Chapter 4
Solution of Fourth Order Elliptic Equations

4.1 Introduction

This chapter describes selected developments in wavelet-Galerkin solutions of fourth order linear elliptic partial differential equations in one dimension. As mentioned earlier, wavelet-Galerkin methods should have a variational formulation of the PDE problem in which the unknown (or the solution) function $u$ is found to be a member of a Sobolev space $H$ which is then approximated in wavelet bases. That is, taking a finite dimensional closed subspace $H_n$ of $H$, we find a basis for $H_n$ by using Daubechies wavelets. Chapter 2 describes elaborately how these approximation functions are obtained in two different approaches, one, by manipulating the scaling functions directly, and, second, by obtaining their integrals. The construction of the approximation (basis) functions for the spaces $H_0^m(a,b)$ and $H^m(a,b)$ in Chapter 2 fulfills the requirements for the solutions of all the problems.
discussed in this chapter.

Fourth order problems can be solved by variational methods in different formulations such as (i) conventional formulation, (ii) Lagrange multiplier formulation, (iii) penalty function formulation, and (iv) mixed formulation. The details can be seen in Reddy [71]. In this thesis, we use the conventional formulation.

In this chapter, we present wavelet-Galerkin solutions of linear elliptic problems of fourth order with two types of boundary conditions. Both the approximation techniques described in Chapter 2 are used. Numerical results are computed and the convergence rates are examined which are found to be better in comparison to finite difference solutions. All the numerical tests are performed in MATLAB 6.1. Onward now, this chapter is organized with four sections. Section 4.2 discusses the formulation of the problems to be solved. In section 4.3, we solve the problems using approximation spaces of 'direct approach' type. Section 4.4 carries out the solutions of the problems using approximation spaces of 'integration approach' type. Section 4.5 concludes the chapter.

4.2 Problem Formulation

The problems to be discussed in this chapter are as follows:

\[ (\alpha(x)u'')'' + \beta(x)u = f(x), \quad a < x < b \]  
(4.1)

with the boundary conditions:

\[
\begin{align*}
(i) \quad & u(a) = c_1, \quad u(b) = c_2, \\
(ii) \quad & u''(a) = d_1, \quad u''(b) = d_2,
\end{align*}
\]  
(4.2)
and,
\[
\begin{cases}
(i) \quad u(a) = c_1, \quad u'(a) = c_2, \\
(ii) \quad u''(b) = d_1, \quad u'''(b) = d_2.
\end{cases}
\] (4.3)

We assume here that \( f \in L^2(a, b) \) and the coefficients \( \alpha \) and \( \beta \) are differentiable in \((a, b)\). The boundary conditions (4.2)(i) and (4.3)(i) are essential boundary conditions and the others are natural boundary conditions.

### 4.2.1 Variational Formulation of the Problems

(i) Problem (4.1)-(4.2) has the weak form:

\[
\begin{cases}
\quad u = u_0 + w, \quad w \in H, \\
A(w, v) = F(v), \text{ for all } v \in H,
\end{cases}
\] (4.4)

with \( H = H_0^2(a, b) \), \( u_0 = \frac{b_0 - a_0}{b - a} + c_0 \), \( A(\cdot, \cdot) \) is defined as

\[
A(u, v) = \int_a^b (\alpha u'' v'' + \beta uv) \, dx,
\] (4.5)

\[
F(v) = \int_a^b f v \, dx + c_0 \alpha(b)v(b) - c_0 \alpha(a)v(a) - A(u_0, v).
\] (4.6)

(ii) Problem (4.1)-(4.3) has the weak form:

\[
\begin{cases}
\quad u = u_0 + w, \quad w \in H, \\
A(w, v) = F(v), \text{ for all } v \in H,
\end{cases}
\] (4.7)

with \( H = H_+^2(a, b) \), \( u_0 = c_1 + c_2(x - a) \), \( A(\cdot, \cdot) \) as defined in (4.5),

\[
F(v) = \int_a^b f v \, dx + c_1 \alpha(b)v'(b) - c_1 \alpha(a)v'(a) - A(u_0, v).
\] (4.8)
Sufficient conditions for the existence of unique solutions of (i.e. for applying Lax-Milgram Lemma to) the above problems consist of $\alpha(x) > 0$ and $\beta(x) > 0$, for all $x \in (a, b)$.

**Remark 4.2.1.** Practically, $\beta$ can be negative or zero and $f$ can be less regular than $L^2$ function.

### 4.3 Solution Using Direct Approach

#### 4.3.1 Approximate Problems and Their Solutions

Taking a suitable value of $N$, let $\phi$ be the $dbN$ scaling function. Then we can approximate problems (4.4) and (4.7) as

\[
\begin{cases}
 w_n \in H_n, \\
 A(w_n, v_n) = F(v_n), \quad \text{for all } v_n \in H_n,
\end{cases}
\]

where $H_n = V^0_n(a, b)$ for problem (4.4) and $H_n = V^-_n(a, b)$ for problem (4.7), where the spaces $V^0_n(a, b)$ and $V^-_n(a, b)$ are as defined in Chapter 2.

**Remark 4.3.1.** By Remark 2.3.1, the error estimates of the approximate solutions for problems (4.4) and (4.7) in $L^2$, $H^1$ and $H^2$ norms can be obtained as follows.

Let $w = u - u_0$ be the solution of the above problems and $w_n$ the solution of the approximate problem (4.9). Then for $w \in H^N(a, b)$, we have

\[
\|w - w_n\|_m \leq C_m h^{N-m}, \quad (m = 0, 1, 2),
\]

where $h = 2^{-n}$, $N$ is the order of the wavelet used and $C_m$ are positive constants.
Now, let
\[ w_n = \sum_{j=1}^{p} \bar{c}_{n,j} \Phi_{n,j-S+1} \]  \hspace{1cm} (4.11)

be the solution of the approximate problem (4.9) at resolution level \( n \geq 0 \), where \( p = 2^n S + S - 2 \). Application of Galerkin method to (4.9) with the approximate solution (4.11) will give a system of linear equations in \( p \) unknowns \( \bar{c}_{n,j}, j = 1, \ldots, p \):
\[ A\bar{c} = F, \]  \hspace{1cm} (4.12)

where \( A \) and \( F \) are the stiffness matrix and the force vector respectively whose elements are given by
\[
\begin{align*}
A &= [A_{ij}], \quad A_{ij} = A(\phi_{n,j-S+1}, \phi_{n,i-S+1}), \\
F &= [F_i], \quad F_i = F(\phi_{n,i-S+1}).
\end{align*}
\hspace{1cm} (4.13)

The solutions of these equations give rise to the approximate solutions of the actual boundary value problems. Here the stiffness matrix \( A \) is always symmetric which is sparse.

### 4.3.2 Numerical Experiments

Here, we perform some numerical tests to justify the quality of the method presented for the solutions of fourth order boundary value problems. All the problems are solved by using \( db4 \), \( db5 \) and \( db6 \) scaling functions successively at resolution levels \( n=0, 1, 2 \) and \( 3 \). The solutions are compared with the exact solutions and \( L^2 \), \( H^1 \) and \( H^2 \) norm errors are obtained. Also we obtain finite difference solutions for comparison with the wavelet solutions.
Test Problem 4.3.1
Here we solve the following problem:

\[
\begin{align*}
\begin{cases}
    u^{iv} + u = (1 + \pi^4) \sin \pi x + 1, & 0 < x < 1; \\
    u(0) = 1, & u(1) = 1; \\
    u''(0) = 0, & u''(1) = 0,
\end{cases}
\end{align*}
\]

whose exact solution is given by

\[u(x) = \sin \pi x + 1.\]  

This problem is solved by using $db4$, $db5$ and $db6$ wavelets at resolution levels $n = 0, 1, 2$ and 3. Also, we approximate the problem using finite difference method with samples of size 5, 10, 20 and 40. The $L^2$ norm error for size 5 is nearly equal to that for $db4$ wavelet solution at $n = 0$. We compare the convergence rates of finite difference solutions and wavelet solutions for $L^2$ norm error in Figure 4.1(a), for $H^1$ norm error in Figure 4.1(b) and for $H^2$ norm error in Figure 4.1(c). In Figure 4.1(c), we can see that the graphs for FD and $db4$ wavelet solutions are (almost) parallel which means that they have the same convergence rate. We know that FDM has quadratic rate of convergence and so $db4$ wavelet solution is quadratic (in $H^2$ norm), which is also predicted by Remark 4.3.1. In the other cases, all the three figures show that the convergence rates of wavelet solutions are higher than that of finite difference solutions in all the three norms. The $db6$ wavelet solution at $n = 3$ is affected by round off errors. The linear systems of equations for this problem are solved by Cholesky factorization method.
Problem 4.3.1: resolution level \( (n) \) versus \( L^2 \) norm error

Figure 4.1(a): Test Problem 4.3.1

Decay in \( L^2 \) norm error with increasing resolution

Problem 4.3.1: resolution level \( (n) \) versus \( H^1 \) norm error

Figure 4.1(b): Test Problem 4.3.1

Decay in \( H^1 \) norm error with increasing resolution
Test Problem 4.3.2

Here we solve the following problem:

\[
\begin{align*}
    u'' + u &= (1 + \pi^4) \sin \pi x + 1, \quad 1 < x < 2; \\
    u(1) &= 1, \quad u'(1) = -\pi; \\
    u''(2) &= 0, \quad u'''(2) = -\pi^3,
\end{align*}
\]  

(4.16)

whose exact solution is given by

\[ u(x) = \sin \pi x + 1. \]  

(4.17)

As for the last problem, this problem is also solved by using \textit{db4}, \textit{db5} and \textit{db6} wavelets at resolution levels \( n = 0, 1, 2 \) and 3. Also, we approximate the problem using finite difference method with samples of size 12, 24, 48 and 96. The \( L^2 \) norm error for size 12
Decay in $L^2$ norm error with increasing resolution is nearly equal to that for $db4$ wavelet solution at $n = 0$. We compare the convergence rates of finite difference solutions and wavelet solutions for $L^2$ norm error in Figure 4.2(a), for $H^1$ norm error in Figure 4.2(b) and for $H^2$ norm error in Figure 4.2(c). As in the last problem, it is established from Figure 4.2(c) that $db4$ wavelet solution is quadratic in $H^2$ norm, which is also predicted by Remark 4.3.1. All the three figures justify the supremacy of the wavelet solutions. The $db6$ wavelet solution at $n = 3$ is affected by round off errors. For this problem also, the system of linear equations is solved by Cholesky factorization method.

Remark 4.3.2. Here we have used Daubechies wavelets of order < 6 for the solutions of fourth order problems which contradicts
Figure 4.2(b): Test Problem 4.3.2
Decay in $H^1$ norm error with increasing resolution

Figure 4.2(c): Test Problem 4.3.2
Decay in $H^2$ norm error with increasing resolution
Remark 2.3.2. This is due the fact that $N \geq 6$ is only sufficient and not necessary.

4.4 Solution Using Integration Approach

4.4.1 Approximate Problems and Their Solutions

Taking a suitable value of $N$, let $\phi$ be the $dbN$ scaling function. Then we can approximate problems (4.4) and (4.7) as the problem in (4.9), where $H_n = S^0_n(a,b)$ for problem (4.4) and $H_n = S^-_n(a,b)$ for problem (4.7), where the spaces $S^0_n(a,b)$ and $S^-_n(a,b)$ are as defined in Chapter 2.

Remark 4.4.1. By the consequence (2.55) of Theorem 2.3.2, the error estimates of the approximate solutions for problems (4.4) and (4.7) in $L^2$, $H^1$ and $H^2$ norms can be obtained as follows.

Let $w (= u - u_0)$ be the solution of the above problems and $w_n$ the solution of the approximate problem (4.9). Then for $w \in H^{N+1}(a,b)$, we have

$$\|w - w_n\|_m \leq C_m h^{N-m+1}, \quad (m = 0, 1, 2),$$

(4.18)

where $h = 2^{-n}$, $N$ is the order of the wavelet used and $C_m$ are positive constants.

Taking solution $w_n$ of the approximate problem (4.9), at resolution level $n \geq 0$, similar to (4.11) with the help of the approximation spaces $S^0_n(a,b)$ and $S^-_n(a,b)$, we get systems of linear equations similar to (4.12).
4.4.2 Numerical Experiments

Here, we perform some numerical tests to justify the quality of the method presented for the solutions of fourth order boundary value problems. All the problems are solved by using $db3$, $db4$ and $db5$ scaling functions successively at resolution levels $n=0$, 1, 2 and 3. The solutions are compared with the exact solutions and $L^2$, $H^1$ and $H^2$ norm errors are obtained. Also we obtain finite difference solutions for comparison with the wavelet solutions.

Test Problem 4.4.1

Here we solve the problem given in Test Problem 4.3.1. This problem is solved by using $db3$, $db4$ and $db5$ wavelets at resolution levels $n = 0$, 1, 2 and 3. Also, we approximate the problem using finite difference method with samples of size 12, 24, 48 and 96. The $L^2$ norm error for size 12 is nearly equal to that for $db3$ wavelet solution at $n = 0$. We compare the convergence rates of finite difference solutions and wavelet solutions for $L^2$ norm error in Figure 4.3(a), for $H^1$ norm error in Figure 4.3(b) and for $H^2$ norm error in Figure 4.3(c). In Figure 4.3(c), we can see that the graphs for FD and $db3$ wavelet solutions are (almost) parallel which means that they have the same convergence rate. We know that FDM has quadratic rate of convergence and so $db3$ wavelet solution is quadratic (in $H^2$ norm), which is also predicted by Remark 4.4.1. In the other cases, all the three figures justify the supremacy of wavelet solutions. The $db5$ wavelet solution at $n = 3$ is affected by round off errors. The linear systems of equations
Problem 4.4.1: resolution level (n) versus $L^2$ norm error

Figure 4.3(a): Test Problem 4.4.1
Decay in $L^2$ norm error with increasing resolution

Problem 4.4.1: resolution level (n) versus $H^1$ norm error

Figure 4.3(b): Test Problem 4.4.1
Decay in $H^1$ norm error with increasing resolution
Problem 4.4.1: resolution level (n) versus $H^2$ norm error

Figure 4.3(c): Test Problem 4.4.1

Decay in $H^2$ norm error with increasing resolution for this problem are solved by Cholesky factorization method.

**Test Problem 4.4.2**

Here we solve the problem given in Test Problem 4.3.2. As for the last problem, this problem is also solved by using $db3$, $db4$ and $db5$ wavelets at resolution levels $n = 0$, 1, 2 and 3. Also, we approximate the problem using finite difference method with samples of size 18, 36, 72 and 144. The $L^2$ norm error for size 18 is nearly equal to that for $db3$ wavelet solution at $n = 0$. We compare the convergence rates of finite difference solutions and wavelet solutions for $L^2$ norm error in Figure 4.4(a), for $H^1$ norm error in Figure 4.4(b) and for $H^2$ norm error in Figure 4.4(c). As in the last problem, it is established from Figure 4.4(c) that $db3$
Problem 4.4.2: resolution level (n) versus $L^2$ norm error

Figure 4.4(a): Test Problem 4.4.2
Decay in $L^2$ norm error with increasing resolution

Problem 4.4.2: resolution level (n) versus $H^1$ norm error

Figure 4.4(b): Test Problem 4.4.2
Decay in $H^1$ norm error with increasing resolution
wavelet solution is quadratic in $H^2$ norm, which is also predicted by Remark 4.4.1. For the other cases, all the three figures justify the supremacy of the wavelet solutions. The $db5$ wavelet solution at $n = 3$ is affected by round off errors. For this problem also, the system of linear equations is solved by Cholesky factorization method.

### 4.5 Conclusion

In this chapter, we have solved linear fourth order elliptic differential equations with two types of boundary conditions using wavelet-Galerkin method as for second order problems in Chapter
3. Both the techniques give fast convergence for fourth order problems also. The underlying systems of linear equations are solved by direct methods only. The iterative methods can be applied under the conditions cited in Chapter 3. The comparison of the method with finite difference method indicates that the present method is a right competitor of the other classical methods.