (3.8.19) after utilizing (3.8.17) and (3.8.20)

$$P_1^*(0, \theta; z) = \frac{z(b(z) + \theta)U_0^*(\theta) - 1}{[q + p \beta^*(\theta + b(z))]\beta_1^*(\theta + \alpha_1 + b(z))\beta_2^*(\theta + \alpha_2 + b(z)) - z}$$

(3.8.21)

Further from equation (3.8.18) for $i=1$ we obtain

$$P_1^*(\theta, z) = \frac{z(b(z) + \theta)U_0^*(\theta) - 1 - \beta_1^*(b(z) + \alpha_1 + \theta)}{[q + p \beta^*(\theta + b(z))]\beta_1^*(\theta + \alpha_1 + b(z))\beta_2^*(\theta + \alpha_2 + b(z)) - z(b(z) + \alpha_1 + \theta)}$$

(3.8.22)

Similarly from equation (3.8.18) for $i=2$ we obtain
Finally from equation (3.8.16)

$$Q^*(\theta; z) = \frac{p[(b(z) + \theta)U_0^*(\theta) - \beta_1^*(\theta + \alpha_1 + b(z) \beta_2^*(\theta + \alpha_2 + b(z))] - z}{[g + p \theta^*(\theta + b(z))] \beta_1^*(\theta + \alpha_1 + b(z)) \beta_2^*(\theta + \alpha_2 + b(z))} \tag{3.8.24}$$

Now consider the coefficient

$$f(z) = \frac{p[(b(z) + \theta)U_0^*(\theta) - \beta_1^*(\theta + \alpha_1 + b(z) \beta_2^*(\theta + \alpha_2 + b(z))] - z}{[g + p \theta^*(\theta + b(z))] \beta_1^*(\theta + \alpha_1 + b(z)) \beta_2^*(\theta + \alpha_2 + b(z))}$$

from which it can be shown that the function $f(z)$ is convex. Hence by Rouche’s theorem $f(z)$ has only exactly one root $w_0(\theta)$ inside the unit circle $|z| = 1$ for Re$(z)$. Therefore we have

$$U_0^*(\theta) = \frac{1}{s + \lambda - \lambda w_0(\theta)}$$

where $w_0(\theta)$ is the unique root of the equation

$$z = \frac{p[(b(z) + \theta)U_0^*(\theta) - \beta_1^*(\theta + \alpha_1 + b(z) \beta_2^*(\theta + \alpha_2 + b(z))] - z}{[g + p \theta^*(\theta + b(z))] \beta_1^*(\theta + \alpha_1 + b(z)) \beta_2^*(\theta + \alpha_2 + b(z))}$$

Hence from equations (3.8.21), (3.8.22) and (3.8.23), we have,

$$R^*(\theta) = U_0^*(\theta) + P_1^*(\theta) + P_2^*(\theta) + Q^*(\theta)$$

$$= U_0^*(\theta) + P_1^*(\theta, 1) + P_2^*(\theta, 1) + Q^*(\theta, 1);$$

From which we get the required expression (3.8.3)
3.9 Numerical Illustration

This section contains some numerical illustrative examples. Note that for the sake of computational convenience; let us assume that the distributions of the service times, the delay times, the repair times and the vacation times are exponential. The distribution of the sizes of successive arriving batches is assumed to be geometric with parameter $a$, so that $a_{[1]} = \frac{1}{a}$ and $a_{[2]} = \frac{(2-a)}{a^2}$.

3.9.1 Optimal design

The optimal design of a queueing system is to determine the optimal system parameters using some cost functions. To illustrate, let $c_h$ be the holding cost per unit time for each customer present in the system, $c_0$ be the cost per unit time for keeping the server on and in operation, $c_s$ be the setup cost per busy cycle, and $c_a$ be the startup cost per unit time for the preparatory work of the server before starting the service. These unit costs can be combined with the performance measures obtained above to write the system total expected cost per unit of time as

$$TC = c_hL_s + c_0\frac{E(T_b)}{E(T_c)} + c_s\frac{1}{E(T_c)} + c_a\frac{E(T_0)}{E(T_c)}$$

$$=c_hL_s + c_0\rho_H + c_s\lambda a_{[1]}(1 - \rho_H) + c_a(1 - \rho_H)$$

Here $E(T_0) = \frac{1}{\lambda a_{[1]}}$ is the expected length of an idle period, while $E(T_b)$ and $E(T_c)$ are the expected length of a busy period and a busy cycle, respectively, and are given by (3.6.1) and (3.6.4) respectively. To show how any parameter can affect the system performance, let us assume that the various distributions involved are exponential. For the following values of the parameters:
\[ p = 0.5, \lambda = 0.325, c_k = 5, c_0 = 100, c_s = 1000, c_a = 100, \alpha_1 = 0.5, \alpha_2 = 0.5, a = 0.5, \beta_1^{(1)} = 0.5, \beta_2^{(1)} = 0.4, \gamma_1^{(1)} = g_1^{(1)} / 5, \gamma_2^{(1)} = g_2^{(1)} / 5, \gamma_{(1)}^{(1)} = g^{(1)} / 2, \gamma_{(2)}^{(1)} = g^{(1)} / 2, \gamma^{(1)} = 0.45 \]

We find a total expected cost per unit of time is \[ TC = 294.769. \]

The effect of the system parameters on the optimal policy can be easily undertaken numerically. For example, Tables 1-5 show the effect on the mean queue size \( L_s \) and on the optimal cost \( TC \) of the parameters of interest to us in this paper, namely, the breakdown rates \( \alpha_1 \) and \( \alpha_2 \), the probability of the Bernoulli vacation \( p \), the parameter of the geometric distribution \( a \) and the arrival rate \( \lambda \), respectively. All the other parameters are kept unchanged.

Again in Table 6 we observe that for higher values of \( \lambda \), the rate of increase in \( L_s \) is faster than the lower values of \( \lambda \) for various values of \( p \) for both the cases, as it should be.

Finally in Table 7 and in Table 8, the numerical results are summarized in which the steady-state server availability \( A_v \) and failure frequency \( M_f \) are calculated with the given data. Clearly, high value of \( \alpha_i \) (i=1, 2) results in low server availability and high failure frequency.

**Table 1. Effect of the FPS breakdown rate \( \alpha_1 \) on the optimal cost**

<table>
<thead>
<tr>
<th>( \alpha_1 )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( TC )</td>
<td>303.569</td>
<td>301.31</td>
<td>299.09</td>
<td>296.908</td>
<td>294.769</td>
<td>292.674</td>
<td>290.628</td>
<td>288.632</td>
<td>286.691</td>
</tr>
</tbody>
</table>

**Table 2. Effect of the SPS breakdown rate \( \alpha_2 \) on the optimal cost**

<table>
<thead>
<tr>
<th>( \alpha_2 )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( TC )</td>
<td>300.352</td>
<td>298.931</td>
<td>297.527</td>
<td>296.139</td>
<td>294.769</td>
<td>293.417</td>
<td>292.084</td>
<td>290.771</td>
<td>289.478</td>
</tr>
</tbody>
</table>
Table 3. Effect of the Bernoulli vacation on the $L_s$ and optimal cost

<table>
<thead>
<tr>
<th>$p$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
</table>

Table 4. Effect of the arriving batch sizes on the $L_s$ and optimal cost

<table>
<thead>
<tr>
<th>$a$</th>
<th>0.45</th>
<th>0.5</th>
<th>0.55</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.85</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>TC</td>
<td>286.41</td>
<td>294.769</td>
<td>308.134</td>
<td>317.274</td>
<td>322.244</td>
<td>324.064</td>
<td>321.707</td>
<td>318.719</td>
</tr>
</tbody>
</table>

Table 5: Effect of the arrival rate on the $L_s$ and optimal cost

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.125</th>
<th>0.15</th>
<th>0.175</th>
<th>0.2</th>
<th>0.225</th>
<th>0.25</th>
<th>0.275</th>
<th>0.3</th>
<th>0.325</th>
<th>0.35</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_s$</td>
<td>2.11775</td>
<td>2.15181</td>
<td>2.18644</td>
<td>2.22168</td>
<td>2.25753</td>
<td>2.29403</td>
<td>2.33119</td>
<td>2.36903</td>
<td>2.4076</td>
<td></td>
</tr>
<tr>
<td>TC</td>
<td>288.443</td>
<td>307.353</td>
<td>320.626</td>
<td>328.362</td>
<td>330.713</td>
<td>327.917</td>
<td>320.378</td>
<td>308.830</td>
<td>294.769</td>
<td></td>
</tr>
</tbody>
</table>

Table 6. Effect of the arrival rate and the Bernoulli vacation on the mean queue size

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>2.21775</td>
<td>2.15181</td>
<td>2.18644</td>
<td>2.22168</td>
<td>2.25753</td>
<td>2.29403</td>
<td>2.33119</td>
<td>2.36903</td>
<td>2.4076</td>
</tr>
<tr>
<td>0.125</td>
<td>2.32</td>
<td>2.346682</td>
<td>2.41755</td>
<td>2.4681</td>
<td>2.5199</td>
<td>2.57303</td>
<td>2.62754</td>
<td>2.6835</td>
<td>2.74098</td>
</tr>
<tr>
<td>0.15</td>
<td>2.54965</td>
<td>2.61582</td>
<td>2.68412</td>
<td>2.75467</td>
<td>2.82761</td>
<td>2.90309</td>
<td>2.98129</td>
<td>3.06238</td>
<td>3.14656</td>
</tr>
<tr>
<td>0.175</td>
<td>2.81308</td>
<td>2.90247</td>
<td>2.99562</td>
<td>3.09283</td>
<td>3.19444</td>
<td>3.30079</td>
<td>3.41231</td>
<td>3.52944</td>
<td>3.65269</td>
</tr>
<tr>
<td>0.2</td>
<td>3.11883</td>
<td>3.23875</td>
<td>3.36521</td>
<td>3.49886</td>
<td>3.64043</td>
<td>3.79076</td>
<td>3.95084</td>
<td>4.12176</td>
<td>4.30482</td>
</tr>
<tr>
<td>0.25</td>
<td>3.90874</td>
<td>4.12629</td>
<td>4.36327</td>
<td>4.62274</td>
<td>4.90842</td>
<td>5.22493</td>
<td>5.578</td>
<td>5.97493</td>
<td>6.42505</td>
</tr>
<tr>
<td>0.275</td>
<td>4.43313</td>
<td>4.73116</td>
<td>5.06326</td>
<td>5.43624</td>
<td>5.85884</td>
<td>6.34247</td>
<td>6.90235</td>
<td>7.55918</td>
<td>8.34197</td>
</tr>
<tr>
<td>0.3</td>
<td>5.08766</td>
<td>5.5046</td>
<td>5.9831</td>
<td>6.53897</td>
<td>7.19396</td>
<td>7.97884</td>
<td>8.93854</td>
<td>10.1414</td>
<td>11.6969</td>
</tr>
</tbody>
</table>

Table 7: Effect of $x_1$ on reliability indices

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$A_y$</th>
<th>$M_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.833275</td>
<td>0.1625</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8284</td>
<td>0.195</td>
</tr>
<tr>
<td>0.3</td>
<td>0.823525</td>
<td>0.2275</td>
</tr>
<tr>
<td>0.4</td>
<td>0.81865</td>
<td>0.26</td>
</tr>
<tr>
<td>0.5</td>
<td>0.813775</td>
<td>0.2925</td>
</tr>
<tr>
<td>0.6</td>
<td>0.8089</td>
<td>0.325</td>
</tr>
<tr>
<td>0.7</td>
<td>0.804025</td>
<td>0.3575</td>
</tr>
<tr>
<td>0.8</td>
<td>0.79915</td>
<td>0.39</td>
</tr>
<tr>
<td>0.9</td>
<td>0.794275</td>
<td>0.4225</td>
</tr>
</tbody>
</table>

Table 8: Effect of $x_2$ on reliability indices

<table>
<thead>
<tr>
<th>$x_2$</th>
<th>$A_y$</th>
<th>$M_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.826255</td>
<td>0.1885</td>
</tr>
<tr>
<td>0.2</td>
<td>0.823135</td>
<td>0.2145</td>
</tr>
<tr>
<td>0.3</td>
<td>0.820015</td>
<td>0.2405</td>
</tr>
<tr>
<td>0.4</td>
<td>0.816895</td>
<td>0.2665</td>
</tr>
<tr>
<td>0.5</td>
<td>0.813775</td>
<td>0.2925</td>
</tr>
<tr>
<td>0.6</td>
<td>0.810655</td>
<td>0.3185</td>
</tr>
<tr>
<td>0.7</td>
<td>0.807535</td>
<td>0.3445</td>
</tr>
<tr>
<td>0.8</td>
<td>0.804415</td>
<td>0.3705</td>
</tr>
<tr>
<td>0.9</td>
<td>0.801295</td>
<td>0.3965</td>
</tr>
</tbody>
</table>
The graphs below show the effect of some of the system parameters on the total expected cost per unit of time and on mean queue size in the system. To investigate the effect of server failures on the total expected cost per unit of time, we give different values to mean failure rates and record the corresponding value of the system total expected cost and mean queue size. Figure 1 and Figure 3 below show that $TC$ decreases as mean failure rates increases. From Figure 2 and Figure 4, we observe that $L_s$ increases as $\alpha_i$ $(i=1,2)$ increases. When breakdown rates increase, the server is unable to provide service for the customers, which leads to the expected number of customers in the system becoming larger and the completion period longer.

Suppose now we are interested in the effect of the Bernoulli vacation schedule on the system performance. Keeping the values of the system parameters unchanged, we vary the probability of a vacation from 0 to 0.9 and again record the corresponding
values of the system total expected cost and mean queue size in the system. Figures 5 and 6 below depict the variations of the cost and mean queue size respectively with the probability of a vacation. We see that $TC$ first decreases as $p$ increases and then becomes stably as $p$ becomes large. Figure 6 reports that mean queue size increases as $p$ increases. As expected, a larger $p$ implies that the number of customers and the completion period becomes larger, due to ongoing preventative maintenance having a higher probability.

![TC and $L_\pi$ graphs](image)

Again we want to see the effect of arriving batch sizes on the system performance. Using the same values of the system parameters as above, we vary the values of the arriving batch sizes from 0.45 to 0.9 and record the corresponding values of the system total expected cost and mean queue size in the system. Figures 7 and 8 below depict the variations of the cost and mean queue size respectively with the variation of arriving batch sizes. From Figure 7 one sees that $TC$ first increases ($\alpha \leq 0.7, \rho_H \leq 0.550875$) and then decreases ($\alpha > 0.7, \rho_H > 0.550875$) with increasing $\alpha$. We find that the maximum cost is $TC = 324.064$, and it is obtained when $\alpha$ is 0.7; on the other hand Figure 8 reveals that $L_\pi$ decreases as batch size $\alpha$ increases.
Finally, we want to see the effect of arrival rate when all the data are kept unchanged. Figure 9 and Figure 10 show the variations of the system total cost and mean queue size in the system respectively when the arrival rate varies from 0.125 to 0.35. Figure 9 reveals that $TC$ first increases ($\lambda \leq 0.225$) and then decreases ($\lambda > 0.225$). We find that the maximum cost is $TC = 330.713$, and it is obtained when $\lambda = 0.225$ and in Figure 10 it appears that $L_s$ increases as $\lambda$ increases.

3.10 Concluding Remarks

In this Chapter we have studied the steady state behavior of an $M^x/G/1$ unreliable queueing system with delayed repair under Bernoulli vacation schedule.
Here arrivals are occurring to the system in batches of variable size. In this investigation we have assume that once system breaks down the repair process does not start immediately and there is a delay in repair. Hence our present model is a generalization of the previous Chapter-II with batch arrivals for two phases of heterogeneous service under Bernoulli vacation schedule where concept of delay time is also introduced. For this model we have developed queue size distribution at departure epoch, busy period distribution, waiting time distribution, some reliability indices and the Laplace transform of the system reliability function. To obtain these results we have applied the most popular classical supplementary variable technique.