CHAPTER 1
INTRODUCTION

1.1 Fluid

The Subject, fluid dynamics is of great significance to man since the earth is 75% covered with water and 100% with air, the scope of fluid mechanics is vast and has numerous applications in engineering and human activities. By fluid we mean a substance which is capable of flowing. Fluids may be divided into two categories: (1) liquids which are incompressible, i.e. their volumes don’t change when the pressure changes, and (2) gases which are compressible fluids i.e. their volumes change whenever the pressure changes. Study of fluid mechanics is important because of the prevalence of fluids and our dependence on them. Among several applications of fluid dynamics, a few are those that relate to the calculation of forces and moments on an aircraft, evaluation of mass flux of mineral oils like petroleum through pipe lines, prediction of weather patterns, rocket science, missile technology, human physiology, food processing, chemical engineering and so on. Further, fluids in motion are potential source of energy that can be converted into useful work, for example, by waterwheel or a watermill. The study of fluid mechanics aims at analyzing and controlling the fluid flow phenomenon and to utilize it for the welfare of the society. The knowledge of temperature distribution is essential in heat transfer studies because of the fact that the heat flow takes place only wherever there is a temperature gradient in a system. The study of temperature distribution and heat transfer is of great importance to engineers because of its almost universal occurrence in many branches of science and engineering. Transient free convection flow of a viscous fluid adjacent to vertical surfaces is a fundamental problem in the fluid dynamics, heat and mass transfer, with significance for a variety of engineering applications. Hence, the motivation of this thesis is to study some convective flow problems along with heat and mass transfer with different physical situations.

1.2 Brief History of Fluid Mechanics

The formal study of fluids began at least 500 hundred years ago with the work of Leonardo da Vinci (1452-1519) who built the first chambered canal lock near Milan. He also made several attempts to study the flight (birds) and developed some concepts on the origin of the forces. After his initial work, the knowledge of fluid mechanics (hydraulic)
increasingly spread by the contributions of Galileo, Torricelli, Euler, Newton, Bernoulli and D'Alembert. At that time theory and experiments had some discrepancy. The fact was acknowledged by D'Alembert who stated that, "The theory of fluids must necessarily be based upon experiment." For example the concept of ideal liquid that leads to motion with no resistance, conflicts with the reality. This discrepancy between theory and practice is called the "D'Alembert paradox".

In the middle of the nineteen century, first Navier in the molecular level and later Stokes from continuous point of view succeeded in creating governing equations for real fluid motion. The Navier-Stokes equations, which describes the flow (or even Euler equations), were considered unsolvable during the mid nineteen century because of the high complexity. Until the present day there exists no general method for solving these equations. Analytical solutions have therefore been attempted only for flows with relatively simple geometry. Even such solutions are based on idealizations such as infinite plates, infinitely long cylinders, fully developed parallel flow in pipe etc. The solutions obtained were interesting but exact solutions for problem which were important from the engineering point of view were not yet available.

At the last of twenty century, the demand for vigorous scientific knowledge that can be applied to various liquids as opposed to formula for every fluid was created by the expansion of many industries. This demand coupled with new several novel concepts like the theoretical and experimental researches of Reynolds, the development of dimensional analysis by Rayleigh, and Froude’s idea of the use of models change the science of the fluid mechanics. Perhaps the most radical concept that effects the fluid mechanics is of Prandtl’s idea of boundary layer which is a combination of the modeling and dimensional analysis that leads to modern fluid mechanics. Therefore Prandtl is known as the father of modern fluid mechanics. Thus, Prandtl and his students Blasius, Von Karman, Meyer, and several other individuals as Nikuradse, Rose, Taylor, Buckingham, Stanton and many others, transformed the fluid mechanics to today modern science.

Towards the end of nineteenth century, researchers in fluid mechanics were divided into two branches of fluid dynamics, namely hydrodynamics and hydraulics. Hydrodynamics, though mathematically elegant, was not able to predict the drag experienced by bodies moving in fluids. This is known as D'Alembert’s paradox. On the other hand, hydraulics offered solutions to practical problems based mainly on empirical data. This situation changed radically when the German scientist Ludwig Prandtl(1875-1953), published perhaps the most important paper ever written on fluid mechanics in 1904. Prandtl’s
paper titled “On fluid motion with small friction”, gave the first description of the boundary layer concept. Prandtl pointed out that fluid flows with small viscosity (water and air flows) can be divided into a thin viscous layer, or boundary layer, near the solid surfaces and interfaces, patched onto a nearly inviscid outer layer, where the Euler and Bernoulli equations apply. In other words, he assumed the no slip condition at the surface and considered that the frictional effects were experienced only in a boundary layer, a thin region near the surface. Outside the boundary layer the flow was essentially the inviscid flow that had been studied for previous two centuries. Thus the shearing stress and the condition of no-slip at solid walls which distinguishes a real fluid from a perfect fluid are to be taken into consideration only in the boundary layer. Since the boundary layer is thin, it plays a vital role in fluid dynamics. It has become a very powerful method of analyzing the complex behavior of real fluids. The concept of a boundary layer can be utilized to simplify the Navier-Stokes equations to such an extent that it becomes possible to tackle many practical problems of great importance. The thinness of the boundary layer enables simplifications like rendering the governing equations parabolic and imposing external pressure on the boundary layer. The theory also showed that separation of flow is decided mainly by a stream wise pressure gradient in the external flow. Though this theory originally developed for laminar flow, but later it was extended to turbulent flow. The most important application of the boundary layer theory is that with the help of this theory, it has been possible to calculate the skin friction or drag and to explain the physical behavior of real fluid around two-dimensional and axial symmetric bodies. The boundary layer concept developed rapidly and is now applied in almost all branches of engineering.

1.3 The Thermal Boundary Layer
The concept of a boundary layer as proposed by German scientist Ludwig Prandtl (1904) from the starting point for the simplification of the equations of motion and energy. It has been successfully used in many a practical problem. In this concept the flow field over a body is divided into two regions: (1) a thin region near the body, called the boundary layer, where the velocity and temperature gradients are large and (2) the region outside the boundary where velocity and temperature gradients are very nearly equal to their free stream value. In general, both the velocity boundary layer and thermal boundary layer will exist simultaneously.
Like the boundary layer in the velocity field, the thermal boundary layer also forms when heat is transferred to the flow field through the surrounding walls. For example, if a hot vertical plate is placed in a fluid medium (which is at rest with uniform temperature lower than that of the plate) then heat transfer will take place initially purely by conduction only. This means the temperature gradient exists in the neighbourhood of the plate and this will generate a density gradient in the fluid medium. This density gradient gives rise to a convective motion in the fluid medium due to buoyancy force. The temperature field in a fluid medium can be divided into two regions: in the region close to the wall where the thermal conductivity plays important role, and the region where the thermal conductivity can be neglected. Further, in the region near the leading edge of the plate, the boundary layer is laminar, but at a certain distance from the leading edge of the plate, transition to a turbulent boundary layer begins.

The thickness of velocity boundary layer $\delta(x)$ and thermal boundary layer $\delta_t(x)$ depends on the magnitude of the Prandtl number $Pr$. Both the velocity and thermal boundary coincide, when the fluid having Prandtl number $Pr=1$. For the fluids having Prandtl number $Pr \gg 1$ (e.g. oil, water), the velocity boundary layer is much thicker than the thermal boundary layer. On the other hand fluid having Prandtl number $Pr << 1$ the velocity boundary layer is much thinner than thermal boundary layer.

### 1.4 Heat Transfer

Heat is defined as a form of energy in transition by virtue of temperature difference, when a system at a given temperature is brought in contact with its surroundings at a different temperature, energy flows from the higher temperature region to the lower temperature region. This energy transfer process is called heat transfer. It is a typical spontaneous irreversible process, which occurs in such a way that the body and surroundings reach a thermal equilibrium state. Heat flow is vectorial in the sense that it is in the direction of negative temperature gradient, i.e. from higher toward lower temperatures. Energy efficiency is one of the most important globally discussed topics. The science of heat transfer studies the mechanisms of heat transfer and extends thermodynamic analysis, through the development of necessary empirical and analytical relations, to calculate heat transfer rates.

The science of heat transfer may be applied to solve various problems relevant to technology and society. Heat transfer problems confront the engineers and researchers in nearly every branch of engineering and science. Although it is generally regarded as most
closely related to mechanical engineering, much work in this field has been done in chemical, nuclear, metallurgical and electrical engineering where heat transfer problems are equally important. There are three different modes of heat transfer. The mechanism of heat transfer between bodies in direct contact is called conduction. When the bodies are separated by empty space then the mechanism is radiation through electromagnetic waves. Heat transfer between a wall and a fluid system in motion is called convection. Fundamental laws governing heat transfer are given below:

(i) First law of thermodynamics gives conservation of energy.
(ii) Second law of thermodynamics gives direction of heat flow.
(iii) Equation of continuity gives conservation of mass.
(iv) Equation of flow gives Newton’s Second law of motion, Navier-Stokes equation.
(v) Rate equations governing the three modes of heat transfer.

1.4.1 Conduction
Conduction is the process in which heat is transferred from regions of higher temperature to regions of lower temperature within a system or between two systems which are in contact physically without any relative motion of the different parts of system or systems. Heat conduction is the mode of heat transfer accomplished via two mechanisms.

(i) By molecular interaction whereby the energy exchange takes place by the kinetic motion or direct impact of molecules. Molecules at relatively high energy level (temperature) impart energy to adjacent molecules at lower energy levels.

(ii) By the drift of free electrons as in the case of metallic solids. The metallic alloys have a different concentration of free electrons and their ability to conduct heat is directly proportional to the concentration of free electrons in them. The free electron concentration of non-metals is very low. Hence materials that are good conductors are also good conductors of heat too. Conduction heat transfer may occur between two bodies or two parts of the same body through molecules which are, more or less stationary, as in the case of solids. In other words, heat conduction takes place in matter which allows the passage for heat energy, even when its parts are not in motion relative to one another. The sequence of materials of decreasing heat conductivity is: pure metals, alloys, non-metallic
Examples of heat transfer by conduction are numerous. For instance, the exposed end of a metal spoon suddenly immersed in a cup of hot tea will become eventually warmed up due to the conduction of thermal energy through the spoon. For another instance, on winter days, there are significant energy losses from heated room to the outside air. This loss may be attributed to conduction heat transfer through the wall that separates the room air from the outside.

The first to give a precise definition of conductivity was Fourier who in his memorable Théorie Analytique de la Chaleur (1882) treated the subject of heat conduction in a masterly way and placed it on firm mathematical basis. Fourier's law of heat conduction states the quantity of heat $Q$ flowing in the positive direction of $x$ in time $dt$ across the isothermal surface of area $A$ at any point $x$ is given

$$Q = -kA \frac{dT}{dx} dt$$

The coefficient $k$ is a quantity depending upon the nature of the substance and is called its thermal conductivity. The thermal conductivity of a material depends upon its state and is a function of its pressure, temperature, humidity and structure. Pure metals have the highest values of thermal conductivities while gases and vapours have the lowest.

### 1.4.2 Convection

When a fluid flows inside a duct or over a solid body and the temperatures of the fluid and the solid surfaces are different, heat transfer between the fluid and solid surface will take place. This is due to the motion of fluid relative to the surface. This type of heat transfer is called convection. Convection is the transference of heat by heated matter which moves carrying its heat with it. Thus it can take place only in fluids. In a system of fluid motion, heat is transferred by conduction and convection. At the surface, heat is first transferred by conduction to the adjacent fluid elements which is turn move to regions of lower temperature (thus convective heat) and impart heat to the neighboring fluid particles by conduction as well.

Convective heat transfer or simply as convection is the transfer of thermal energy from one place to another by the movement of fluids or gases. Convection is usually the dominant form of heat transfer in liquids and gases. Although often discussed as a distinct method of heat transfer, convection describes the combined effects of conduction and fluid flow or mass exchange. Convection is the mode of heat transfer between a surface and a fluid moving over it. The energy transfer in convection is predominantly due to
bulk motion of the fluid particles, though the molecular conduction within the fluid itself also contributes to some extent. If we assume the validity of the no-slip condition (i.e. the assumption that the fluid layer in contact with a solid surface sticks to the surface of contact), then the heat transfer between the solid surface and the fluid at the surface of contact can take place only by conduction. Convection of heat is classified into two types, namely, free or natural convection and forced or assisted convection.

1.4.3 Forced Convection
If the fluid motion is artificially created by means of an external agency like a blower or fan, the heat transfer is termed as forced convection. The theoretical treatment of convection is rather complicated though the problem of forced convection is a little simpler. In forced convection a steady stream of fluid is forced past the hot body by external means. The ventilation of rooms and central heating of buildings in winter are some examples of forced convection in everyday life. Low speed laminar forced convection flows are of wide interest because of their importance in several engineering studies. In forced convection the velocity field is independent of the temperature field though the temperature field is dependent on the velocity field. Mathematically, the problem reduces to finding the temperature field due to heated or cooled boundaries in a given velocity field. In forced convection, the driving force is external to the fluid and the flow velocities are high. Forced convection occurs in a variety of flow configuration, such as flow of liquids through pipe, flow of fluids over flat surfaces, flow of liquid heating tubes of a solar collector etc.

1.4.4 Free or Natural Convection
If the fluid motion is set up by buoyancy effects resulting from the density variation caused by the temperature difference in the fluid, the heat transfer is said to be free or natural convection. Free or natural convection always takes place vertically and is caused by gravity as a consequence of change in density resulting from the rise in temperature and consequent expansion. When a heated body is cooled in air, all the three methods of heat transference are acting simultaneously. But the air is a very poor conductor of heat, and radiation is important only for large differences of temperature; thus the chief means of heat loss is convection. The mechanism of this heat loss is easy to understand but the derivation of a theoretical formula is extremely difficult. The land and sea breezes, the trade winds, the fall of temperature with height in the atmosphere are all examples of convection on a huge scale in nature. In natural convection, an increase in temperature
produces a reduction in density, which causes fluid motion due to pressures and forces when fluids of different densities are affected by gravity (or any g-force). For example, when water is heated on a stove, hot water from the bottom of the pan rises, displacing the colder denser liquid which falls. After heating has stopped, mixing and conduction from this natural convection eventually result in a nearly homogeneous density, and even temperature. In the case of natural convection, it is found experimentally (an elaborate series of experiments on natural convection was performed by Dulong and Petit) that the rate of loss of heat from the body is proportional, not to the temperature difference between the body and the surrounding fluid, but very nearly to the $5/4$ power of it. Thus the rate of loss of heat due to natural convection can be written in the form

$$\frac{dT}{dt} = k(T - T_0)^{5/4}$$

where $k$ is a constant depending upon the nature of the surface and $T, T_0$ the temperature of the thermometer and the enclosure respectively.

### 1.4.5 Mixed Convection

Combined (or mixed) convective heat transfer is the term applied to the case where both mechanisms, buoyancy force effects and forced flow effects, are important, the heat transfer then being the result of a combination or mixture of free and forced convection. In such flows the buoyancy forces and the forced flow play an important role in determining the flow pattern and hence, the heat transfer rate. For example, when air is flowing over a heated vertical surface at a relatively low velocity but at a relatively high heating rate, the resulting density changes can give rise to a superimposed natural convection process. It is a well known fact that whenever there is an unstable temperature gradient free convection is bound to take place. In most practical situations, the free and forced convection processes arise simultaneously and it becomes necessary to evaluate the total heat transfer by adding the heat transfer due to forced and free convections.

### 1.4.6 Radiation

When a heated body is placed in vacuum, it loses heat. In this case no heat can be lost by conduction or convection since matter, which is absolutely essential for both these processes, is absent. In such cases the heat is lost by radiation. Radiation or radiant energy is an energy transfer process from material into surrounding space by electromagnetic
waves. The propagation of heat by radiation consists merely in a transference of energy. But the radiation or radiant energy in the processes of transference does not make itself evident unless it falls on matter. When it falls on matter and is absorbed, it is converted to heat and can be thereby detected. Radiation and light obey identical laws and have identical properties. Some of the properties are given below.

(i) Radiation, like light, can travel through vacuum
(ii) Radiation, like light, travels in straight lines
(iii) Radiation travels with the velocity of light
(iv) Radiation follows the law of inverse square like light
(v) Radiation obeys the laws of reflection of light.

Every hot body emits radiation from its surface which depends upon the nature of the surface, its size and its temperature. This is known as emission. When the emitted radiation falls on matter, a part is reflected regularly or irregularly, another part is absorbed while the remainder is transmitted. Radiation is the only form of heat transfer that can occur in the absence of any form of medium. In the context of space technology and in process involving high temperature, the effects of radiation are of vital importance.

1.5 Mass Transfer

In a system comprising two or more components having different concentrations at different points, the molecular motions will cause the transfer of mass by diffusion to minimize the concentration differences within the system. The transport of one constituent from a region of higher concentration to that of a lower concentration is known as mass transfer. Mass transfer will take place as long as a difference in species concentration exists. Therefore, the species concentration gradient acts as the driving potential in mass transfer just as the temperature gradient does in heat transfer. Mass transfer occurs in many processes, such as absorption, evaporation, adsorption, drying, precipitation, membrane filtration and distillation. Mass transfer is used by different scientific disciplines for different processes and mechanisms. Some common examples of mass transfer are the evaporation of water from ponds, lakes, rivers and oceans to the atmosphere, the purification of blood in the kidneys and liver and the distillation of alcohol. Investigations of mass transfer carry special significance in energy transfer since all heat transfer processes within one fluid or within different fluids which are mixing with each other are associated with mass transfer due to the mass transport of the fluids.
There are two distinct modes of mass transfer namely, molecular mass transfer (mass diffusion) and convective mass transfer.

1.5.1 Mass Diffusion

Diffusion means the spontaneous process of spreading or scattering of matter in binary medium or two component system under the influence of concentration. In a mixture homogeneous in respect of temperature and pressure, diffusion is directed towards equalizing the concentration in the system and is accompanied by transfer of mass from the region of higher concentration to the region of lower concentration. Mass transfer by molecular diffusion is analogous to the conduction of heat in solids. Mass diffusion occurs in liquids and solids, as well as in gases. However, since mass transfer is greatly influenced by molecular spacing, diffusion occurs more readily in gases than in liquids, and more easily in liquids than in solids. Since the mechanism of molecular diffusion and conduction heat transfer are analogous, their corresponding rate equations are also similar. A relation between the flux of the diffusing substance and the concentration gradient responsible for this mass transfer was first postulated by Fick (1855) and is therefore referred to as Fick's first law of diffusion. Mathematically, this law can be defined as

\[ J = -D \frac{dC}{dx} \]

Where \( J \) = molar flux in the x direction relative to the molar average velocity

\[ \frac{dC}{dx} \] = concentration gradient in the x direction

\( D \) = proportionality constant, diffusion coefficient or mass diffusivity.

1.5.2 Convective Mass Transfer

Mass transfer may take place due to convection between a moving fluid and a surface and between two relatively immiscible moving fluids both under natural or free convection and forced convection. In fact there exists a similarity between mass transfer and heat transfer by convective processes. Mass transfer by convection occurs in cases where the bulk velocity is appreciable or when both the species, in a binary mixture, are moving with significant velocities. The buoyancy force causing circulation in free convection mass transfer results from the differences in density of the vapour air mixtures of varying compositions. An example of free convection mass transfer is the evaporation of alcohol.
Similarly, the example of forced convection is the evaporation of water from an ocean when air blows over it. Depending upon the fluid stream through which mass transfer takes place it may be defined as laminar or turbulent.

The combined effects of heat and mass transfer processes are frequently encountered in many engineering and practical situations. The humidifiers, cooling towers, absorbers, evaporative condensers, wet bulb thermometer are some examples of where heat and mass transfer take place simultaneously. Free convective flows driven by temperature and concentration difference have been studied extensively. When both the temperature and concentration differences occur simultaneously, the free convective flow can become quite complex. A common example of simultaneous diffusion of thermal energy and chemical species is the evaporation of lake water into the wind flowing over it. This example may also be cited as a case of low-rate heat and mass transfer at low or moderate temperatures.

1.6 Chemical Reaction

In nature, the presence of pure air or water is rather impossible. It is always possible that a foreign mass is either present naturally in air or water or foreign masses (as impurities) are mixed with air or water. Due to the presence of such foreign masses (as impurities) in fluid in the process of convection, chemical reaction also takes place. This type of chemical reaction may change the temperature and heat content of the fluid and may affect the convection processes.

Chemical reactions can be codified as either heterogeneous or homogeneous processes. This depends on whether they occur at an interface or as a single phase volume reaction. In well-mixed systems, the reaction is heterogeneous if it takes place at an interface, and homogeneous if it takes place in solution. In most chemical reactions, the reaction rate depends on the concentration of the species itself. A reaction is said to be of zero order if the rate of reaction is constant (i.e. independent of concentration). In general, a reaction is said to be of order n, if the reaction rate is proportional to the nth power of the concentration. In particular, a reaction is said to be of first order, if the rate of reaction is directly proportional to concentration itself. Mass transfer with chemical reaction plays an important role in chemical industries as well as in physical and biological sciences. Processes involving mass transfer with chemical reaction occur frequently in nature. It occurs not only due to the temperature difference but also due to concentration difference or combination of two. In many chemical engineering processes, there is a chemical
reaction between a foreign mass and a fluid. These processes take place in numerous industrial applications such as manufacturing of ceramics, food processing and polymer production.

1.7 Basic Terminology

1.7.1 Viscosity

Viscosity is that fluid property by virtue of which a fluid offers resistance to shear stresses, i.e., sliding movement of one particle past or near another particle. Viscosity is also known as internal friction of fluid. All real fluids exhibit viscosity but in many cases, such as arise when the rates of variation of fluid velocity with distances are small, viscous effects may be ignored. Viscosity of Glycerin and oil is large in comparison to viscosity of water or gases. To understand the causes of viscosity of a fluid, consider the observed effects of temperature on the viscosity of a gas and a liquid. It is seen that for gases, viscosity increases with increasing temperature and for liquids, viscosity decreases with increasing temperature. The reason is that viscosity appears to depend on two phenomena, namely the transfer of momentum between molecules and intermolecular forces between molecules of the fluid.

From experiments with various fluids, Sir Isaac Newton (1713) postulated that for the straight and parallel motion of a given fluid, the tangential stress between two fluid layers is proportional to the velocity gradient in a direction perpendicular to the layers. This Newton's law of viscosity is expressible as under

$$\tau = \mu \left( \frac{du}{dy} \right)$$

Here \( \mu \) is a constant for a particular fluid at a particular temperature. The coefficient of proportionality is the absolute viscosity. Shear stress(\( \tau \)) varies with the rate of deformation. In deriving the above relation, Newton assumed that, the fluid to be confined in between two horizontal plates and y axis is at right angle to the plates.

Fluids which obey the equation \( \tau = \mu \left( \frac{du}{dy} \right) \) referred to as Newtonian fluids, if its viscosity \( \mu \) does not change with the rate of deformation. In this case the equation \( \tau = \mu \left( \frac{du}{dy} \right) \) is

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similar to the equation $y = mx$, where $y = \tau$, $m = \mu$ and $x = \frac{du}{dy}$. Hence Newtonian fluid is represented by a straight line. On the other hand, the fluids which do not obey this law are called non-Newtonian fluid, if its viscosity varies with the rate of deformation. $T$, $\mu$, $\frac{du}{dy}$ all are variables in the equation $\tau = \mu \left( \frac{du}{dy} \right)$. Therefore non-Newtonian fluid is represented by a curve.

### 1.7.2 Kinematic Viscosity

Kinematic viscosity $\nu$ is defined as the ratio of absolute viscosity to density. Mathematically, it is $\nu = \frac{\mu}{\rho}$. For most fluids $\mu$ depends on the pressure and temperature,

(i) $\mu$ is very small for water, gases, alcohol but not negligible but $\mu$ is very large in case of oil, glycerine. (ii) $\mu$ depends upon both pressure and temperature for a large number of fluids whereas $\mu$ is inversely proportional to temperature and is independent of pressure for gases.

Its use in turbulent and laminar flows where the ratio of viscous forces (which is proportional to $\mu$), to the inertial forces (which is proportional to $\rho$) is involved.

### 1.7.3 Compressibility of fluids

At normal conditions, there cannot be much variation in volume (i.e. density) on varying pressure or temperature of a liquid and hence liquids are considered as incompressible fluids. On the other hand gases and vapours show variation in volume (i.e. density) for even a small change in pressure or temperature. Therefore, gases and vapours are considered as compressible fluids. The degree of compressibility of substance is measured by the bulk modulus of elasticity $K$, defined as $K = -\frac{dP}{(dV/V)}$ where $(dV/V)$ represents the volumetric strain for an infinitesimal pressure change $dP$. Since a rise in pressure always causes a decrease in volume, $dV$ is always negative, and the negative sign indicates that $K$ is always positive. The value of $K$ depends on the relation between pressure and density under which the compression occurs. For liquids, $K$ is very high and so there is very little change of density with pressure. For this reason the density of liquids can be assumed to be constant. On the other hand, gases are very compressible.
When the compression is carried out in a reversible adiabatic manner, then $K = K_s$ and is known as the isentropic bulk modulus. Since specific volume is the reciprocal of density,

\[ \frac{dV}{V} = -\frac{dP}{\rho} \]

The equation for isentropic bulk modulus can be written in terms of the corresponding density change,

\[ K_s = \frac{dP}{(d\rho/\rho)} \]

While analyzing the fluid in motion, it is quite natural to classify the flow as compressible and incompressible based on the fluid which is flowing, namely, gases and liquids. But, this assumption is not absolutely correct because one cannot reach the conclusion about the compressibility of the fluid system without analyzing the processes involved in the system.

### 1.7.4 Incompressible flows

An incompressible fluid is that which requires a large variation in pressure to produce some appreciable change in its density. Incompressible flows concern with the motion of incompressible fluids. Thus, an incompressible flow is a kind of flow where the density is constant for the fluid flow. Liquids are generally incompressible while gases may be considered to be incompressible as long as their flow velocities remain below a certain level (small compared with the velocity of sound in case of gases under consideration). Their change in volumes is less than 0.5% and practically irrelevant as long as their velocities are less than 0.5% of the sound.

### 1.7.5 Fluid Pressure

Let $P$ be a point in an inviscid fluid moving locally with velocity $\mathbf{q}$. Insert a small rigid plane area $\delta A$ into the fluid at $P$, but instead of having it at rest let it move with the local fluid velocity $\mathbf{q}$ of $P$. Then the only momentum interchange between fluid and plane is on the molecular scale and if $\delta F$ denotes the force exerted on one side of $\delta A$ by this momentum interchange, then this force is normal to $\delta A$ and, assuming the limit to exist uniquely. It is defined as

\[ p = \lim_{\delta t \to 0} \frac{\delta F}{\delta A} \]

In the case of an inviscid fluid when $\delta A$ is situated tangentially to the local fluid velocity at $P$, the only momentum interchange between fluid and surface is of the random

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molecular type, so that the force \( \delta F \) would be same, whether \( \delta A \) is at rest, moving with the local fluid velocity \( q \).

1.7.6 Fluid Density
The density \( \rho \) of a fluid at a point is a quantitative expression and may be defined as its mass per unit volume. Consider \( \delta m \) is the mass of the fluid in a small volume \( \delta V \) surrounding that point. Mathematically, the density \( \rho \) at the given point may be defined as, \( \rho = \frac{\delta m}{\delta V} \)

The mass density of a fluid varies with temperature and pressure. The density is an index of the internal characteristics. The density of a fluid depends on the space coordinates and the temperature.

1.7.7 Specific Heat
The specific heat of a fluid is defined as the heat capacity of the fluid mass. Thus, specific heat is the quantity of heat per unit mass necessary to raise the temperature of a system by unit temperature. The amount of heat added for a given temperature differential depends on the process by which heat is added. The value of specific heat depends on two well-known process-the constant volume process and the constant pressure process. For the constant volume process, the specific heat is defined as \( C_v = (\frac{\partial Q}{\partial T})_v \). Specific heat at constant pressure is defined as \( C_p = (\frac{\partial Q}{\partial T})_p \) where \( \partial Q \) is the amount of heat added to raise the temperature by \( \partial T \).

1.7.8 Steady and Unsteady Motion
A fluid motion is said to be steady if the condition at any point in the fluid at any time remains the same for all time. Mathematically, it can be expressed as

\[
\frac{\partial A}{\partial t} = 0
\]

where \( A \) represents the characteristic of the fluid, e.g., velocity, density, temperature and pressure. Thus in steady motion various field quantities become function of the space coordinates. Water being pumped through a fixed system at a constant rate is an example of steady flow.
A flow motion is said to be unsteady when conditions at any point change with regards to the time. In unsteady flow, the stream line pattern changes from instant to instant whereas in steady flow, the stream lines are constant with time and also represent the trajectories of fluid particles (path lines). Water being pumped through a fixed system at an increasing rate is an example of unsteady flow. Unsteady flows are important in technological applications like metallurgy, nuclear reactors, turbo machinery, aerospace technology and chemical processes industries. Steady flows are significant not only because of their own interest, but also because of their wide applications in geophysics and engineering.

1.7.9 Laminar (streamline) and Turbulent Flow
A flow, in which each fluid particle traces out a definite curve and the curves traced out by any two different fluid particles do not intersect, is said to be laminar. Laminar flow occurs when a fluid flows in parallel layers, with no disruption between the layers. At low velocities the fluid tends to flow without lateral mixing, and adjacent layers slide past one another like playing cards. In fluid dynamics, laminar flow is a flow regime characterized by high momentum diffusion and low momentum convection. For instance, when a stream of water flows inside a pipe with low velocity, it moves in parallel layers. These type of flow may be termed as laminar flow. Investigations regarding laminar flow are of great practical significance in fields such as aeronautical where the emphasis is on the reduction of viscous drag on an airplane in its flight and thereby improve its flight range and fuel usage efficiency. On the other hand, a flow, in which each fluid particle does not trace out a definite curve and the curves traced out by fluid particles intersect, is said to be turbulent. Most practical flows are turbulent in nature. The turbulent flow in characterized by the existence of random fluctuations in the fluid. A stable flow may become unsteady if the small disturbances due to fluid stream or surface roughness get amplified in the direction of fluid flow. The transition from laminar to turbulent flow takes place if the inertia forces are large in comparison with the viscous forces or the Reynolds number is large enough to amplify the small disturbances.

1.8 Basic Equations
1.8.1 Equation of Continuity (Conservation of Mass)
Physical quantities are said to be conserved when they do not change with regard to time during a process. The mathematical form of the law of conservation of mass is known as equation of continuity. This equation expresses that the rate of generation of mass within
a given volume is entirely due to the net inflow of mass through the surface enclosing the given volume. Its amounts to the basic physical law that the matter is conserved; i.e. it cannot be created nor destroyed. Mathematically, the equation of continuity can be defined as

\[ \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{q} = 0 \]

Here \( \mathbf{q} = (u, v, w) \) is the velocity vector, \( \rho \) is the fluid density and \( \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \nabla \cdot \mathbf{q} \) is the material derivative. For an incompressible fluid in which \( \rho \) is constant and hence the equation of continuity reduces to \( \nabla \cdot \mathbf{q} = 0 \). In cylindrical coordinate \((r, \theta, z)\) system, the equation of continuity is

\[ \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}( \rho q_r ) + \frac{1}{r} \frac{\partial}{\partial \theta}( \rho q_\theta ) + \frac{\partial}{\partial z}( \rho q_z ) = 0 \]

with velocity vector \( \mathbf{q} = (q_r, q_\theta, q_z) \).

### 1.8.2 Equation of Motion (Conservation of Momentum)

The momentum of a body is defined as the product of the mass of the body and its velocity. The dynamical behavior of fluid motion is governed by a set of equations called the momentum equations or the equations of motion. These equations are obtained by applying either Newton’s second law of motion to an infinitesimal mass of fluid or the law of conservation of linear momentum, to an infinitesimal control volume in the fluid. From the Newton’s law, the Euler’s equation of motion for inviscid fluid can be derived as

\[ \frac{D\mathbf{q}}{Dt} = \frac{F}{\rho} - \frac{1}{\rho} \nabla p \]

Navier-Stokes equation for a viscous compressible fluid with constant viscosity becomes

\[ \frac{D\mathbf{q}}{Dt} = \frac{F}{\rho} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{q} + \frac{\nu}{3} \left( \nabla \cdot \mathbf{q} \right) \]

Here \( F \) is the body force vector per unit mass, \( \nabla^2 \) is the Laplacian operator and other symbols have their usual meaning.

For incompressible fluid, the Navier-Stokes equation reduces to,

\[ \rho \frac{D\mathbf{q}}{Dt} = \rho \left( F - \nabla p + \mu \nabla^2 \mathbf{q} \right) \]

(17)
For viscous compressible fluid with constant viscosity, in cylindrical coordinate 
(r, θ, z) system, the Navier-Stokes equations are

\[
\rho \left( \frac{Dq_r}{Dt} - \frac{q_x^2}{r} \right) = \rho F_r - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[ \nabla^2 q_r - \frac{q_r}{r^2} - \frac{2}{r^2} \frac{\partial q_\theta}{\partial \theta} + \frac{1}{3} \frac{\partial}{\partial r} \left( \vec{\nabla} \cdot \vec{q} \right) \right]
\]

\[
\rho \left( \frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r} \right) = \rho F_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[ \nabla^2 q_\theta - \frac{q_\theta}{r^2} + \frac{2}{r^2} \frac{\partial q_r}{\partial \theta} + \frac{1}{3} \frac{\partial}{\partial \theta} \left( \vec{\nabla} \cdot \vec{q} \right) \right]
\]

\[
\rho \frac{Dq_z}{Dt} = \rho F_z - \frac{\partial p}{\partial z} + \mu \left[ \nabla^2 q_z + \frac{1}{3} \frac{\partial}{\partial z} \left( \vec{\nabla} \cdot \vec{q} \right) \right]
\]

Here \( \vec{\nabla} \equiv \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial z} \) and \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \) with \( (q_r, q_\theta, q_z) \) be the velocity components and \((F_r, F_\theta, F_z)\) be the components of body force in the r, θ, z direction respectively.

### 1.8.3 Energy Equation (Conservation of Energy)

The principle of conservation of energy is based entirely on experience and it states that energy may be neither created nor destroyed: it can only be transformed from one state to another, i.e. total amount of energy in a system always remains same. The first law of thermodynamics is merely one statement of this principle with reference to heat energy and mechanical energy. The science of thermodynamics deals with the relation between heat and other forms of energy, but the science of heat transfer is concerned with the analysis of the rate of heat transfer that takes place in a system. The energy transfer by heat flow cannot be measured directly, but the concept has the physical meaning because it is related with temperature. Therefore the knowledge of temperature distribution is essential to analyze the problems in heat transfer. For example, in air conditioning of buildings, in aerospace technology, in the design of nuclear reactor cores, in chemical and other industries (in the design of heat exchangers such as boilers, condensers, radiators etc) proper analysis and treatment of heat transfer is necessary.

The energy equation can be derived by applying the first law of thermodynamics. According to the first law of thermodynamics, the gain in total energy \( DE \) in unit time \( Dt \) is equal to the heat supplied to the mass element \( \dot{Q} Dt \) and the work done on the element \( \dot{W} Dt \).

(18)
Mathematically \( \frac{DE}{Dt} = \dot{Q} + \dot{W} \)

It can be expressed in terms of temperature \( T \), without radiation effect as follows,

\[
\rho C_p \frac{DT}{Dt} = \nabla \left( k \nabla T \right) + q^{iii} + \beta T \frac{Dp}{Dt} + \mu \phi
\]

Here \( q^{iii} \) is the dissipation rate or the rate of internal heat generation, \( \frac{Dp}{Dt} \) is the pressure work term, \( \phi \) is the viscous dissipation function and other symbols have their usual meanings.

For ideal gas \( (\beta = \frac{1}{T}) \), the energy equation reduces to,

\[
\rho C_p \frac{DT}{Dt} = \nabla \left( k \nabla T \right) + q^{iii} + \frac{Dp}{Dt} + \mu \phi
\]

For viscous incompressible \( (\beta=0) \) fluid with constant conductivity \( k \) and neglecting viscous dissipation term, the energy equation becomes,

\[
\rho C_p \frac{DT}{Dt} = k \nabla^2 T
\]

1.8.4 Equation of State

In thermodynamics, the state of a given system can be fully defined by a certain minimum number of thermodynamic properties. The relationship between the properties which define the thermodynamic state is known as the equation of state. If two independent properties are known, then the third can be evaluated using the equation of state. The equation of state of a substance is a relation between its states namely, pressure, temperature and density. There exists an equation of state corresponding to a given homogeneous substance, solid, liquid or gas. The relationship may be expressed as

\[
f(p, \rho, T) = 0
\]

The function \( f \) is a single valued function of \( p, \rho \) and \( T \). The exact nature of the function \( f \) is, in general, very complicated and varies from fluid to fluid. The kinetic theory of gases, which is based on perfect gases, states that for equilibrium conditions, the absolute pressure, \( p \), the volume \( V \) occupied by mass \( m \), and absolute temperature \( T \) would be related as
\[ pV = mRT \]

or \[ p = \rho RT \]

where \( \rho \) is the density and \( R \) the gas constant. This equation is known as equation of state for perfect gas.

### 1.9 Dimensional analysis and dimensionless numbers

The dimensional analysis is a mathematical technique, which we deduce information about a phenomenon from the single premise that the phenomenon can be described by a dimensionally correct equation among certain variables. There are four basic dimensions, namely mass, length, time and temperature. The dimensions of all other physical variables of any phenomenon is expressible in terms of these four fundamental dimensions. There are two techniques of dimensional analysis:

- Rayleigh’s technique.
- Buckingham \( \pi \)-theorem.

With the help of dimensional analysis one can combine the variables of any given equation into non-dimensional quantities. A dimensionless quantity is a quantity without an associated physical dimensions. It is a pure number. Dimensionless quantities are widely used in mathematics, physics, engineering and economics. Dimensionless quantities are often defined as products or ratios of quantities that are not dimensionless.

Properties of dimensionless quantities:

- Even though a dimensionless quantity has no physical dimension associated with it, it can still have dimensionless units.
- The ratio of two quantities with the same dimensions is dimensionless and has the same value regardless of the units used to calculate them.

There are some useful dimensionless quantities of great significance. A few of them are discussed below.

#### 1.9.1 Reynolds Number (Re)

Reynolds number is a dimensionless number and it is defined as the ratio of inertial forces to viscous forces and consequently quantifies the relative importance of these two types of forces for given flow condition. Reynolds numbers frequently arise when performing...
dimensional analysis of fluid dynamics problems. The Reynolds number is named in honour of the British mathematician Osborne Reynolds (1842-1912). Mathematically, the Reynolds number \( Re \) is defined as

\[
Re = \frac{\text{inertial forces}}{\text{viscous forces}} = \frac{UL}{\nu}
\]

Here \( U \) = characteristic velocity, \( L \) = characteristic length and \( \nu \) = kinematic viscosity.

Obviously this number has no units. Hence for geometrical similarity of two flows, it is necessary that their Reynolds number should be the same and the boundary conditions are satisfied. Osborne Reynolds established its importance in determining the nature of fluid flow through pipes. In all the problems where viscous resistance to flow occurs, this number plays an important role. When the Reynolds number is small, the viscous force is predominant and the effect of viscosity is important in the whole velocity field. When the Reynolds number is large, the inertial force is predominant. The effect of viscosity is important only in a narrow region near the solid wall which gives rise to Prandtl boundary layer theory. When the Reynolds number is enormously large, the flow becomes turbulent.

### 1.9.2 Prandtl Number (\( Pr \))

Prandtl number is a dimensionless number, named after the German scientist Ludwig Prandtl (1875-1953). The Prandtl number is a measure of relative effectiveness of momentum and energy diffusion in the velocity and thermal boundary layers, respectively. It physically relates the relative thickness of the hydrodynamic boundary layer and thermal boundary layer. It is defined as

\[
Pr = \frac{\text{Kinematic viscosity}}{\text{Thermal diffusivity}} = \frac{\nu}{\alpha} = \frac{\mu/\rho}{\kappa/\rho C_p}
\]

Most of the gases Prandtl number is near unity indicating that momentum and energy transfer by diffusion are of the same order. For small values of the Prandtl number (i.e. less than 1) in a given fluid indicates that thermal diffusion occurs at a greater rate than momentum diffusion and hence heat conduction is more effective than convection, but for fluid with Prandtl number greater than one, the situation is reversed.

### 1.9.3 Thermal Grashof Number (\( Gr \))

(21)
The thermal Grashof number or simply Grashof number is a dimensionless quantity used in analyzing the velocity distribution in free convection systems. It is named after the German engineer Franz Grashof (1826-1893). The Grashof number is the ratio of the buoyancy force to the viscous force in the fluid. This number is of great importance and plays the same role as the Reynolds number \(Re\) in forced convection. A critical value of the Grashof number is used to indicate transition from laminar to turbulent flow in free convection. Similarly, the critical value of the Reynolds number is used to indicate transition from laminar to turbulent flow in force convection.

The combined effects of free and forced convection must be considered when \(\frac{Gr}{Re^2} = 1\)

For \(\frac{Gr}{Re^2} \gg 1\) forced convection is negligible and for \(\frac{Gr}{Re^2} \ll 1\) free convection effects are to be neglected. The product of the Grashof number \(Gr\) and the Prandtl number \(Pr\) are called Rayleigh number \(Ra\). It is defined as \(Ra = Gr.Pr\). It is a dimensionless number that characterizes convection problems in heat transfer.

For a vertical circular cylinder of radius \(r_0\), the thermal Grashof number is defined as

\[
Gr = g\beta r_0 \frac{T_w' - T_\infty'}{v^2}
\]

### 1.9.4 Mass Grashof Number (\(Gc\))

Mass Grashof number \(Gc\) is a dimensionless number defined as the ratio of buoyancy force due to concentration difference to viscous force. It is significant in free convection flows involving mass transfer. This is because the density variation in free convection mass transfer is due to species concentration difference and hence the buoyancy force in natural convection mass transfer is incorporated into \(Gc\).

For a vertical circular cylinder of radius \(r_0\), the mass Grashof number \(Gc\) is defined as,

\[
Gc = g\beta^* r_0 \frac{C'_w - C'_\infty}{v^2}
\]

### 1.9.5 Nusselt Number (\(Nu\))

In fluid dynamics the quantity of heat exchanged between the solid body and the fluid is derived from using a coefficient of heat transfer \(\alpha(x)\), and is defined by Newton’s law of cooling as \(q(x) = \alpha(x)(T_w' - T_\infty')\)

(22)
But the quantity of heat transfer between the solid body and fluid takes place due to conduction. Therefore, according to Fourier's law \( q(x) = -k \left( \frac{\partial T}{\partial n} \right) \), where \( n \) is the direction of the normal to the surface of the solid body. From these two laws, non-dimensional coefficient of heat transfer, known as Nusselt number \( Nu \) and is defined by

\[
Nu = \frac{L \alpha(x)}{k} = \frac{L}{(T_w' - T_o') \left( \frac{\partial T}{\partial n} \right)_{n=0}}.
\]

The characteristic length (\( L \)) is determined by the direction of the growth (thickness) of the boundary layer. Nusselt number is a non-dimensional number. To honour the inventor German engineer Wilhelm Nusselt (1882-1957) of this number, therefore this dimensionless number is called Nusselt number.

Nusselt number also can be defined as the ratio of convective to conductive heat transfer across the boundary. If \( Nu=1 \), this indicates that the convection and conduction terms have relatively similar magnitude and hence it is characterized by the laminar flow. For large value of \( Nu \), the convective term is dominant and it is characterized by turbulent flows (usually \( Nu \) lies between 100 and 1000). For free convection, the Nusselt number can be defined as a function of the Reyleigh number and Prandtl number. But for forced convection, the dimensionless number can be expressed as a function of the Reynolds number and Prandtl number.

**1.9.6 Schmidt Number (Sc)**

The dimensionless number i.e. Schmidt number was named after the German engineer Ernst Heinrich Wilhelm Schmidt (1892-1975). It is defined as the ratio of momentum diffusivity (viscosity) and mass diffusivity. This dimensionless number can be expressed as the ratio of the shear component for the diffusivity to the diffusivity for mass transfer \( D \). This number physically signifies the relative thickness of the hydrodynamic layer and mass transfer boundary layer. Mathematically, the Schmidt number \( Sc \) is defined as

\[
Sc = \frac{\nu}{D} = \frac{\mu}{\rho D}.
\]

The Schmidt number is important in problems involving both momentum and convection mass transfer. It assumes the same importance in mass transfer as the Prandtl number in convection heat transfer problems.

**1.9.7 Sherwood Number (Sh)**

(23)
Sherwood number is a non-dimensional number which is used in mass transfer system. This dimensionless number is defined as the ratio of convective to diffusive mass transport, and it is named in honour of American chemical engineer Thomas Kilgore Sherwood (1903-1976). It is defined as

$$Sh = \frac{\text{Convective mass transfer coefficient}}{\text{Diffusive mass transfer coefficient}}$$

Using dimensional analysis, it can be expressed as a function of the Reynolds and Schmidt numbers. i.e. \(Sh=f(Re, Sc)\). Sherwood number represents the ratio between mass transfer by convection and mass transfer by diffusion. It is analogous to the Nusselt number in heat transfer.

1.9.8 Skin-Friction

A body of arbitrary shape and orientation when immersed in a fluid stream experiences two types of forces from the flow. One of the forces on the body in the direction of motion of fluid is called Drag force. The other force on the body in the direction normal to the flow direction is called Lift force. These two forces are produced by tangential and normal stresses. The drag due to tangential stress (shearing stress) is called friction or skin friction or viscous drag. The drag due to normal stress is called form or pressure drag.

Concerning to the velocity boundary layer, the shearing stress at the wall is given by

$$\tau = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}$$

according to the Newton’s law of friction.

The most significant applications of boundary layer theory is the calculation of the skin friction which acts on a body as if moved through a fluid. Some examples are drag experienced by flat plate at zero incidences, the drag of a ship, of an aero plane wing etc.

1.10 Laplace Transformation Method

The knowledge of Laplace Transforms is an essential part of mathematical background required by Scientists and Engineers. This is because the Laplace Transform Methods provide an easy and effective means for the solution of many problems arising in Science and Engineering. The Laplace transform is named after French Mathematician and astronomer Pierre-Simon Laplace (1749-1827) who used a similar transform (now called Z-transform) in the year 1790 in his work on probability theorem. The current use of the
transform came after World War II. The Laplace transformation replaces a given function 
$F(t)$ by another function $f(s)$. Then Laplace transformation reduces an ordinary 
differential equation with some given initial conditions into an algebraic equation in terms 
of $f(s)$. Finally, using inverse Laplace transformation we recover the original function. 
Hence the Laplace transform Method is especially useful for initial value problems, as it enables us to solve the problem without the trouble of finding the general solution first 
and then evaluating the arbitrary constants. The Laplace transform of a function $f(t)$, 
defined for all real numbers $t \geq 0$, the Laplace transform of $f(t)$ is a function of a new 
variable $s$ given by 
$$F(s) = L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) dt$$ 
Here new variable $s$ is real or complex, the function $F(s)$ is the Laplace transform of $f(t)$ and $L$ is Laplace transform operator. The operation of multiplying $f(t)$ by $e^{-st}$ and integrating from 0 to $\infty$ is called Laplace 
transformation. The inverse Laplace transform is given by 
$$f(t) = L^{-1}\{F(s)\}$$ 
Here $f(t)$ is called an inverse Laplace transform of $F(s)$ and $L^{-1}$ is known as the inverse Laplace 
transformation operator.

1.11 Bessel Function 
Bessel function first introduced by Bessel, in 1824, while discussing a problem in 
astronomy, the Bessel functions have received the most extensive treatment among all the 
special functions. Bessel function is usually defined as a particular solution of a linear 
differential equation of the second kind known as Bessel's equation. Bessel's equation 
arises when finding separable solutions to Laplace's equation and the Helmholtz equation 
in cylindrical or spherical co-ordinates. Bessel functions are therefore basically important 
for many problems of wave propagation and static potentials. The Bessel's differential 
equation of order $n$ is defined as, 
$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$ 
where $n$ is a non-negative constant. The two independent solutions of the Bessel's 
equation are called Bessel function of first kind of order $n$ and Bessel function of second 
kind of order $n$ respectively. 

(25)
Bessel's function of the first kind of order $n$ is denoted by $J_n(x)$ and is defined as

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}, \quad n \text{ is not integer.}$$

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!(n+r)!} \left(\frac{x}{2}\right)^{2r+n}, \quad n \text{ is an integer.}$$

Bessel's function of the second kind of order $n$ is denoted by $Y_n(x)$ and is defined by

$$Y_n(x) = \frac{J_n(x) \cos(n\pi) - J_{-n}(x)}{\sin(n\pi)}, \quad n \text{ is not an integer.}$$

$$Y_n(x) = \frac{L_{-n}}{\pi} \frac{J_v(x) \cos(v\pi) - J_{-v}(x)}{\sin(v\pi)}, \quad n \text{ is an integer.}$$

Modified Bessel Functions

The Bessel functions are valid for complex arguments $x$ as well as purely imaginary argument. We obtain the modified Bessel's equation as follows

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2)y = 0$$

In this case, the solutions to the Bessel equation are called the modified Bessel functions of the first kind and second kind of order $n$ and are denoted by $I_n(x)$ and $K_n(x)$ respectively and defined by

$$I_n(x) = i^{-n}J_n(ix) = \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}$$

$$K_n(x) = L_{-n} \pi \frac{I_{-v}(x) - I_v(x)}{\sin(v\pi)}$$

Relations satisfied by the Bessel functions

$$J_0'(x) = -J_1(x), \quad Y_0'(x) = -Y_1(x)$$

$$2 \frac{d}{dx} [J_1(x)] = J_0(x) - J_2(x), \quad 2 \frac{d}{dx} [Y_1(x)] = Y_0(x) - Y_2(x)$$

(26)
where \( J_0(x), J_1(x), J_2(x), Y_0(x), Y_1(x) \) and \( Y_2(x) \) are the Bessel functions of first and second kind of order zero, one and two respectively.

\[
J_{-n}(x) = (-1)^n J_n(x), \quad Y_{-n}(x) = (-1)^n Y_n(x)
\]

\[
\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x), \quad \frac{d}{dx}[x^n Y_n(x)] = x^n Y_{n-1}(x)
\]

\[
\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x), \quad \frac{d}{dx}[x^{-n} Y_n(x)] = -x^{-n} Y_{n+1}(x)
\]

\[
x J'_n(x) + n J_n(x) = x J_{n-1}(x), \quad x Y'_n(x) + n Y_n(x) = x Y_{n-1}(x)
\]

\[
x J'_n(x) - n J_n(x) = -x J_{n+1}(x), \quad x Y'_n(x) - n Y_n(x) = -x Y_{n+1}(x)
\]

\[
J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x), \quad Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)
\]

\[
J_{n-1}(x) - J_{n+1}(x) = 2 J'_n(x), \quad Y_{n-1}(x) - Y_{n+1}(x) = 2 Y'_n(x)
\]

\[
J_n(x) Y'_n(x) - J'_n(x) Y_n(x) = \frac{2}{x\pi}
\]

**Relations satisfied by the Modified Bessel Functions**

\[
I'_0(x) = I_1(x), \quad K'_0(x) = -K_1(x)
\]

Where \( I_0(x), I_1(x), K_0(x) \) and \( K_1(x) \) are called the modified Bessel functions of first and second kind of order zero and one respectively.

\[
I_n(x) = I_{-n}(x), \quad K_n(x) = K_{-n}(x)
\]

\[
\frac{d}{dx}[x^n I_n(x)] = x^n I_{n-1}(x), \quad \frac{d}{dx}[x^n K_n(x)] = -x^n K_{n-1}(x)
\]

\[
\frac{d}{dx}[x^{-n} I_n(x)] = x^{-n} I_{n+1}(x), \quad \frac{d}{dx}[x^{-n} K_n(x)] = -x^{-n} K_{n+1}(x)
\]

\[
I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x} I_n(x), \quad K_{n-1}(x) - K_{n+1}(x) = -\frac{2n}{x} K_n(x)
\]

\[
I_{n-1}(x) + I_{n+1}(x) = 2I'_n(x), \quad K_{n-1}(x) + K_{n+1}(x) = -2K'_n(x)
\]

(27)
\[ xI'_n(x) + nI_n(x) = xI_{n-1}(x) \quad xK'_n(x) + nK_n(x) = -xK_{n-1}(x) \]

\[ xI'_n(x) - nI_n(x) = xI_{n+1}(x) \quad xK'_n(x) - nK_n(x) = -xK_{n+1}(x) \]

\[ I_n(x)K'_n(x) - I'_n(x)K_n(x) = -\frac{1}{x} \quad I_n(x)K_{n+1}(x) + I_{n+1}(x)K_n(x) = \frac{1}{x} \]

Relations between Bessel Functions and Modified Bessel Functions

\[ I_0(ix) = J_0(x) \quad K_0(ix) = -\frac{\pi}{2i}[J_0(x) - iY_0(x)] \]

\[ I_n(x) = (-i)^n J_n(ix) \quad K_n(x) = \frac{\pi}{2i}[J_n(x) + iY_n(x)] \]

\[ I_n(xe^{\pm\pi i/2}) = e^{\mp in\pi/2} J_n(x) \]

\[ K_n(xe^{\pm\pi i/2}) = \pm \frac{i\pi}{2} e^{\mp in\pi/2} [-J_n(x) \pm iY_n(x)] \]

Applications of Bessel Functions

There is a wide range of different applications of Bessel functions including mathematical physics and various technical problems. Bessel function theory are used for solving problem of acoustics, radio physics, hydrodynamics, atomic and nuclear physics. Its application to heat conduction theory, including dynamical and linked problems. Bessel’s equation arises when solving boundary value problems such as whose solution connected with integration of the Laplace equation, the wave equation, the heat equation, especially when working in cylindrical or spherical coordinates. In solving problems in cylindrical coordinate systems and in spherical coordinate systems, one gets Bessel functions of integer order and half integer order respectively. For examples

- Electromagnetic waves in a cylindrical waveguide
- Heat conduction in a cylindrical object
- Modes of vibration of a thin circular( or annular) artificial membrane( such as a drum or other membranophone)
- Diffusion problems on a lattice
• Solutions to the radial Schrödinger equation (in spherical and cylindrical coordinates) for a free particle.

• Frequency-dependent friction in circular pipelines.

1.12 Objective of the present study:
Heat transfer is commonly associated with fluid dynamics. The transfer and conversion of energy from one form to another is basic to all heat transfer processes and hence they are governed by the first as well as the second law of thermodynamics. The knowledge of temperature distribution is essential in heat transfer studies because of the fact that the heat flow takes place only whenever there is a temperature gradient in a system. The study of temperature distribution and heat transfer is of great importance to engineers because of its almost universal occurrence in many branches of science and engineering. Free or natural convection is the principal mode of heat transfer from pipes, transmission lines, refrigerating coils, hot radiations and many other practical situations in everyday life. Mass transfer may also take place due to convection between a moving fluid and a surface and between two relatively immiscible moving fluids both under natural and forced conditions. In fact there exists a similarity between mass transfer and heat transfer by convective processes. Our discussion is mainly restricted to convective flow past vertical cylinder. In this thesis, we have considered some unsteady free convection flows past vertical circular cylinder under different physical situations. In each problem of this thesis, the closed form solutions of the governing partial differential equations are obtained in terms of Bessel functions and modified Bessel functions by Laplace transform technique.

1.13 Literature Review/ Survey of Relevant Literature
1.13.1 On Natural Convection
Natural convection flows under the influence of gravitational force have been investigated most extensively because they occur frequently in nature as well as in science and engineering applications. When a heated surface is in contact with the fluid, the result of temperature difference causes buoyancy force, which induces the natural convection. Before 1948, there was less experimental work on the vertical cylinder than on the horizontal cylinder. One of the reasons was probably that there was an additional variable in this case, namely the height of the vertical cylinder. It is true that the length of
the horizontal cylinder also plays its part, but since the various zones are cooled by parallel currents, they do not influence one another in sections that are far enough away from the ends, and the coefficient of heat transmission thus reaches a limit with increasing length, in contradiction to the vertical cylinder, where coefficient of heat transfer decreases with increasing height of the cylinder. Elenbaas [29] used Langmuir's stagnant film model to evaluate the heat transfer coefficient for vertical cylinders under the uniform wall temperature. Sparrow and Gregg [96] refined this method and reworked this problem using a power series solution. LeFrve and Ede [61] executed the integral method to the laminar boundary layer equations in cylindrical coordinates. The numerical results were almost in agreement with those by Sparrow and Gregg [96]. Subsequently, Yang [108] made a study of unsteady laminar free convection on vertical plates and cylinders to establish necessary and sufficient conditions under which similarity solutions are possible. Goldstein and Briggs [46] analyzed the transient free convection about vertical plates and circular cylinders to a surrounding initially quiescent fluid. They presented analytical solutions for the infinite cylinders. Dring and Gebhart [28] presented the transient natural convection results from thin wires in liquids. An experimental and analytical study was reported by Evans et al. [31] for transient free convection in a vertical cylinder. Fujii and Uehara [32] studied the heat transfer of non-isothermal vertical cylinders with large curvatures by series solutions. A boundary layer analysis of free convection heat transfer from a vertical cylinder with uniform heat flux at its surface was studied by Nagendra et al. [76].

Bejan [6], Gebhart [44], Sachdeva [90] and Ostrach [80] presented a detailed survey on the free convection heat transfer problem from vertical elements. Bottemanne [8] discussed the experimental results of pure and simultaneous heat and mass transfer by free convection about a vertical cylinder for $Pr=0.71$ and $Sc=0.63$. Minkowycz and Sparrow [69] studied the natural convection along a vertical cylinder with a constant surface temperature by a local non-similarity solution. Minkowycz and Cheng [70] made an analysis for natural convection flow about a vertical cylinder embedded in a saturated porous medium, where surface temperature of the cylinder varies as a power function of distance from the leading edge. Cebeci [12] had given a numerical results for Prandtl numbers from 0.01 to 100, extending earlier work that had been for $Pr = 1$ and 0.72. The effects of Prandtl was further studied by Crane [17].

Chen and Yuh [15] dealt with steady combined heat and mass transfer processes for both conditions of uniform wall temperature and concentration and with uniform heat and
mass fluxes. Their study covered a wide range of radii and Prandtl number. Aziz and Na [4] used the perturbation technique to solve the laminar natural convection flow from an isothermal thin vertical cylinder and established the efficiency over the popular local non-similarity and finite difference methods. Chen [16] investigated the steady free convection from a vertical needle with variable wall heat flux and found that there was significant influence of its shape, size and wall temperature variation upon the flow and heat transfer. Bejan and Khair [7] studied the vertical natural convection boundary layer flow due to the combined buoyancy of thermal and mass diffusion. Surma Devi et al. [98] considered the power-law variation in wall temperature along slender vertical cylinders and needles, for large curvature parameter. Lee et al. [62, 63] investigated the effect of buoyancy force on the mixed convection flow configuration under uniform surface temperature and heat flux conditions for the entire range of mixed convection, from pure forced convection at one end to pure free convection at the other end. Gorla [48] carried out the problem of combined forced and free convection in the boundary layer flow of a micro polar fluid on continuous moving cylinder. Similarly, mixed convection along slender vertical cylinder with variable surface temperature was analyzed by Heckel et al. [52]. Also Lee et al. [64] investigated, the problem of natural convection in laminar boundary layer flow along slender vertical cylinders and needles for power law variations in wall temperature.

Further, Takhar et al. [100] had considered the mixed convection flow over a vertical thin cylinder due to the combined effect of the thermal and mass diffusion. Both uniform wall temperature and uniform wall heat flux conditions were included in the analysis. The governing boundary layer equations were solved by using an implicit finite-difference Scheme. Velusamy and Garg [102] presented the numerical solution for transient natural convection over heat generating vertical cylinders of various thermal capacities and radii. The rate of propagation of the leading edge effect has been given special consideration by them. Ganesan and Loganathan [39, 40, 41] had studied the effects of heat and mass transfer on the natural convection flow of an incompressible viscous fluid past a semi-infinite isothermal vertical cylinder under different physical situations. Kwang Hyo Chung and Jae Min Hyun [60] analyzed the flow of an incompressible Boussinesq-fluid in a vertical cylinder. The effect of circumferential variation of sidewall temperature on buoyant convection in a vertical cylinder was given special consideration. Ganesan and Rani [34, 36] studied the transient flow along vertical cylinder for uniform wall temperature and concentration. Hata et al. [50] had studied the experimental data of

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natural convection heat transfer from a vertical cylinder with two cylinders of different dimensions for a wide range of heat flux and also analyzed the numerical solution of local Nusselt numbers on a vertical cylinder from a theoretical laminar natural convection equations for the same conditions as the experimental ones. The effect of lateral mass flux on mixed convection heat and mass transfer in a saturated porous medium adjacent to an inclined permeable surface was analyzed by Singh and Sharma [93]. Das et al. [21] studied the vertical plate convection problem with periodic temperature variations. Rani [85] carried out to study the effects of variable surface temperature and concentration along a vertical cylinder. A Crank- Nicolson type of implicit finite-difference method was used to solve the governing equations. Later on, the laminar upward mixed convection heat transfer for thermally developing air flow in the entrance region of a vertical circular cylinder under buoyancy effect and wall heat flux boundary condition had been numerically investigated by Mohammed and Yasin [73]. Typical of the experimental studies of natural convective heat transfer from vertical cylinders were considered by those of Fukusawa and Iguchi [33], Welling et al. [106] and Jarall and Campo [58]. Gori et al. [47] considered theoretically the natural convection around a vertical thin cylinder with constant wall heat flux. Heat transfer from a short cylinder with an exposed top surface had been considered by Oosthuizen [78, 79]. The steady mixed convection boundary layer flow along a vertical cylinder with prescribed surface heat flux was investigated by Ishak Anuar [57]. Recently, Deka and Paul [24] had studied on transient natural convection flow about an infinite moving vertical circular cylinder with constant temperature. Again, Deka and Paul [25, 26, 27] had studied unsteady natural convective flow past an infinite vertical cylinder with combined effects of heat and mass transfer. And they also discussed an analytical treatment for the unsteady one-dimensional natural convective flow past an infinite moving vertical cylinder in the presence of thermal stratification. They use Laplace transform technique to solve the governing boundary layer equations.

1.13.2 On Radiation

When the temperature of the surrounding fluid is rather high, the effects of radiation are of vital importance in the study of geological formations, in the exploration and thermal recovery of oil, and in the assessment of aquifers, geothermal reservoirs and underground nuclear waste storage sites. In these cases, it is necessary to consider the radiation effects in free convection flows. Studies of free convection flow along a vertical or horizontal
cylinder are important in the field of geothermal power generation and drilling operations where the free stream and buoyancy induced fluid velocities are roughly the same order of magnitude. Studies with interaction of thermal radiation and free convection were made Arpaci [3]; Cess [11], Cheng and Ozisik [14], Bankston et al. [5], Rapits [87], Hossian and Takhar [54, 55]. In all these papers, flow is considered to be steady. Yih [109] investigated the combined radiation and free convection flow over a vertical cylinder. Ramanaiah and Kumaran [84] discussed free convection about a permeable cone and a cylinder subjected to radiation boundary condition. A numerical study of radiation effect on MHD transient mixed-convection flow over a moving vertical cylinder with constant heat flux through a porous medium was analyzed by Elgazery and Hassan [30]. Sokovishin and Shapiro [94] considered the effect of radiation on free convective heat liberation from the surface of a vertical cylinder and found that the radiative component of thermal flux equalizes the surface temperature. The unsteady flow past a moving vertical plate in the presence of free convection and radiation were studied by Das et al. [19]. Hossain et al. [56] discussed the radiation-conduction interaction on mixed convection from a horizontal circular cylinder. The authors discuss the Nusselt number, Sherwood number for both generative and destructive reaction. Ganesan and Loganathan [43] studied the radiation effects of heat and mass transfer on the natural convection of an incompressible viscous fluid past a moving semi-infinite isothermal vertical cylinder in the vertically upward direction. They were using an implicit finite-difference Scheme of Crank-Nicolson type to solve the governing equations. Reddy and Reddy [89] investigated the radiation and mass transfer effects on hydro magnetic free convection flow of a viscous incompressible optically thick fluid past a moving vertical cylinder. Suneetha and Reddy [97] analyzed the radiation and mass transfer effects on MHD free convection flow of a viscous incompressible fluid past a moving vertical cylinder in a porous medium. To solve the non-linear coupled equations, they used finite difference scheme of Crank-Nicolson type.. Loganathan et al. [65, 66] carried out works to study the influence of radiation and magneto hydrodynamic effects on the unsteady laminar free convection flow over a moving vertical cylinder. Also they analyzed the non-linear MHD flow with heat and mass transfer of an incompressible, viscous and electrically conducting fluid on moving semi-infinite vertical cylinder in the presence of chemical reaction. For both these problem they solved the governing boundary layer equations using an implicit finite difference scheme of Crank-Nicolson method.
1.13.3 On Chemical Reaction

In nature, the presence of pure air or water is very difficult. It is always possible that a foreign mass is either present naturally in air or water or foreign masses (as impurities) are mixed with air or water. Again, due to presence of foreign masses in fluid in the process of convection, chemical reaction also takes place. Therefore, buoyancy driven heat and mass transfer flows with chemical reactions under different physical situation has been studied among many researchers. Chambre and Young [13] had analysed a first order chemical reaction in the neighbourhood of a horizontal plate. The analytical solution for mass transfer with chemical reaction of first order was discussed by Apelblat [2]. Das et al. [18, 20] presented an analytic investigation of the effect of a homogeneous first order chemical reaction on the flow past an impulsively started infinite vertical plate with uniform heat flux and mass transfer. Later on they studied the mass transfer effect on moving isothermal vertical plate in the presence of chemical reaction. The dimensionless governing equations were solved by the usual Laplace transform technique and the solutions are valid only at lower time level. The numerical solution of transient natural convection flow over a vertical cylinder under the combined buoyancy effects of heat and mass transfer along with chemical reaction was analyzed by Ganesan and Rani [37]. Ganesan and Loganathan [42] studied free convection boundary layer flow of a viscous and incompressible fluid past an impulsively started semi-infinite vertical cylinder with uniform heat and mass fluxes and chemically reactive species. Numerical solutions were derived by the finite-difference scheme of Crank-Nicolson type. Kabeir and Abdou [59] presented the effect of non linear MHD flow with combined heat and mass transfer of an incompressible, viscous, electrically conducting fluid with chemical reaction and heat generation. Vijayalakshmi and Muthucumaraswamy [104] studied the combined radiation and chemical reaction effects on isothermal moving vertical plate with variable mass diffusion. Further, Mahapatra et al. [68] investigated the heat and mass transfer effect on a steady flow of viscous fluid through a porous medium bounded by a porous surface in the presence of homogeneous chemical reaction of first order. Muthucumaraswamy and Shankar [75] studied the first order chemical reaction on unsteady flow past a uniformly accelerated isothermal infinite vertical plate with heat and mass transfer in the presence of thermal radiation.