CHAPTER 8
TRANSIENT FREE CONVECTION FLOW PAST AN ACCELERATED VERTICAL CYLINDER IN A ROTATING FLUID

8.1 Introduction
The rotating fluid flow plays a vital role in industry, namely, in geophysical, astrophysical and cosmical fluid dynamics. Many researchers have studied the problem of flow past rotating fluid under various physical aspects. Deka et al. [22] studied the flow of viscous incompressible rotating fluid induced by a uniformly accelerated plate. They observed that in presence of rotation the velocity profiles for varying time are not similar in contrast to the velocity profiles which are similar in absence of rotation.

Xu et al. [107] presented experimentally the flow of a homogeneous, incompressible, rotating fluid past a vertical circular cylinder oscillating laterally in a uniform free stream. The numerical simulation of the two-dimensional incompressible unsteady Navier-Stokes equations for streaming flow past a rotating circular cylinder was studied by Padrino and Joseph [81]. Mittal and Kumar [71] initiated the flow past a spinning circular cylinder placed in a uniform stream. A stabilized finite element method is utilized to solve the incompressible Navier-Stokes equations in the primitive variables formulation. The study of the dynamics of the spin-up time of an incompressible viscous rotating fluid was illustrated by Greenspan and Howard [49]. Hayat et al. [51] presented an analysis of an electrically conducting viscous fluid over a porous plate in a rotating system. They used Laplace Transform technique to solve the governing mathematical problem.

However, the flow past vertical cylinders in a rotating fluid has never been considered in the literature as far as analytical treatment is concerned. This may be due to the complicated mathematical calculation involved in the solution of such problems. The aim of the present paper is to investigate the boundary layer flow past an infinite vertical cylinder in a rotating fluid. It would be of interest to see how the flow past a vertical cylinder in a rotating fluid gets modified. This is because of the coriolis force due to the rotation manifests itself in changing the pattern of flows. We have discussed the subsequent flow when the cylinder started impulsively from rest (relative to the rotating fluid) moves with uniform acceleration in its own plane. The closed form solutions of the boundary layer governing partial
differential equations are obtained in terms of Bessel functions and modified Bessel functions by usual Laplace transform technique. The effects of various significant parameters on the flow and heat transfer characteristics are presented in graphical form. The mathematical formulation of the problem is presented in Section 8.2, followed by the analytic solution procedure in Section 8.3. Results and discussion are presented in Section 8.4 and conclusions in Section 8.5.

8.2 Mathematical Formulation

Consider the unsteady free convection flow of an incompressible viscous fluid past an infinite vertical cylinder of radius $r_0$. The x-axis is taken vertically upward along the axis of the cylinder and the radial coordinate $r$ is taken normal to it. Initially at $t' \leq 0$, it is assumed that the cylinder is at rest and the cylinder and fluid are at the same temperature $T_\infty$. At $t' > 0$, the temperature of the cylinder raised to constant temperature $T_v$ and the cylinder starts accelerate with acceleration $ct'$ in its own plane relative to the fluid which is rotating in unison with uniform angular velocity $\omega$ about the axis of the cylinder. It is also assumed that all the fluid properties are constant except for the density in the buoyancy term, which is given by the usual Boussinesq's approximation. Under these assumptions the governing boundary layer equations are given by,

$$\frac{\partial u}{\partial t'} = 
\nu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + 2\omega v + g\beta(T' - T_\infty) \cos \omega t' \quad (8.1)$$

$$\frac{\partial v}{\partial t'} = \nu \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) - 2\omega u - g\beta(T' - T_\infty) \sin \omega t' \quad (8.2)$$

$$\frac{\partial T'}{\partial t'} = \alpha \left( \frac{\partial^2 T'}{\partial r^2} + \frac{1}{r} \frac{\partial T'}{\partial r} \right) \quad (8.3)$$

with initial and boundary conditions,

$$\begin{align*}
t' \leq 0 : & \quad u = 0, \quad v = 0, \quad T' = T_\infty \quad \text{for all} \quad r \\
 t' > 0 : & \quad u = ct', \quad v = 0, \quad T' = T_\infty \quad \text{at} \quad r = r_0 \\
 & \quad u \to 0, \quad v \to 0, \quad T' \to T_\infty \quad \text{as} \quad r \to \infty
\end{align*} \quad (8.4)$$

Introducing the non-dimensional quantities,
the governing equations (8.1) – (8.3) reduce to,

\[
\begin{align*}
\frac{\partial U}{\partial t} &= \frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} + 2\Omega V + Gr \cos \Omega t, \\
\frac{\partial V}{\partial t} &= \frac{\partial^2 V}{\partial R^2} + \frac{1}{R} \frac{\partial V}{\partial R} - 2\Omega U - Gr \sin \Omega t, \\
\frac{\partial T}{\partial t} &= \frac{1}{Pr} \left( \frac{\partial^2 T}{\partial R^2} + \frac{1}{R} \frac{\partial T}{\partial R} \right).
\end{align*}
\]

with the corresponding initial and boundary conditions,

\[
\begin{align*}
t &\leq 0 : \quad U = 0, \quad V = 0, \quad T = 0 \quad \text{for all} \quad R, \\
t &> 0 : \quad U = t, \quad V = 0, \quad T = 1 \quad \text{at} \quad R = 1, \\
& \quad U \to 0, \quad V \to 0, \quad T \to 0 \quad \text{as} \quad R \to \infty.
\end{align*}
\]

Equations (8.6) and (8.7) can be combined into a single equation,

\[
\frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial R^2} + \frac{1}{R} \frac{\partial W}{\partial R} - 2i\Omega W + GrTe^{-\Omega t}.
\]

where \( W(R, t) = U(R, t) + iV(R, t) \) with

\[
\begin{align*}
t &\leq 0 : \quad W = 0, \quad T = 0 \quad \text{for all} \quad R, \\
t &> 0 : \quad W = t, \quad T = 1 \quad \text{at} \quad R = 1, \\
& \quad W \to 0, \quad T \to 0 \quad \text{as} \quad R \to \infty.
\end{align*}
\]

8.3 Solution Procedure

To solve the governing non-dimensional equations (8.10) and (8.8) subject to initial and boundary conditions (8.11), we apply Laplace transform technique. Laplace transforms of equations (8.10) and (8.8) subject to initial conditions (8.11) give,

\[
\frac{d^2 \tilde{W}(R, s)}{dR^2} + \frac{1}{R} \frac{d \tilde{W}(R, s)}{dR} - (s + 2i\Omega) \tilde{W}(R, s) = -Gr \tilde{T}(R, s + i\Omega).
\]
\[
\frac{d^2 \overline{T}(R,s)}{dR^2} + \frac{1}{R} \frac{dT(R,s)}{dR} - sPr \overline{T}(R,s) = 0
\]  
(8.13)

where \( \overline{W}(R,s) \) and \( \overline{T}(R,s) \) are the Laplace transforms of \( W(R,t) \) and \( T(R,t) \) respectively, \( s \) being the Laplace transform parameter. Solutions of the equation (8.12) subject to the transformed initial and boundary conditions for \( T \) in (8.11) we have,

\[
\overline{T} = \frac{K_0(\sqrt{sPr} R)}{sK_0(\sqrt{sPr})}
\]  
(8.14)

Using (8.14), the equation (8.12) reduces to,

\[
\frac{d^2 \overline{W}(R,s)}{dR^2} + \frac{1}{R} \frac{d\overline{W}(R,s)}{dR} - (s + 2i\Omega) \overline{W}(R,s) = -Gr \frac{K_0(R\sqrt{(s + i\Omega)Pr})}{(s + i\Omega)K_0(\sqrt{(s + i\Omega)Pr})}
\]  
(8.15)

To solve (8.15), we have applied variation of parameter technique and subject to the transformed initial and boundary conditions for \( W \) from (8.11) is obtained as,

\[
\overline{W} = \frac{K_0(\sqrt{s + 2i\Omega} R)}{s^2 K_0(\sqrt{s + 2i\Omega})} + \frac{Gr}{K_0(\sqrt{s + 2i\Omega})} \frac{K_0(\sqrt{(s + i\Omega)Pr} - (s + 2i\Omega))}{(s + i\Omega)(s + 2i\Omega)}
\]  
(8.16)

We use the complex inversion theorem [cf. Carslaw and Jaeger [9]] to obtain the inverse Laplace transform of (8.14) and (8.16). The inverse Laplace transform of (8.16) gives

\[
W = Gr e^{2i\Omega t} \frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{sR} \frac{1}{K_0(\sqrt{s})} \frac{1}{(s + i\Omega)(s + 2i\Omega)Pr - s} ds
\]

\[
-Gr e^{-i\Omega t} \frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{sR} \frac{1}{K_0(\sqrt{sPr})} \frac{1}{s(sPr - (s + i\Omega))} ds
\]

\[
+ e^{-2i\Omega t} \frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{sR} \frac{1}{K_0(\sqrt{s})} \frac{1}{(s - 2i\Omega)^2} ds
\]

\[
W = Gr e^{2i\Omega t} \frac{1}{2i\pi} I_1 - Gr e^{-i\Omega t} \frac{1}{2i\pi} I_2 + e^{-2i\Omega t} \frac{1}{2i\pi} I_3
\]  
(8.17)

where,

\[
(126)
\]
The integrand in $I_1$ has a branch point at $s = 0$ and simple poles at $s = i\Omega$ and $s = \frac{i\Omega Pr}{Pr-1}$.

Now $K_0(\sqrt{s})$ do not have zero at any point in the real and imaginary plane, if the branch cut is made along the negative real axis. To obtain $W(t, R)$ from $\bar{W}(s, R)$, we use the adjoining contour shown in figure 8.1. Therefore the line integral $I_1$ may be replaced by the limit of the sum of the integrals over FE, ED, DC, CB and BA as $S_1 \to \infty$ and $S_0 \to 0$. Since the integral approaches zero along the paths FE, DC, BA and FE as $S_1 \to \infty$ and $S_0 \to 0$; accordingly the integral alongs paths CB and EB has to be evaluated.

![Figure 8.1: path of contour integration for the inverse integral.](image)

Along the path CB and ED we choose $s = \nu^2 e^{i\pi}$ and $s = \nu^2 e^{-i\pi}$ respectively. Then on the path CB and ED, the integral $I_1$ assumes the values,
\[ I_{CB} = 2 \int_0^\infty e^{-V^2} \frac{J_0(RV) - i Y_0(RV)}{J_0(V) - i Y_0(V)} \frac{VdV}{(V^2 + i\Omega)(V^2 - V^2Pr - i\Omega Pr)} \]  \hfill (8.22)

On the path ED,

\[ I_{BD} = -2 \int_0^\infty e^{-V^2} \frac{J_0(RV) + i Y_0(RV)}{J_0(V) + i Y_0(V)} \frac{VdV}{(V^2 + i\Omega)(V^2 - V^2Pr - i\Omega Pr)} \]  \hfill (8.23)

Now, the residue of the integral of \( I_1 \) at the simple poles \( s = i\Omega \) and \( s = i\Omega Pr/(Pr - 1) \) are,

\[ \frac{e^{i\Omega}}{i\Omega} K_0\left( \frac{R\sqrt{i\Omega}}{i\Omega} \right) \] and \( \frac{(Pr - 1)e^{i\Omega(K_0((Pr - 1)))}}{i\Omega} K_0\left( \frac{R\sqrt{i\Omega Pr/(Pr - 1)}}{i\Omega} \right) \)

respectively.

Thus from the theory of residues we have,

\[ I_1 = I_{CB} + I_{BD} + 2\pi i \text{(sum of residues of } I_1) \]

\[ = 4i \int_0^\infty e^{-V^2} \frac{VdV}{(V^2 + i\Omega)(V^2 - V^2Pr - i\Omega Pr)} + 2i\pi \left\{ - \frac{e^{i\Omega}}{i\Omega} K_0\left( \frac{R\sqrt{i\Omega}}{i\Omega} \right) - \frac{(Pr - 1)e^{i\Omega(K_0((Pr - 1)))}}{i\Omega} K_0\left( \frac{R\sqrt{i\Omega Pr/(Pr - 1)}}{i\Omega} \right) \right\} \]  \hfill (8.24)

Similarly \( I_2 \) and \( I_3 \) are obtained as

\[ I_2 = 4iPr \int_0^\infty e^{-V^2/iPr} \frac{dV}{(V^2 + V^2Pr - i\Omega Pr)} \frac{dV}{(V^2 - V^2Pr - i\Omega Pr)} \] + \[ 2i\pi \left\{ - \frac{1}{i\Omega} - \frac{(Pr - 1)e^{i\Omega(K_0((Pr - 1)))}}{i\Omega} K_0\left( \frac{R\sqrt{i\Omega Pr/(Pr - 1)}}{i\Omega} \right) \right\} \]  \hfill (8.25)

\[ I_3 = -4i \int_0^\infty e^{-V^2} \frac{VdV}{(V^2 + i\Omega)^2} \] + \[ \frac{e^{i\Omega}}{2\sqrt{2i\Omega}} K_0\left( \sqrt{2\Omega} R \right) K_0\left( \sqrt{2\Omega} \right) + \frac{K_0\left( \sqrt{2\Omega} R \right) K_0\left( \sqrt{2\Omega} \right)}{2\sqrt{2i\Omega}} \]  \hfill (8.26)

Using (8.24), (8.25) and (8.26), we have solution for \( W \) as,

\[ (128) \]
\[ W = \frac{Gr e^{-i\Omega t}}{i\Omega} \left\{ -\frac{K_0(\sqrt{i\Omega})}{K_1(\sqrt{i\Omega})} \right\} \\
+ \frac{2Gr e^{-2i\Omega t}}{\pi} \int_0^{2\pi} \frac{e^{-i\Omega t} \Gamma_1(R,V) V}{(V^2 + i\Omega)(V^2 - V^2 Pr - i\Omega Pr)} dV \\
+ \frac{2Gr e^{-2i\Omega t}}{\pi} \int_0^{2\pi} \frac{e^{-i\Omega t} \Gamma_1(R,V) V}{V(V^2 - V^2 Pr - i\Omega Pr)} dV \\
\frac{2Gr e^{-i\Omega t}}{\pi} \int_0^{2\pi} \frac{e^{-i\Omega t} \Gamma_1(R,V) V}{(V^2 + 2i\Omega)^3} dV \]

(8.27)

In a similar manner, the inverse Laplace transform of \( \bar{T} \) is obtained as,

\[ T = 1 + \frac{2}{\pi} \int_0^{2\pi} e^{-\psi^2} \Gamma_1(R,V) \frac{dV}{V} \]

(8.28)

The non dimensional skin friction, \( \tau = \tau_x + i \tau_y = \frac{\partial W}{\partial R} \bigg|_{R=1} \) is obtained from the equation (8.27) as,

\[ \tau = \frac{-Gr e^{-i\Omega t}}{\sqrt{i\Omega}} \left\{ \frac{K_1(\sqrt{i\Omega})}{K_1(\sqrt{i\Omega})} \right\} \]

\[ + \frac{2Gr e^{-2i\Omega t}}{\pi} \int_0^{2\pi} e^{-i\Omega t} \Gamma_1(\sqrt{i\Omega}) V^2 \frac{dV}{(V^2 + i\Omega)(V^2 - V^2 Pr - i\Omega Pr)} \\
- \frac{2Gr Pr e^{-i\Omega t}}{\pi} \int_0^{2\pi} e^{-i\Omega t} \frac{\Gamma_2(V)}{(V^2 - V^2 Pr - i\Omega Pr)} dV \\
- \frac{2Gr e^{-2i\Omega t}}{\pi} \int_0^{2\pi} e^{-i\Omega t} \frac{\Gamma_2(V) V^2}{(V^2 + 2i\Omega)^3} dV \]

(8.29)

Also, the non dimensional Nusselt number, \( Nu = \frac{\partial T}{\partial R} \bigg|_{R=1} \) is obtained from the equation (8.28) as,

\[ Nu = \frac{2}{\pi} \int_0^{2\pi} e^{-\psi^2} \Gamma_2(V) dV \]

(8.30)

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\[ \Gamma_1(R,V) = \frac{J_0(RV)Y_0(V) - Y_0(RV)J_0(V)}{J_0^2(V) + Y_0^2(V)} \quad \text{and} \quad \Gamma_2(V) = \frac{J_1(V)Y_1(V) - Y_1(V)J_1(V)}{J_0^2(V) + Y_0^2(V)} \]

### 8.4 Results and Discussions

In order to get an physical insight into the problem, quantities such as velocity (axial and transverse), temperature, skin friction (axial and transverse), Nusselt number have been computed by assigning numerical values of the various parameters and the results are graphically shown in Figures 8.2-8.16.

The figures 8.2-8.3 shows the effects of rotation parameter \( \Omega \) and Prandtl number on axial and transverse velocity profiles at \( Gr = 5 \) and \( t = 0.4 \) against the radial distance from the surface of the cylinder. It is observed from the figures that the axial velocity decreases with increase in rotation parameter, whereas transverse velocity increases. Both axial and transverse velocity decreases with an increase in Prandtl number. As the Prandtl number increases the viscous forces tend to suppress the buoyancy force leads to a decrease in fluid velocity in the hydrodynamic boundary layer. The negative sign for \( V \) in the figures indicate that this component is transverse to the main flow direction in the clockwise direction.

The effect of Grashof number on axial and transverse velocity profiles at \( Pr = 0.71, \Omega = 1.4 \) and \( t = 0.4 \) are depicted in figure 8.4 and 8.5 respectively. It is clear from the figures that the axial velocity as well as the transverse velocity increases with increasing values of Grashof number.

Figures 8.6 and 8.7 depict the axial and transverse velocity profiles against time at \( R = 1.4, Gr = 5 \) and \( Pr = 0.71 \). It is observed from the figures that the velocity profiles show oscillatory behavior with increasing time. Also, the oscillation of axial velocity and transverse velocity decreases with increasing rotation parameter. Deka et al. [22] pointed out that for the flow past an accelerated plate in a rotating fluid, the axial and transverse velocities shows inertial oscillations at smaller times, while in our present study, at smaller time no significant oscillatory behavior is seen for axial component, but as time progresses, the oscillatory behavior is more prominent. On the other hand the transverse component is more oscillatory as compared to axial one. Further, it is observed that for both components, the oscillatory behavior is dying as the values of rotation parameter increases.

Figures 8.8 and 8.9 represent the effect of rotation parameter on axial and transverse skin friction against time at \( Gr=5 \) and \( Pr=0.71 \). It is observed from the figures that with increasing
time both the axial and transverse skin friction increases and the behavior is oscillatory. Also, magnitude of oscillation of the axial and transverse skin friction increase with increasing values of rotation parameter.

Figures 8.10 and 8.11 and figures 8.12 and 8.13 depict the effect of Grashof number and Prandtl number on skin friction against time. It is observed from the figure that oscillation of axial skin friction increases with increasing Grashof Number or decreasing Prandtl number, whereas the oscillation of transverse skin friction increases with decreasing Grashof Number or increasing Prandtl number. It is important to see that at smaller Grashof number and time, the axial skin friction is negative and then changes to positive value as time progresses. However, the trend is reversed for transverse component. This indicates that as time increases, the axial flow gets reversed and the flow is in the transverse direction and this reversed behavior with respect to $Gr$ is obviously seen from figure 8.4 and 8.5 showing the axial and transverse velocity components for different $Gr$.

The effect of Prandtl number plays an important role in temperature field. The Prandtl number $Pr$ physically relates the relative thickness of the viscous boundary layer and thermal boundary layer. Fig.8.14 shows the effects of $Pr$ on temperature field. It is seen that the magnitude of temperature is maximum at the surface of the cylinder and approaches to zero asymptotically. It is also seen that the temperature decreases as $Pr$ increases. Figure 8.15 depicts the temperature field for various values of $Pr$ with respect to time. Temperature increases as $Pr$ decreases. Though, initially temperature increases sharply but for larger time it becomes steady. Rate of heat transfer i.e. Nusselt number $Nu$ is presented in Figure 8.16. Initially the rate of heat transfer decreases, but after certain time it approaches to a fixed value. It is also observed that Nusselt number increases with increasing values of $Pr$.

8.5 Conclusions

In this paper an analytical observation is made for study on the effect of convective flow past an accelerated infinite vertical cylinder in rotating fluid. Solutions of the governing boundary layer equations of the flow model have been obtained by using Laplace transform technique. The conclusions of our study are as follows:

i. Rotation of the fluid causes oscillation to the flow.

ii. Axial velocity component increases with $\Omega$ and $Gr$ but decreases with increase in $Pr$.

iii. Transverse velocity component increases with $Gr$ and $\Omega$ but decreases with increase in $Pr$.

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iv. Axial skin friction increases with $Pr$ and $\Omega$ but decreases with increase in $Gr$.

v. Transverse skin friction increases with the increase in $Gr$ but decreases with increase in $Pr$ and $\Omega$.

vi. Temperature decreases with increase in Prandtl number but increases with time. Initially, it increases sharply with time but for large time it becomes steady.

vii. Nusselt number increases with $Pr$, but decreases for smaller time and attains a fixed value for larger time.

Fig. 8.2 Effects of $Q$ and $Pr$ on axial velocity profiles when $Gr=5$ and $t=0.4$. 

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Fig. 8.3 Effects of $\Omega$ and $Pr$ on transverse velocity profiles when $Gr=5$ and $t=0.4$

Fig. 8.4 Axial velocity profiles for different $Gr$ at $Pr=0.71$, $Q=1.4$ and $t=0.4$
Fig. 8.5 Transverse velocity profiles for different $Gr$ at $Pr=0.71$, $\Omega=1.4$ and $t=0.4$.

Fig. 8.6 Axial velocity profiles with respect to time when $R=1.4$, $Pr=0.71$, $Gr=5$. 

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Fig. 8.7 Transverse velocity profiles with respect to time when \( \Omega = 1.4, \, Pr = 0.71, \, Gr = 5 \).

Fig. 8.8 Effects of \( \Omega \) on axial Skin friction at \( Gr = 5 \) and \( Pr = 0.71 \).
Fig. 8.9 Effects of $\Omega$ on transverse Skin friction at $Gr=5$ and $Pr=0.71$

Fig. 8.10 Effects of $Gr$ on axial Skin friction at $\Omega=0.4$ and $Pr=0.71$
Fig. 8.11 Effects of $Gr$ on transverse skin friction at $\Omega = 0.4$ and $Pr = 0.71$

Fig. 8.12 Effects of $Pr$ on axial skin friction at $\Omega = 0.6$ and $Gr = 5$
Fig. 8.13 Effects of $Pr$ on transverse Skin friction at $\Omega = 0.6$ and $Gr = 5$.

Fig. 8.14 Effects of $Pr$ on temperature profile at $t = 1.2$. 

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Fig. 8.15 Temperature profile for different Prandtl number at $R=1.4$.

Fig. 8.16. Effects of Nusselt number with respect to time for various Prandtl number.

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