Chapter 7

Allocation for Combined Ratio Estimator

Summary

In chapter 6, we have investigated the problem of optimum allocation of sample size to strata for the strategies - stratified SRSWR and SRSWOR with separate ratio estimator. We have noted in chapter 2 that if the stratum ratios do not vary much from stratum to stratum, then the combined ratio estimator is preferable to separate ratio estimator. Therefore the problem of optimum allocation of sample size to strata for the strategies \{Stratified SRSWOR: \( \hat{Y}_{CR} \)\} and \{Stratified SRSWR: \( \hat{Y}_{CR} \)\} is studied under appropriate superpopulation model in this chapter.

Tschuprow - Neyman optimum allocation is derived under an appropriate superpopulation model for the strategies stratified SRSWR and SRSWOR with combined ratio estimator. Approximate allocations have been
worked out under different approximations such as large stratum size approxi-
mation. Approximate allocations under the assumption of equal SCV have also been obtained. Several particular cases of the various model based allocations have been deduced for certain values of superpopulation parameter $g$. The results have been illustrated with 2 live populations for stratified SRSWOR design.

7.1 Introduction

Consider once again a finite population $U = \{U_1, U_2, \ldots, U_n\}$ of N units divided into k strata of sizes $N_i$, $i = 1, 2, \ldots, k$ i.e.,

$$U = \{U_{11}, \ldots, U_{1N_1}, U_{21}, \ldots, U_{2N_2}, \ldots, U_{k1}, \ldots, U_{kN_k}\}$$

Suppose parameter of our interest is the population total or mean of the study variable $Y$. When information on a closely related variable is available, we can make use of it for improving the precision of the estimation by using ratio estimators. When the population is heterogenous, more precise estimates of the parameter of interest can be obtained by using the technique of stratification. Both the techniques viz., stratification and ratio estimation can be fruitfully simultaneously used compared to using either only i.e., Ratio estimation in stratified sampling can provide better estimates than stratified sampling without ratio estimates or ratio estimators without stratification.
We have already noted in Chapter 2 that there are two types of ratio estimators in stratified sampling viz., Separate Ratio (SR) Estimator (SRE) and Combined Ratio (CR) Estimator (CRE). In Chapter 6, we have investigated the problem of optimum allocation of sample size for the strategies stratified SRSWR and SRSWOR using SRE when the interest lies in the estimation of population parameters such as population total or mean. In this chapter, we will consider the problem of optimum allocation of total sample size to strata for the stratified SRSWR and SRSWOR strategies using CRE specifically the Tschuprow-Neyman optimum allocation (TNOA) under a superpopulation model.

If the ratios $R_i$'s for the strata are not significantly different from one another, CRE is preferable to the SRE as already mentioned in chapter 2. The statistical tests for testing the equality of the stratum ratios can be found in P.S.R.S.Rao (2000) [47]. Besides, the regression of $y$ on $x$ may not pass through the origin in each stratum and the sample size in each stratum may not be large enough rendering the approximate variance expression for SRE no more valid. Due to all these reasons, CRE may be preferred to SRE.

In the preceding chapter, we have investigated the problem of optimum allocation in stratified sampling using SRE for estimation of population parameters such as total or mean. In the present chapter, the problem of optimum allocation is studied in stratified sampling with both SRSWR and SRSWOR within each stratum using CRE when the interest lies in the estimation of population total or mean.
7.2 Stratified SRSWR Design

7.2.1 CRE, TNOA and Superpopulation Model

Consider SRSWR design within each stratum. The CRE under the strategy (St.SRSWR: $\hat{Y}_{CR}$) for estimating the population total

$$Y = \sum_{i=1}^{k} \sum_{j=1}^{N_i} Y_{ij}$$

is given by

$$\hat{Y}_{CR} = \frac{\hat{Y}_{st}}{\hat{X}_{st}} \cdot \frac{\sum_{i=1}^{k} \frac{N_i}{n_i} \sum_{j=1}^{n_i} y_{ij}}{\sum_{i=1}^{k} \frac{N_i}{n_i} \sum_{j=1}^{n_i} x_{ij}} \cdot \frac{\sum_{i=1}^{k} N_i \bar{y}_i}{\sum_{i=1}^{k} N_i \bar{x}_i} \cdot X$$

(7.2.1)

Where $\bar{y}_i$ and $\bar{x}_i$ are the sample means of $y$ and $x$ respectively and

$$X = \sum_{i=1}^{k} \sum_{j=1}^{N_i} X_{ij}$$

is the population total of $x$.

Let us assume that the regression of $y$ on $x$ is linear and passing through the origin. Further assume that the total sample size is so large that
the assumption

\[ \left| \frac{\bar{x}_{st}}{x} - 1 \right| < 1 \text{ holds } \quad (7.2.2) \]

and hence

\[ (1 - \frac{\bar{x}_{st}}{X})^{-1} \text{ can be expanded.} \]

The bias of \( \hat{Y}_{CR} \), \( B(\hat{Y}_{CR}) \), to the second order of approximation is given by

\[
B(\hat{Y}_{CR}) = \frac{1}{X} \sum_{i=1}^{k} \left\{ R V(\hat{X}_i) - Cov(\hat{X}_i, \hat{Y}_i) \right\}
\]

\[
= \frac{1}{X} \sum_{i=1}^{k} \frac{N_i^2}{n_i} \left\{ R \sigma_i^2(x) - \sigma_i(x, y) \right\}
\]

\[
= \frac{1}{X} \sum_{i=1}^{k} \frac{N_i^2}{n_i} \left\{ R \sigma_i^2(x) - \rho_i(x, y) \sigma_i(x) \sigma_i(y) \right\}.
\]

Under large sample approximation, the bias vanishes. Therefore the approximate variance of the strategy (St.SRSWR: \( \hat{Y}_{CR} \)), which is identical with its MSE, is given by

\[
V(\hat{Y}_{CR}) = \sum_{i=1}^{k} \frac{N_i^2}{n_i} \sigma_i^2(v) \quad (7.2.3)
\]

where
\[
\sigma^2_i(v) = \frac{1}{N_i} \sum_j (V_{ij} - \bar{V}_i)^2
\]

and

\[
V_{ij} = Y_{ij} - RX_{ij}.
\]

Therefore,

\[
\sigma^2_i(v) = \sigma^2_i(y) + R^2 \sigma^2_i(x) - 2R \sigma_i(x, y).
\]

The TNOA of fixed total sample size \( n \) which minimises the variance in (7.2.3) is given by

\[
\text{TNOA: } n_i \propto N_i \sigma_i(v).
\]

This allocation needs atleast the proportionate values of \( \sigma_i(v) \) which is not easy to speculate. Therefore we try to get some estimate of \( \sigma_i(v) \) based on the known auxiliary variable using a superpopulation model approach. For the purpose, we explicitly define the superpopulation model \( \Delta_g \) i.e., a class of distributions \( \delta_g \) which satisfies the following properties:
\[(i) \quad \mathcal{E}_{\mathbf{g}} (Y_{ij} | X_{ij}) = \beta X_{ij} \]

\[(ii) \quad \mathcal{N}_{\mathbf{g}} (Y_{ij} | X_{ij}) = \sigma^2 X_{ij}^2 \]

\[(iii) \quad \mathcal{C}_{\mathbf{g}} (Y_{ij}, Y_{ij'} | X_{ij}, X_{ij'}) = 0 \]

\[(iv) \quad \mathcal{C}_{\mathbf{g}} (Y_{ij}, Y_{ij'} | X_{ij}, X_{ij'}) = 0. \]

\[\text{(7.2.4)}\]

### 7.2.2 TNOA under the Superpopulation Model \(\Delta_g\)

**Derivation of Expectation of \(\sigma_i^2(v)\)**

The expectation of \(\sigma_i^2(v)\) given \(X\) will be derived under the model \(\Delta_g\) in (7.2.4) and the same may be substituted in place of the unknown \(\sigma_i^2(v)\) in order to obtain the TNOA under the model \(\Delta_g\).

\[
\mathcal{E}_{\mathbf{g}}[\sigma_i^2(v)]_i^A \mathcal{L} = \mathcal{E}_{\mathbf{g}}[\{\sigma_i^2(y) + R^2 \sigma_i^2(x) - 2R\sigma_i(x,y)\}]_i^A \mathcal{L}
\]

\[
= \mathcal{E}_{\mathbf{g}}[\sigma_i^2(y)]_i^A \mathcal{L} + \frac{\sigma_i^2(y)}{x^2} \mathcal{E}_{\mathbf{g}}[Y^2]_i^A \mathcal{L}
\]

\[
- \frac{2}{x} \mathcal{E}_{\mathbf{g}} \left[ \{Y_{\frac{1}{N_i}} \sum \limits_j (X_{ij} - \bar{X}_i)(Y_{ij} - \bar{Y}_i) \} \right]_i^A \mathcal{L}
\]

**We will now evaluate the three terms one by one. Taking the first term**
\[ N_i \mathcal{E}_{\delta_y} \{ \sigma^2_i(y) \mid X \} = \mathcal{E}_{\delta_y} \left[ \sum_j (Y_{ij} - \bar{Y}_i)^2 \mid X \right] \]

\[ = \left[ 1 - \frac{1}{N_i} \mathcal{E}_{\delta_y} \left\{ \left( \sum_j Y^2_{ij} \mid X \right) \right\} - \frac{1}{N_i} \mathcal{E}_{\delta_y} \left\{ \left( \sum_{j \neq j'} Y_{ij} Y_{ij'} \right) \mid X \right\} \right] \]

\[ = \frac{N_i - 1}{N_i} \sum_j \mathcal{E}_{\delta_y} (Y^2_{ij} \mid X_{ij}) - \frac{1}{N_i} \sum_{j \neq j'} \mathcal{E}_{\delta_y} (Y_{ij} Y_{ij'} \mid X_{ij}, X_{ij'}) \cdot \]

\[ N_i \mathcal{E}_{\delta_y} \{ \sigma^2_i(y) \mid X \} = \frac{N_i - 1}{N_i} \sum_j \left[ \mathcal{E}_{\delta_y} (Y_{ij} \mid X_{ij}) + \{ \mathcal{E}_{\delta_y} (Y_{ij} \mid X_{ij}) \}^2 \right] \]

\[ - \frac{1}{N_i} \left[ \mathcal{E}_{\delta_y} (Y_{ij} Y_{ij'} \mid X_{ij}, X_{ij'}) + \mathcal{E}_{\delta_y} (Y_{ij} \mid X_{ij}) \mathcal{E}_{\delta_y} (Y_{ij'} \mid X_{ij'}) \right] \]

\[ = \frac{N_i - 1}{N_i} \sum_j (\sigma^2_i \chi_{ij}^2 + \beta^2 \chi_{ij}^2) - \frac{1}{N_i} \sum_{j \neq j'} (0 + \beta^2 \chi_{ij} \chi_{ij'}) \cdot \]

On simplification, we get

\[ \mathcal{E}_{\delta_y} \{ \sigma^2_i(y) \mid X \} = \sigma^2 \frac{N_i - 1}{N_i} \frac{1}{N_i} \sum_j \chi_{ij}^2 + \beta^2 \sigma_i^2(x). \quad (7.2.6) \]
Now
\[ Y^2 = \left( \sum_{i=1}^{k} Y_i \right)^2 = \sum_{i} Y_i^2 + \sum_{i \neq i'} Y_i Y_{i'} \]

\[ = \sum_{i} \left( \sum_{j} Y_{ij} \right)^2 + \sum_{i \neq i'} \sum_{j} \left( \sum_{j} Y_{ij} \right) \left( \sum_{l} Y_{jl} \right) \]

\[ = \sum_{i} \sum_{j} Y_{ij}^2 + \sum_{i} \sum_{j \neq j'} Y_{ij} Y_{ij'} + \sum_{i \neq i'} \sum_{j} \sum_{l=1}^{N_a} \sum_{l=1}^{N_{i'}} Y_{ij} Y_{jl} . \]

Hence,
\[ \mathcal{E}_{\delta_y}(Y^2|X) = \sum_{i} \sum_{j} \mathcal{E}_{\delta_y}(Y_{ij}^2|X_{ij}) + \sum_{i} \sum_{j \neq j'} \mathcal{E}_{\delta_y}(Y_{ij} Y_{ij'}|X_{ij}, X_{ij'}) \]

\[ + \sum_{i \neq i'} \sum_{j=1}^{N_a} \sum_{l=1}^{N_{i'}} \mathcal{E}_{\delta_y}(Y_{ij} Y_{jl}|X_{ij}, X_{jl}) \]

\[ = \sum_{i} \sum_{j} \left( \sigma^2 X_{ij}^2 + \beta^2 X_{ij}^2 \right) + \sum_{i} \sum_{j \neq j'} \{ \mathcal{C}_{\delta_y}(Y_{ij}, Y_{ij'}|X_{ij}, X_{ij'}) \}
\]

\[ + \mathcal{E}_{\delta_y}(Y_{ij}|X_{ij}) \mathcal{E}_{\delta_y}(Y_{ij'}|X_{ij'}) \}
\]

\[ + \sum_{i \neq i'} \sum_{j=1}^{N_a} \sum_{l=1}^{N_{i'}} \{ \mathcal{C}_{\delta_y}(Y_{ij}, Y_{jl}|X_{ij}, X_{jl}) \}
\]

\[ + \mathcal{E}_{\delta_y}(Y_{ij}|X_{ij}) \mathcal{E}_{\delta_y}(Y_{jl}|X_{jl}) \}
\].

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\[ \mathcal{E}_\delta(y^2 | X) = \sum_i \left( \sigma^2 \sum_j X_{ij}^2 + \beta^2 \sum_j X_{ij}^2 \right) + \sum_i \sum_{j \neq j'} (0 + \beta^2 X_{ij} X_{ij'}) \]

\[ + \sum_{i \neq i'} \sum_{j=1}^{N_i} \sum_{l=1}^{N_{i'}} (0 + \beta^2 X_{ij} X_{ij'}) \]

\[ = \sigma^2 \sum_i \sum_j X_{ij}^2 + \beta^2 \left\{ \sum_i \left( \sum_j X_{ij}^2 + \sum_j \sum_{j \neq j'} X_{ij} X_{ij'} \right) \right\} \]

\[ + \beta^2 \sum_{i \neq i'} \sum_{j=1}^{N_i} \sum_{l=1}^{N_{i'}} X_{ij} X_{ijl} \]

\[ = \sigma^2 \sum_i \sum_j X_{ij}^2 + \beta^2 \left\{ \sum_i \left( \sum_j X_{ij}^2 + \sum_j \sum_{j \neq j'} X_{ij} X_{ij'} \right) \right\} \]

\[ + \beta^2 \sum_{i \neq i'} \sum \left( \sum_{j} X_{ij} \right) \left( \sum_{l} X_{ijl} \right) \]

\[ = \sigma^2 \sum_i \sum_j X_{ij}^2 + \beta^2 \sum_i \left( \sum_j X_{ij}^2 + \sum_j \sum_{j \neq j'} X_{ij} X_{ij'} \right) \]

\[ + \beta^2 \sum_{i \neq i'} \sum X_{ij} X_{ij'} \]

\[ \mathcal{E}_\delta(y^2 | X) = \sigma^2 \sum_i \sum_j X_{ij}^2 + \beta^2 \sum_i \left( \sum_j X_{ij} \right)^2 \]

\[ + \beta^2 \sum_{i \neq i'} \sum X_{ij} X_{ij'} . \]
$$E_{g} (Y^2 | X) = \sigma^2 \sum_i \sum_j X_{ij}^g + \beta^2 \left( \sum_i X_i^2 + \sum_{i \neq i'} X_{i}X_{i'} \right)$$

$$= \sigma^2 \sum_i \sum_j X_{ij}^g + \beta^2 \left( \sum_i X_i \right)^2$$

$$E_{g} (Y^2 | X) = \sigma^2 \sum_i \sum_j X_{ij}^g + \beta^2 X^2. \quad (7.2.7)$$

Now, taking the third term of (7.2.5),

$$\sum_j (X_{ij} - \bar{X}_i)(Y_{ij} - \bar{Y}_i)Y = \sum_j (X_{ij} - \bar{X}_i)(Y_{ij} - \bar{Y}_i)(Y_i + \sum_{i' \neq i} Y_{i'})$$

$$= \sum_j (X_{ij} - \bar{X}_i)(Y_{ij} - \bar{Y}_i)Y_i + \left\{ \sum_j (X_{ij} - \bar{X}_i)(Y_{ij} - \bar{Y}_i) \right\} \left( \sum_{i' \neq i} Y_{i'} \right). \quad (7.2.8)$$

Now, simplifying and evaluating expectation of the two terms one by one, we take up the first term first.

$$\sum_j (X_{ij} - \bar{X}_i)(Y_{ij} - \bar{Y}_i)Y_i = \sum_j (X_{ij} - \bar{X}_i)\{Y_{ij}Y_i - \frac{Y_i^2}{N_i}\}$$

$$= \sum_j (X_{ij} - \bar{X}_i)\{Y_{ij}(\sum_j Y_{ij}) - \frac{1}{N_i}(\sum_j Y_{ij})^2\}$$

$$= \sum_j (X_{ij} - \bar{X}_i)\left\{ \left( Y_{ij}^2 + Y_{ij} \sum_{j' \neq j} Y_{ij'} \right) - \frac{1}{N_i} \left( \sum_j Y_{ij}^2 + \sum_{j \neq j'} \sum_j Y_{ij}Y_{ij'} \right) \right\}.$$
\[ \epsilon_{\delta_y} \left[ \left( X_{ij} - \bar{X}_i \right) (Y_{ij} - \bar{Y}_i) Y_i \big| X \right] \]

\[ = \sum_j (X_{ij} - \bar{X}_i) \left[ \epsilon_{\delta_y} \left( Y_{ij}^2 \big| X_{ij} \right) + \sum_{j' \neq j} \epsilon_{\delta_y} \left( Y_{ij} Y_{ij'} \big| X_{ij}, X_{ij'} \right) \right] \]

\[ - \frac{1}{N_i} \left\{ \sum_j \epsilon_{\delta_y} \left( Y_{ij}^2 \big| X_{ij} \right) + \sum_{j \neq j'} \sum_{j'} \epsilon_{\delta_y} \left( Y_{ij} Y_{ij'} \big| X_{ij}, X_{ij'} \right) \right\} \]

\[ = \sum_j (X_{ij} - \bar{X}_i) \left[ \left\{ \epsilon_{\delta_y} \left( Y_{ij} \big| X_{ij} \right) + \left( \epsilon_{\delta_y} \left( Y_{ij} \big| X_{ij} \right) \right)^2 \right\} \right] \]

\[ + \left\{ \sum_{j' \neq j} \epsilon_{\delta_y} (Y_{ij} \big| X_{ij}) \epsilon_{\delta_y} (Y_{ij'} \big| X_{ij'}) \right\} \]

\[ - \frac{1}{N_i} \sum_j (X_{ij} - \bar{X}_i) \left[ \sum_j \left\{ \epsilon_{\delta_y} \left( Y_{ij} \big| X_{ij} \right) + \left( \epsilon_{\delta_y} \left( Y_{ij} \big| X_{ij} \right) \right)^2 \right\} \right] \]

\[ + \sum_{j \neq j'} \sum_{j'} \left\{ \epsilon_{\delta_y} \left( Y_{ij} \big| X_{ij} \right) \epsilon_{\delta_y} \left( Y_{ij'} \big| X_{ij'} \right) \right\} \]

\[ = \sum_j (X_{ij} - \bar{X}_i) \left[ \left\{ \sigma^2 X_{ij}^2 + \beta^2 X_{ij}^2 \right\} + \beta^2 \sum_{j' \neq j} X_{ij} X_{ij'} \right] \]

\[ - \frac{1}{N_i} \left\{ \sum_j \left( \sigma^2 X_{ij}^2 + \beta^2 X_{ij}^2 \right) + \beta^2 \sum_{j \neq j'} X_{ij} X_{ij'} \right\} \]
\[ 
\sum_{j} (X_{ij} - \bar{X}_i) \left[ \left\{ \sigma^2 X_{ij}^g + \beta^2 X_{ij} \left( X_{ij} + \sum_{j' \neq j} X_{ij'} \right) \right\} \\
- \frac{1}{N_i} \left\{ \sigma^2 \sum_{j} X_{ij}^g + \beta^2 \left( \sum_{j} X_{ij}^2 + \sum_{j \neq j'} \sum_{j'} X_{ij} X_{ij'} \right) \right\} \right] \\
= \sigma^2 \sum_{j} (X_{ij} - \bar{X}_i) \left( X_{ij}^g - \frac{1}{N_i} \sum_{j} X_{ij}^g \right) \\
+ \beta^2 \sum_{j} (X_{ij} - \bar{X}_i) \left( X_{ij} X_i - \frac{X_i^2}{N_i} \right) \\
= \sigma^2 \sum_{j} (X_{ij} - \bar{X}_i) \left( X_{ij}^g - \bar{X}_i^g \right) + \beta^2 \sum_{j} (X_{ij} - \bar{X}_i)^2 X_i . 
\]

Thus

\[ E_{\delta_g} \left[ \{(X_{ij} - \bar{X}_i)(Y_{ij} - \bar{Y}_i)Y_i \} | X \right] = \sigma^2 \sum_{j} (X_{ij} - \bar{X}_i) \left( X_{ij}^g - \bar{X}_i^g \right) \\
+ \beta^2 \frac{1}{N_i} \sum_{j} (X_{ij} - \bar{X}_i)^2 . \tag{7.2.9} \]

Now, simplifying and finding the expectation of the second term of (7.2.8) we have

\[ \left\{ \sum_{j} (X_{ij} - \bar{X}_i) \left( Y_{ij} - \bar{Y}_i \right) \right\} \left( \sum_{i' \neq i} Y_{i'} \right) \]

\[ = \sum_{j} (X_{ij} - \bar{X}_i) \left( Y_{ij} \left( \sum_{i' \neq i} Y_{i'} \right) - \frac{Y_i}{N_i} \left( \sum_{i' \neq i} Y_{i'} \right) \right) \]
\[
\sum_j (X_{ij} - \bar{X}_i) \left\{ Y_{ij} \sum_{l \neq i} \sum_{l} Y_{il} \frac{1}{N_i} \sum_{l \neq i} \left( \sum_j Y_{ij} \right) \left( \sum_l Y_{il} \right) \right\} = \sum_j (X_{ij} - \bar{X}_i) \left\{ \sum_{l \neq i} \sum_{l} Y_{ij} Y_{il} \frac{1}{N_i} \sum_{l \neq i} \sum_j \sum_l Y_{ij} Y_{il} \right\}.
\]

Expectation of this term under the model $\Delta_g$ will be

\[
\mathbb{E}_{\delta_g} \left[ \left\{ \sum_j (X_{ij} - \bar{X}_i) Y_{ij} (Y_{ij} - \bar{Y}_i) \right\} \left( \sum Y_{it} | X \right) \right]
\]

\[
= \sum_j (X_{ij} - \bar{X}_i) \left\{ \sum_{l \neq i} \sum_{l} \mathbb{E}_{\delta_g} (Y_{ij} Y_{il} | X_{ij}, X_{il}) \right\}
\]

\[
- \frac{1}{N_i} \sum_{l \neq i} \sum_j \sum_l \mathbb{E}_{\delta_g} (Y_{ij} Y_{il} | X_{ij}, X_{il}) \right\}
\]

\[
= \sum_j (X_{ij} - \bar{X}_i) \left\{ \sum_{l \neq i} \sum_{l} \mathbb{E}_{\delta_g} (Y_{ij} | X_{ij}) \mathbb{E}_{\delta_g} (Y_{il} | X_{il}) \right\}
\]

\[
- \frac{1}{N_i} \sum_{l \neq i} \sum_j \sum_l \mathbb{E}_{\delta_g} (Y_{ij} | X_{ij}) \mathbb{E}_{\delta_g} (Y_{il} | X_{il}) \right\}
\]

\[
= \sum_j (X_{ij} - \bar{X}_i) \left\{ \sum_{l \neq i} \sum_{l} \beta^2 X_{ij} X_{il} \frac{1}{N_i} \sum_{l \neq i} \sum_j \sum_l \beta^2 X_{ij} X_{il} \right\}
\]

\[
= \beta^2 \sum_j (X_{ij} - \bar{X}_i) \left\{ X_{ij} \sum_{l \neq i} X_{il} \frac{1}{N_i} X_i \sum_{l \neq i} X_{il} \right\}
\]

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Now from (7.2.8) using (7.2.9) and (7.2.10), we have:

\[ E_{\delta_y} \left[ \left\{ \sum_{j} (X_{ij} - \bar{X}_i) (Y_{ij} - \bar{Y}_i) \right\} \right] \]

\[ = E_{\delta_y} \left[ \left\{ \sum_{j} (X_{ij} - \bar{X}_i) (Y_{ij} - \bar{Y}_i) \right\} Y_{i} | X \right] \]

\[ + E_{\delta_y} \left[ \left\{ \sum_{j} (X_{ij} - \bar{X}_i) (Y_{ij} - \bar{Y}_i) (\sum_{i'} Y_{i'}) \right\} X \right] \]

\[ = \sigma^2 \sum_{j} (X_{ij} - \bar{X}_i) (X_{ij}^2 - \bar{X}_i^2) + \beta^2 \sum_{i} (X_{ij} - \bar{X}_i)^2 \]

\[ + \beta^2 \sum_{j} (X_{ij} - \bar{X}_i)^2 \sum_{i'} X_{i'} \]

\[ = \sigma^2 \sum_{j} (X_{ij} - \bar{X}_i) (X_{ij}^2 - \bar{X}_i^2) + \beta^2 \sum_{j} (X_{ij} - \bar{X}_i)^2. \] (7.2.11)

Now substituting from equations (7.2.6), (7.2.7) and (7.2.11) in (7.2.5), we get

\[ E_{\delta_y} \left[ \sigma^2(v) | X \right] = \left\{ \sigma^2 \frac{N_i-1}{N_i} X_{ij}^2 + \beta^2 \sigma^2(x) \right\} \]

\[ + \frac{\sigma^2(x)}{X^2} \left\{ \sigma^2 \sum_{i} \sum_{j} X_{ij}^2 + \beta^2 X^2 \right\} \]
\[-\frac{2}{X} \left\{ \sigma^2 \frac{1}{N_i} \sum_j (X_{ij} - \bar{X}_i) \left( X_{ij}^g - \overline{X}_i^g \right) + \beta^2 X \frac{1}{N_i} \sum_j (X_{ij} - \bar{X}_i)^2 \right\} \]

\[= \sigma^2 \left\{ N_i^{-1} \overline{X}_i^g + \left( \sum_i \sum_j X_{ij}^g \right) \frac{\sigma_i^2(x)}{X^2} - \frac{2}{X} \sigma_i(x, x^g) \right\} \]

\[E_{g_i} \left[ \sigma_i^2(v | X) \right] = \sigma^2 b_i(g) \quad (7.2.12)\]

where

\[b_i(g) = \kappa(g) \sigma_i^2(x) + q_i(g) \quad (7.2.13)\]

with

\[\kappa(g) = \frac{\sum_i \sum_j X_{ij}^g}{X^2} \quad (7.2.14)\]

\[q_i(g) = \frac{N_i - 1}{N_i} \overline{X}_i^g - \frac{2}{X} \sigma_i(x, x^g) \quad (7.2.15)\]

and

\[\overline{X}_i^g = \frac{1}{N_i} \sum_j X_{ij}^g .\]
Model-based Allocations under the Model $\Delta_g$

From (7.2.12), it follows that

$$\mathcal{E}_{b_i}[\sigma_i^2(v)|X] \propto b_i(g).$$

Thus $\sigma_i^2(v)$ is expected to be proportional to $b_i(g)$ under the model $\Delta_g$.

TNOA reduces under the model $\Delta_g$ to

$$n_i(g) \propto N_i \sqrt{b_i(g)} \quad (7.2.16)$$

**Definition 7.2.1.** The allocation (7.2.16) is termed $\Delta_g$-Model Optimum Allocation ($\Delta_g$-MOA) under the superpopulation model $\Delta_g$ for the strategy (St.SRSWR: $\hat{Y}_{CR}$).

Now

$$\mathcal{E}_{b_i}[\tau_i^2(v)|X] = \sigma^2 \{\kappa(g)\sigma_i^2(x) + q_i(g)\}$$

or

$$\mathcal{E}_{b_i}[\tau_i^2(v)|X] \propto \sigma_i^2(x) \propto q_i(g)$$

if $\sigma_i^2(x) \propto q_i(g)$
Therefore two more allocations are obtained under the condition that $\sigma_i^2(x) \propto q_i(g)$ as follows:

1. 
   \[ n_i \propto N_i \sigma_i(x) \quad (7.2.17) \]

2. 
   \[ n_i(g) \propto N_i \sqrt{q_i(g)} \quad (7.2.18) \]

provided

\[ \theta_i(g) = \frac{\sigma_i(x)}{\sqrt{q_i(g)}}, \quad i = 1, 2, \ldots, k \quad (7.2.19) \]

are equal in all strata.

**Definition 7.2.2.** The allocations (7.2.17) and (7.2.18) under the condition (7.2.19) are termed $\Delta_g$-Conditional Model Auxiliary Variable Optimum Allocation ($\Delta_g$-CMAVOA) and $\Delta_g$-Conditional Model Optimum Allocation ($\Delta_g$-CMOA) respectively under the superpopulation model $\Delta_g$ for the strategy (St.SRSWR: $\hat{Y}_{CR}$).

### 7.2.3 Approximate Allocations

Some quick approximations to $\Delta_g$-MOA and $\Delta_g$-CMOA will be obtained in this subsection. As the overall population total $X$ of $\mathcal{X}$ is expected to be much larger than the individual stratum covariance between $x$ and $x^g$ i.e.,
$\frac{2}{X} \sigma_i(x, x^g)$ can be neglected compared to $q_i(g)$.

The Relative Neglected Term (RNT) $t'_i(g)$ in this approximation is given by

$$t'_i(g) = \frac{\frac{2}{X} \sigma_i(x, x^g)}{q_i(g)}.$$  \hfill (7.2.20)

Thus $q_i(g)$ may be approximated by

$$q'_i(g) = \left(1 - \frac{1}{N_i}\right) \left(\overline{X_i^g}\right).$$  \hfill (7.2.21)

Further if the stratum sizes are so large that the terms in $\frac{1}{N_i}$ can be neglected compared to unity i.e., $\frac{1}{N_i} \approx 0 \ll 1$, then $q''_i(g)$ may be approximated by

$$q''_i(g) = \overline{X_i^g}.$$  \hfill (7.2.22)

The RNT in the approximation is given by

$$t''_i(g) = \frac{\frac{2}{X} \sigma_i(x, x^g) + \overline{X_i^g}}{q_i(g)}.$$  \hfill (7.2.23)

Consequent upon the approximation of $q_i(g)$ by $q'_i(g)$ and $q''_i(g)$, $b_i(g)$ may be approximated by $b'_i(g)$ and $b''_i(g)$ respectively, which are given below:

$$b'_i(g) = \kappa(g) \sigma^2_i(x) + q'_i(g)$$  \hfill (7.2.24)

and

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\[ b''_i(g) = \kappa(g) \sigma_i^2(x) + q''_i(g). \quad (7.2.25) \]

The RNT's in the two approximations are given by

\[ m'_i(g) = \frac{\frac{2}{X} \sigma_i(x, x^o)}{b_i(g)} = \frac{b'_i(g)}{b_i(g)} - 1 \quad (7.2.26) \]

and

\[ m''_i(g) = \frac{\frac{2}{X} \sigma_i(x, x^o) + \left(\frac{x^o_i}{N_i}\right)}{b_i(g)}. \quad (7.2.27) \]

Since \( \kappa(g) \sigma_i^2(x) \geq 0 \),

hence \( b_i(g) \geq q_i(g) \),

\[ b'_i(g) \geq q'_i(g) \]

and

\[ b'_i(g) - b_i(g) = \frac{2}{X} \sigma_i(x, x^o) = q'_i(g) - q_i(g). \]

Now

\[ m'_i(g) = \frac{\frac{2}{X} \sigma_i(x, x^o)}{b_i(g)}. \]
Similarly

\[ t'_i(g) = \frac{\frac{2}{X} \sigma_i(x, x^g)}{q_i(g)}. \]

We know from (7.3.17) that

\[ m'_i(g) \leq t'_i(g). \]

Now

\[ m''_i(g) = \frac{\left( \frac{X_i^2}{N_i} \right) + \frac{2}{X} \sigma_i(x, x^g)}{b_i(g)} = \frac{\left( \frac{X_i^2}{N_i} \right)}{b_i(g)} + m'_i(g). \]

\[ m''_i(g) - m'_i(g) = \frac{\left( \frac{X_i^2}{N_i} \right)}{b_i(g)} \geq 0, \]

i.e., \( m'_i(g) \leq m''_i(g). \) \hspace{1cm} (7.2.28)

Similarly

\[ t'_i(g) \leq t''_i(g). \] \hspace{1cm} (7.2.29)
\[ m''_i(g) = \frac{b''_i(g) - b'_i(g)}{b_i(g)} = \frac{q''_i(g) - q'_i(g)}{b_i(g)} \leq \frac{q''_i(g) - q'_i(g)}{q_i(g)} = t''_i(g). \]

Since

\[ b_i(g) \geq q_i(g), \]

hence

\[ m''_i(g) \leq t''_i(g). \quad (7.2.30) \]

If the RNT \( m'_i(g) \) is such that

\[ m'_i(g) \approx 0 \ll 1 \quad (7.2.31) \]

then the \( \Delta_g \)-MOA (7.2.16) may be approximated by the following approximate allocation

\[ n_i(g) \propto N_i \sqrt{b'_i(g)}. \quad (7.2.32) \]

**Definition 7.2.3.** The allocation (7.2.32) is called \( \Delta_g \)-Approximate Model Optimum Allocation (\( \Delta_g \)-AMOA) under the superpopulation model \( \Delta_g \) for the strategy (St.SRSWR: \( \hat{Y}_{CR} \)).

Similarly the RNT \( m''_i(g) \) is such that
then the $\Delta_g$-MOA may be approximated by

$$n_i(g) \propto N_i \sqrt{b_i(g)}.$$  \hfill (7.2.34)

**Definition 7.2.4.** The allocation (7.2.34) obtained from $\Delta_g$-MOA under large strata approximations is termed the $\Delta_g$-Large Strata Approximate Model Optimum Allocation ($\Delta_g$-LSAMOA) under superpopulation model $\Delta_g$ for the strategy (St.SRSWR: $\hat{Y}_{CR}$).

Further, if the RNT $t'_i(g)$ is such that

$$t'_i(g) \approx 0 \ll 1$$ \hfill (7.2.35)

then the $\Delta_g$-CMOA (7.2.18) may be approximated by the following approximate allocation:

$$n_i(g) \propto N_i \sqrt{q'_i(g)} = N_i \sqrt{\left(1 - \frac{1}{N_i}\right) \frac{X_i^q}{X_i^q}}$$ \hfill (7.2.36)

provided

$$q'_i(g) = \frac{\sigma_i(x)}{\sqrt{q'_i(g)}} = \frac{\sigma_i(x)}{\sqrt{\frac{N_i-1}{N_i} X_i^q}} = \frac{S_i(x)}{\sqrt{X_i^q}}, \quad i = 1, 2, \ldots, k$$ \hfill (7.2.37)

are equal in all strata.
Definition 7.2.5. The allocation (7.2.36) obtained from $\Delta_g$-CMOA is called $\Delta_g$-Approximate Conditional Model Optimum Allocation ($\Delta_g$-ACMOA) under the superpopulation model $\Delta_g$ for the strategy (St.SRSWR: $\hat{Y}_{CR}$).

In addition if the stratum sizes are also large so that the term in $\frac{1}{N_i}$ is negligible compared to $q_i'(g)$, then the $\Delta_g$-ACMOA may be approximated by the approximate allocation given below.

In other words, if the RNT $t''_i(g)$ is such that

$$t''_i(g) \approx 0 \ll 1$$  \hspace{1cm} (7.2.38)

then the $\Delta_g$-CMOA can straight away be approximated by the following approximate allocation:

$$n_i(g) \propto N_i \sqrt{q''_i(g)} = N_i \sqrt{X_i^g}$$  \hspace{1cm} (7.2.39)

provided

$$\theta''_i(g) = \frac{\sigma_i(x)}{\sqrt{X_i^g}}$$  \hspace{1cm} (7.2.40)

$i=1, 2, \ldots, k$ are equal in all strata.
Definition 7.2.6. The allocation (7.2.39) obtained from $\Delta_g$-ACMOA under large strata approximation is called $\Delta_g$-Large Strata Approximate Conditional MOA ($\Delta_g$-LSACMOA) under the superpopulation model $\Delta_g$ for the strategy (St.SRSWR: $\hat{Y}_{CR}$).

Approximate Allocations under the ESCV-Assumption

The quantity $q'_i(g)$ in (7.2.21) may be written as

$$q'_i(g) = \left( \frac{N_i - 1}{N_i} \right) \left( \overline{X}_i^g \right)$$

$$= \left( \frac{N_i - 1}{N_i} \right) \left\{ \sigma_i^2(x^{g/2}) + \left( \overline{X}_i^{g/2} \right)^2 \right\}$$

(7.2.41)

$$= \left( \frac{N_i - 1}{N_i} \right) \left( \overline{X}_i^{g/2} \right)^2 \left\{ C_i^2(x^{g/2}) + 1 \right\}. \quad (7.2.42)$$

Therefore if $C_i^2(x^{g/2}) \propto$ constant, then $\Delta_g$-ACMOA under the assumption of equal SCV$(x^{g/2})$ reduces to approximate allocation given by

$$n_i(g) \propto N_i \left( \overline{X}_i^{g/2} \right) \sqrt{\frac{N_i - 1}{N_i}}$$

$$\propto \sum_j X_{ij}^{g/2} \sqrt{\frac{N_i - 1}{N_i}} \quad (7.2.43)$$

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provided

\[ \eta_i(g) = \frac{\sigma_i(x)}{\left( X_i^{g/2} \right) \sqrt{\frac{N_i - 1}{N_i}}} = \frac{S_i(x)}{\left( X_i^{g/2} \right)}, \quad i = 1, 2, \ldots, k \]

are equal in all strata.

**Definition 7.2.7.** The allocation (7.2.43) obtained from approximate allocation \( \Delta_g\)-ACMOA and based on mean of \( x^{g/2} \) under the assumption of equal \( \text{SCV}(x^{g/2}) \) is called \( \Delta_g\)-Mean-based ESCVACMOA (\( \Delta_g\)-MESCVACMOA) under the superpopulation model \( \Delta_g \) for the strategy (St.SRSWR: \( \hat{Y}_{CR} \)).

Again from (7.2.41),

\[ q_i(g) = \left( \frac{N_i - 1}{N_i} \right) \sigma_i^2(x^{g/2}) \left\{ 1 + \frac{1}{C_i^2(x^{g/2})} \right\} \quad (7.2.44) \]

under the assumption of equal \( \text{SCV}(x^{g/2}) \), \( \Delta_g\)-ACMOA reduces also to allocation based on standard deviation of \( x^{g/2} \) given by

\[ n_i(g) \propto N_i \sigma_i(x^{g/2}) \sqrt{\frac{N_i - 1}{N_i}} \quad (7.2.45) \]

provided
\[ \zeta_i(g) = \frac{\sigma_i(x)}{\sigma_i(x^{\eta/2})} = \frac{S_i(x)}{\sigma_i(x^{\eta/2})} = \frac{S_i(x)}{\left(\frac{x_i^{\eta/2}}{\sigma_i^2(x)}\right)} \]

\[ = \frac{S_i(x)}{(x_i^{\eta/2})} \frac{1}{C_i(x^{\eta/2})} \propto \eta_i(g) \]

are equal in all strata.

**Definition 7.2.8.** The allocation (7.2.45) derived from approximate allocation \( \Delta_g\)-ACMOA and based on standard deviation of \( x^{\eta/2} \) under the assumption of equal SCV \( x^{\eta/2} \) is called \( \Delta_g\)-Standard Deviation-based ESCVACMOA-(\( \Delta_g\)-SDESCVACMOA) under the superpopulation model \( \Delta_g \) for the strategy (St.SRSWR: \( \hat{Y}_{\text{CR}} \)).

If the stratum sizes are so large that \( \frac{1}{N_i} \) can be neglected compared to unity, then \( \Delta_g\)-MESCVCMAOA reduces to the following allocation:

\[ \eta_i(g) \propto \sum_j x_{ij}^{\eta/2} \quad (7.2.46) \]

provided

\[ \eta_i(g) = \frac{\sigma_i(x)}{(x_i^{\eta/2})}, \quad i = 1, 2, \ldots, k \]

are equal in all strata.
**Definition 7.2.9.** The allocation (7.2.46) obtained from $\Delta_g$-ACMOA under the approximation of large stratum sizes and under the assumption of equal $SCV(x^{g/2})$ is called $\Delta_g$-Mean-based ESCVLSACMOA ($\Delta_g$-MESCVLSACMOA) under the superpopulation model $\Delta_g$ for the strategy (St:SRSWR: $\hat{Y}_{CR}$). The allocation $\Delta_g$-MESCVLSACMOA $\propto \sum_j x_{ij}^{g/2}$ may also be called Generalized Auxiliary Variable Proportional Allocation (GAVPA).

Similarly under the approximation of large stratum sizes the $\Delta_g$-SDESCVACMOA reduces to the following allocation:

$$n_i(g) \propto N_i \sigma_i(x^{g/2}) \quad (7.2.47)$$

provided

$$c_i(g) = \frac{\sigma_i(x)}{\hat{\sigma}_i(x^{g/2})} = \frac{\sigma_i(x)}{\left(\frac{x_i^{g/2}}{\hat{\sigma}_i(x^{g/2})}\right)} = \frac{\eta_i(g)}{\hat{\sigma}_i(x^{g/2})} \propto \eta_i(g)$$

are equal in all strata.

**Definition 7.2.10.** The allocation (7.2.47) deduced from $\Delta_g$-ACMOA under the approximation of large stratum sizes and under the assumption of equal $SCV(x^{g/2})$ is called $\Delta_g$-standard deviation-based ESCVLSACMOA ($\Delta_g$-SDESCVLSACMOA) under the superpopulation model $\Delta_g$ for the strategy (St:SRSWR: $\hat{Y}_{CR}$). The allocation $\Delta_g$-SDESCVLSACMOA $\propto N_i \sigma_i(x^{g/2})$
may also be called Generalized Auxiliary Variable Optimum Allocation (GAVOA).

7.2.4 Summary of the Results of Subsections 7.2.2 and 7.2.3

The results obtained in the previous two subsections 7.2.1 and 7.2.3 are summarised in this subsection in the following three theorems:

Theorem 7.2.11. The Tschuprow-Neyman optimum allocation for combined ratio estimator under stratified sampling design with SRSWR within each stratum i.e., for the strategy (St.SRSWR: \( \hat{Y}_{CR} \)) under the superpopulation model \( \Delta_g \) defined by (7.2.4) reduces to the following \( \Delta_g \)-model optimum allocation (\( \Delta_g \)-MOA):

\[
n_i(g) \propto N_i \sqrt{b_i(g)}
\]

where

\[
b_i(g) = \kappa(g) \sigma_i(x)^2 + q_i(g),
\]

with

\[
\kappa(g) = \frac{\sum_i \sum_j X_{ij}^2}{X^2}
\]

and

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Corollary 7.2.12. Since the population total is expected to be very large compared to the individual stratum covariances, hence \( \frac{2}{X} \sigma_i(x, x^g) \) can be considered negligible relative to \( b_i(g) \)

i.e., \( m_i^l(g) = \frac{\frac{2}{X} \sigma_i(x, x^g)}{b_i(g)} \approx 0 \ll 1. \)

Therefore, \( \Delta_g \)-MOA of the theorem 7.2.11 can be approximated by \( \Delta_g \)-Approximate MOA i.e., (\( \Delta_g \)-AMOA)

\[
q_i(g) = \left( \frac{X_i}{N_i} \right) \left( \frac{N_i - 1}{N_i} \right) - \frac{2}{X} \sigma_i(x, x^g).
\]

where

\[
b_i'(g) = \kappa(g) \sigma_i^2(x) + q_i'(g)
\]

and

\[
q_i'(g) = \left( \frac{N_i - 1}{N_i} \right) \left( \frac{X_i^g}{X_i} \right).
\]
Corollary 7.2.13. In addition to the approximation considered in Cor. 7.2.12 i.e.,

\[ m'_i(g) \approx 0 \ll 1, \]

if the stratum sizes are also assumed to be large such that

\[ \frac{N_i - 1}{N_i} \approx 1, \]

then \( \Delta_g\text{-AMOA} \) in the Cor. 7.2.12 may, under the large stratum sizes, further be approximated by \( \Delta_g\text{-Large Strata AMOA (}\Delta_g\text{-LSAMOA)} \)

\[ n_i(g) \propto N_i \sqrt{b''_i(g)} \]

where \( b''_i(g) = \kappa(g) \sigma_i^2(x) + q''_i(g) \)

and \( q''_i(g) = X_{i_1}^g \).

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Theorem 7.2.14. The Tschuprow-Neyman optimum allocation for combined ratio estimator under stratified SRSWR design i.e., for the strategy (St.SRSWR: $\hat{Y}_{CR}$) reduces under the model $\Delta_g$ either to $\Delta_g$-Conditional Model Auxiliary Variable Optimum Allocation ($\Delta_g$-CMAVOA):

$$n_i \propto N_i \sigma_i(x)$$

or to $\Delta_g$-Conditional Model Optimum Allocation ($\Delta_g$-CMOA):

$$n_i(g) \propto N_i \sqrt{q_i(g)}$$

provided

$$\theta_i(g) = \frac{\sigma_i(x)}{\sqrt{q_i(g)}}, \quad i = 1, 2, \ldots, k$$

are equal in all strata,

where

$$q_i(g) = \left(\frac{N_i - 1}{N_i}\right) \left(\frac{X_i^{g}}{X}ight) - \frac{2}{X} \sigma_i(x, x^g).$$

Corollary 7.2.15. The individual stratum covariances between $x$ and $x^g$ are expected to be negligible compared to the overall population total $X$. Moreover
\( \frac{2}{N} \sigma_i(x, x^g) \) can be considered negligible relative to \( q_i(g) \) i.e.,

\[
t'_i(g) = \frac{2}{N} \sigma_i(x, x^g) / q_i(g) \approx 0 \ll 1,
\]

therefore \( q_i(g) \) may be approximated by

\[
q_i'(g) = \left( \frac{N_i - 1}{N_i} \right) \left( \bar{X}_i^g \right)
\]

in the \( \Delta_g \)-CMOA of the above theorem 7.2.14 and consequently \( \Delta_g \)-CMOA may be approximated by \( \Delta_g \)-Approximate CMOA (\( \Delta_g \)-ACMOA) given by

\[
n_i(g) \propto N_i \sqrt{q_i'(g)} = N_i \sqrt{\frac{N_i - 1}{N_i} \left( \bar{X}_i^g \right)}
\]

provided

\[
\theta'_i(g) = \frac{\sigma_i(x)}{\sqrt{q_i'(g)}}, \quad i = 1, 2, \ldots, k
\]

are equal in all the strata.

**Corollary 7.2.16.** In addition to the approximation on \( t'_i(g) \) in cor. 7.2.15, if it is assumed that the stratum sizes are so large that

\[
\frac{N_i - 1}{N_i} \approx 1,
\]

...
then $q''_i(g)$ in the cor. 7.2.15, may further be approximated by

$$q''_i(g) = \overline{X_i^g}$$

and $\Delta_g$ - ACMOA of cor. 7.2.15 under the large stratum approximation (LSA) may be approximated by $\Delta_g$-Large Strata ACMOA ($\Delta_g$-LSACMOA):

$$n_1(g) \propto N_i \sqrt{q''_i(g)} = N_i \sqrt{X_i^g}$$

provided

$$q''_i(g) = \frac{\sigma_i(x)}{\sqrt{q''_i(g)}} = \frac{\sigma_i(x)}{\left(X_i^g\right)}, \quad i = 1, 2, \ldots, k$$

are equal in all strata.

**Theorem 7.2.17.** If (i) $\frac{2}{X} \sigma_i(x, x_g)$ can be neglected compared to

$$q_i(g) = \left(\frac{N_i - 1}{N_i}\right) \left(\overline{X_i^g}\right) - \frac{2}{X} \sigma_i(x, x_g)$$

i.e., $t'_i(g) = \frac{2\sigma_i(x, x_g)}{u_i(g)} \approx 0 \ll 1$ and

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(ii) $C_i(x^{g/2}) \propto$ constant, i.e., the stratum coefficients of variation of $x^{g/2}$ are equal in all strata [ESCV($x^{g/2}$)-assumption] then Tschuprow-Neyman optimum allocation for combined ratio estimator under stratified SRSWR design i.e., for the strategy (St.SRSWR: $\hat{Y}_{CR}$) reduces under the model $\Delta_g$ to the following two ESCV($x^{g/2}$) Approximate Conditional Model Optimum Allocations (ESCVACMOAs):

1. $\Delta_g$-Mean-based ESCVACMOA ($\Delta_g$-MESCVCAMOA):

\[
n_i(g) \propto \sum_j X_{ij}^{g/2} \sqrt{\frac{N_i - 1}{N_i}}\]

provided

\[
\eta_i(g) = \frac{\sigma_i(x)}{(X_i^{g/2}) \sqrt{\frac{N_i - 1}{N_i}}} = \frac{S_i(x)}{(X_i^{g/2})}, \quad i = 1, 2, \ldots, k
\]

are equal in all strata.

2. $\Delta_g$-Standard Deviation-based ESCVACMOA ($\Delta_g$-SDESCVACMOA):

$(\Delta_g$-SDESCVACMOA):
\[ n_i(g) \propto N_i \sigma_i(x^{g/2}) \sqrt{\frac{N_i - 1}{N_i}} \]

provided

\[ \zeta_i(g) = \frac{\sigma_i(x)}{\sigma_i(x^{g/2}) \sqrt{\frac{N_i - 1}{N_i}}} = \frac{S_i(x)}{\sigma_i(x^{g/2})} \propto \eta_i(g), \quad i = 1, 2, \ldots, k \]

are equal in all strata.

Corollary 7.2.18. In addition to the (i) approximation on \( t'_i(g) \) and (ii) assumption of equal \( SCV(x^{g/2}) \), if (iii) the stratum sizes are assumed large so that

\[ \frac{N_i - 1}{N_i} \approx 1, \]

then the two \( \Delta_g\)-ESCVA\textsc{CMOAs} in the theorem 7.2.17 may be approximated under the large strata approximation (LSA) by the following allocations:

1. \( \Delta_g\)-MSCV Large Strata ACMOA (\( \Delta_g\)-MSCVLSACMOA)

\[ n_i(g) \propto \sum_j X_{ij}^g \]
provided

\[ r_i(g) = \frac{\sigma_i(x)}{\left( X_i^{2/2} \right)}, \quad i = 1, 2, \ldots, k \]

are equal in all strata.

2. $\Delta_g$-SDESCV Large Strata ACMOA ($\Delta_g$-SDESCVLSACMOA)

\[ n_i(g) \propto N_i \sigma_i(x^{g/2}) \]

provided

\[ \zeta_i(g) = \frac{\sigma_i(x)}{\sigma_i(x^{g/2})}, \quad i = 1, 2, \ldots, k \]

are equal in all strata.

Various model-based allocations obtained in subsections 7.2.2 and 7.2.3 are summarized in the following table 7.2.4.1.
### Table - 7.2.4.1

Summary of Allocations for CUE under St. SRSWR

<table>
<thead>
<tr>
<th>Condition</th>
<th>without any condition</th>
<th>Conditional Allocation</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓Approx.</td>
<td>↓Assump.</td>
<td></td>
</tr>
<tr>
<td>Without any approx.</td>
<td>MOA ( \propto N_i \sqrt{b_i(g)} )</td>
<td>CMOA Allocation: ( \propto N_i \sqrt{q_i(g)} )</td>
</tr>
<tr>
<td></td>
<td>Condition: ( \theta_i(g) = \frac{\sigma_i(x)}{\sqrt{q_i(g)}} )</td>
<td>Condition: ( \theta_i(g) = \frac{\sigma_i(x)}{\sqrt{q_i(g)}} )</td>
</tr>
<tr>
<td>( \frac{N_i-1}{N_i} \approx 1 ) Under Approx.: Without ESCV Assump. ( \frac{2}{N_i} \sigma_i(x, \sigma^2) \approx 0 )</td>
<td>WithoutLSA AMOA Allocation: ( \propto N_i \sqrt{b_i(g)} )</td>
<td>LSACMOA Allocation: ( \propto N_i \sqrt{q_i(g)} )</td>
</tr>
<tr>
<td></td>
<td>RNT: ( m_i(g) \approx 0 ) Under LSA: LSAMOA: ( \propto N_i \sqrt{b_i''(g)} ) ( \propto N_i \sqrt{q_i''(g)} )</td>
<td>Condition: ( \theta_i'(g) = \frac{\sigma_i(x)}{\sqrt{q_i(g)}} )</td>
</tr>
<tr>
<td></td>
<td>1. MESCVA- ACMOA Allocation: ( \propto \sum_j X_{ij}^{\sigma_i^2/2} \sqrt{\frac{N_i-1}{N_i}} )</td>
<td>1. MESCVA- ACMOA Allocation: ( \propto \sum_j X_{ij}^{\sigma_i^2/2} )</td>
</tr>
<tr>
<td></td>
<td>Condition: ( \eta_i(g) = \frac{S_i(x)}{X_i^{\sigma_i^2/2}} )</td>
<td>Condition: ( \eta_i(g) = \frac{S_i(x)}{X_i^{\sigma_i^2/2}} )</td>
</tr>
<tr>
<td></td>
<td>2. SDESCVA- ACMOA Allocation: ( \propto N_i \sigma_i(x^{\sigma_i^2/2}) \sqrt{\frac{N_i-1}{N_i}} )</td>
<td>2. SDESCVA- ACMOA Allocation: ( \propto N_i \sigma_i(x^{\sigma_i^2/2}) )</td>
</tr>
<tr>
<td></td>
<td>Condition: ( \zeta_i(g) = \frac{S_i(x)}{\sigma_i(x^{\sigma_i^2/2})} )</td>
<td>Con: ( \zeta_i(g) = \frac{\sigma_i(x)}{\sigma_i(x^{\sigma_i^2/2})} )</td>
</tr>
<tr>
<td>Under ESCV assum: ( C_i(x^{\sigma_i^2/2}) ) equal</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Particular Cases

Particular cases of various model-based allocations in subsections 7.2.1 to 7.2.3 can easily be deduced for different values of $g$ of the model $\Delta_g$. However, these allocations for certain values of $g$ of special interest viz., $g=0$, 1 and 2 are presented in the following tables 7.2.4.2, 7.2.4.3 and 7.2.4.4.

Table - 7.2.4.2

<table>
<thead>
<tr>
<th>Allocation</th>
<th>$g = 0$</th>
<th>$g = 1$</th>
<th>$g = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOA</td>
<td>$N_i\sqrt{\frac{N}{X^2}\sigma^2(x) + \frac{N_i-1}{N_i}X}$</td>
<td>$N_i\sqrt{\frac{N_i-1}{N_i}X_i - \frac{\sigma^2(x)}{X}}$</td>
<td>$N_i\sqrt{\frac{\kappa(2}\sigma^2(x) + q(2)}{X}}$</td>
</tr>
<tr>
<td>Approx.</td>
<td>Not required</td>
<td>$m_i(1) = \frac{2\sigma^2(x)}{b_i(1)}$</td>
<td>$m_i(2) = \frac{2\sigma^2(x)}{b_i(2)}$</td>
</tr>
<tr>
<td>LSAMOA</td>
<td>$N_i\sqrt{\frac{N}{X^2}\sigma^2(x) + \frac{N_i-1}{N_i}X}$</td>
<td>$N_i\sqrt{\frac{\sigma^2(x)}{X} + \frac{N_i-1}{N_i}X_i}$</td>
<td>$N_i\sqrt{\frac{\kappa(2}\sigma^2(x) + q(2)}{X}}$</td>
</tr>
</tbody>
</table>

App.

(i) $\frac{2}{X}\sigma^2(x, x^2)$

(ii) $N_i\approx 1$

Allocn.:

$n_i(g) \propto \frac{N_i}{X^2}\sigma^2(x) + 1$
Table - 7.2.43

Particular Cases of $\Delta_g$-CMOA and its Approximates for Different $g$ for CRE under St. SRSWR.

<table>
<thead>
<tr>
<th>Allocation</th>
<th>$g$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CMOA Cond:</td>
<td>$S_i(x)$</td>
<td>$\theta_i(1) = \frac{\sigma_i(x)}{\sqrt{N_i}}$</td>
<td>$\theta_i(2) = \frac{\sigma_i(x)}{\sqrt{N_i}}$</td>
<td>$\theta_i(2) = \frac{\sigma_i(x)}{\sqrt{N_i}}$</td>
</tr>
<tr>
<td>Equality of</td>
<td>$N_i \sqrt{\frac{N_i - 1}{N_i}}$</td>
<td>$N_i \sqrt{q_i(1)}$</td>
<td>$N_i \sqrt{q_i(2)}$</td>
<td>$N_i \sqrt{q_i(2)}$</td>
</tr>
<tr>
<td>ACMOA Approx.:</td>
<td>$\frac{2}{\bar{x}} \sigma_i(x, x^g)$</td>
<td>Not required</td>
<td>$\frac{2}{\bar{x}} \sigma_i^2(x)$</td>
<td>$\frac{2}{\bar{x}} \sigma_i(x, x^g)$</td>
</tr>
<tr>
<td>Allocn.: $n_i'(g) \propto N_i \sqrt{\frac{N_i - 1}{N_i}}$</td>
<td>$\theta_i(1) = \frac{S_i(x)}{\sqrt{N_i}}$</td>
<td>$\theta_i(2) = \frac{S_i(x)}{\sqrt{N_i}}$</td>
<td>$\theta_i(2) = \frac{S_i(x)}{\sqrt{N_i}}$</td>
<td></td>
</tr>
<tr>
<td>Cond.: Equality of</td>
<td>$\theta_i(0) = S_i(x)$</td>
<td>$\theta_i(1) = \frac{S_i(x)}{\sqrt{N_i}}$</td>
<td>$\theta_i(2) = \frac{S_i(x)}{\sqrt{N_i}}$</td>
<td></td>
</tr>
<tr>
<td>LSACMOA Approx.:</td>
<td>$\frac{2}{\bar{x}} \sigma_i(x, x^g)$</td>
<td>Not required</td>
<td>$\frac{2}{\bar{x}} \sigma_i^2(x)$</td>
<td>$\frac{2}{\bar{x}} \sigma_i(x, x^g)$</td>
</tr>
<tr>
<td>Allocn.: $n_i'(g) \propto N_i \sqrt{\frac{N_i - 1}{N_i}}$</td>
<td>$\theta_i(1) = \frac{S_i(x)}{\sqrt{N_i}}$</td>
<td>$\theta_i(2) = \frac{S_i(x)}{\sqrt{N_i}}$</td>
<td>$\theta_i(2) = \frac{S_i(x)}{\sqrt{N_i}}$</td>
<td></td>
</tr>
<tr>
<td>Cond.: Equality of</td>
<td>$\theta_i'(0) = \sigma_i(x)$</td>
<td>$\theta_i'(1) = \frac{\sigma_i(x)}{\sqrt{N_i}}$</td>
<td>$\theta_i'(2) = \frac{\sigma_i(x)}{\sqrt{N_i}}$</td>
<td></td>
</tr>
</tbody>
</table>

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Table - 7.2.4.4
Particular Cases of $\Delta(g)$-ESCVACMOAs
for Different $g$ for CRE under St. SRSWR

<table>
<thead>
<tr>
<th>Allocation</th>
<th>$g$</th>
<th>Assump.:</th>
<th>Approx.:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>ESCV($x^{qg/2}$)</td>
<td>$\frac{2}{N} \sigma_i(x, x^g)$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>Not required</td>
<td>Not required</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$C_i(\sqrt{x})$</td>
<td>$\frac{2}{N} \sigma_i(x, x^g)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\approx 0 &lt; q_i(g)$</td>
<td>$\approx 0 &lt; q_i(1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\approx 0 &lt; q_i(2)$</td>
<td>$\approx 0 &lt; q_i(1)$</td>
</tr>
</tbody>
</table>

**MESCVCACMOA:**
- Alloc.: $n_i(g) \propto N_i \sqrt{\frac{N_i - 1}{N_i}}$
- Cond.: $\eta_i(0) = S_i(x)$
- $\eta_i(1) = \frac{S_i(x)}{x_i^{1/2}}$
- $\eta_i(2) = \frac{S_i(x)}{X_i} = C_i(x)$

**MESCVLSACMOA**
- Alloc.: $n_i(g) \propto N_i$
- Cond.: $\eta_i(0) = \sigma_i(x)$
- $\eta_i(1) = \frac{\sigma_i(x)}{x_i^{1/2}}$
- $\eta_i(2) = \frac{\sigma_i(x)}{X_i} = C_i(x)$

**SDESCVCACMOA**
- Alloc.: $n_i(g) \propto \sqrt{N_i}$
- Cond.: $\zeta_i(g)$ does not exist
- $\zeta_i(1) = \frac{S_i(x)}{\sigma_i(\sqrt{x})}$
- $\zeta_i(2) = \frac{S_i(x)}{\sigma_i(\sqrt{x})} = \sqrt{\frac{N_i}{N_i - 1}}$

**SDESCVLSACMOA**
- Alloc.: $n_i(g) \propto \sqrt{N_i}$
- Cond.: $\zeta_i(g)$ does not exist
- $\zeta_i(1) = \frac{\sigma_i(\sqrt{x})}{\sigma_i(\sqrt{x})}$
- $\zeta_i(2) = \frac{\sigma_i(\sqrt{x})}{\sigma_i(\sqrt{x})} = 1$
7.3 Stratified SRSWOR Design

7.3.1 Tschuprow-Neyman Optimum Allocation

If simple random sampling without replacement (SRSWOR) is used within each stratum, then the bias of the strategy (St.SRSWOR: \( \hat{Y}_{CR} \)) using combined ratio estimator (CRE) under the large sample assumption (7.2.2) mentioned in subsection 7.2.1 to the second order of approximation is given by

\[
B(\hat{Y}_{CR}) = \frac{1}{k} \sum_{i=1}^{k} \left\{ R \cdot V(\hat{X}_i) - \text{Cov}(\hat{X}_i, \hat{Y}_i) \right\}
\]

\[
= \frac{1}{k} \sum_{i=1}^{k} N_i^2 \left( \frac{1-f_i}{n_i} \right) \left\{ R \cdot S_i^2(x) - S_i(x, y) \right\}
\]

\[
= \frac{1}{k} \sum_{i=1}^{k} N_i^2 \left( \frac{1-f_i}{n_i} \right) \left\{ R \cdot S_i^2(x) - \rho_i(x, y)S_i(x)S_i(y) \right\}
\]

where \( f_i = \frac{n_i}{N_i} \) is the sampling fraction in the \( i^{th} \) stratum.

The bias \( B(\hat{Y}_{CR}) \) becomes negligible for large samples. Therefore the approximate variance to the second order of approximation, which is identical with \( \text{MSE}(\hat{Y}_{CR}) \), is given by

\[
V(\hat{Y}_{CR}) = \sum_{i=1}^{k} N_i^2 \left( \frac{1-f_i}{n_i} \right) S_i^2(v)
\]

(7.3.1)

where

\[
S_i^2(v) = \frac{1}{N_i - 1} \sum_j (V_{ij} - \bar{V}_i)^2
\]

\[
\text{and}
\]

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Therefore,

\[ V_{ij} = Y_{ij} - RX_{ij} \]

Therefore,

\[ S_i^2(v) = S_i^2(y) + R^2 S_i^2(x) - 2R S_i(x, y) \]  \hspace{1cm} (7.3.2)

The TNOA of fixed total sample size \( n \) which minimises the variance in (7.3.1) is given by

\[ \text{TNOA: } n_i \propto N_i S_i(v) \]  \hspace{1cm} (7.3.3)

where \( S_i^2(v) \) is given by (7.3.2) above. All the quantities in \( S_i^2(v) \) except \( S_i^2(x) \) are unknown parameters. We need at least proportionate values of \( S_i^2(v) \) in order to use the TNOA in practice. Therefore an estimate of \( S_i^2(v) \) is obtained using the superpopulation model approach based on the auxiliary variable \( X \). Thus the expectation of \( S_i^2(v) \) under the model (7.2.4) is derived and the same is substituted in place of \( S_i^2(v) \) in the unknown TNOA in (7.3.3).

### 7.3.2 TNOA under Superpopulation Model \( \Delta_g \)

Using the expression of

\[ E_{\delta_g}[\sigma_i^2(v)|X] \]

from (7.2.12), we have,
\[ \mathcal{E}_{\bar{x}}[S_i^2(v) | X] = \frac{N_i}{N_{i-1}} \mathcal{E}_{\bar{x}}[\sigma_i^2(v) | X] \]

\[ = \sigma^2 \frac{N_i}{N_{i-1}} b_i(g) = \sigma^2 \frac{N_i}{N_{i-1}} \{ \kappa(g) \sigma_i^2(x) + q_i(g) \} \]
\[ = \sigma^2 [\kappa(g) S_i^2(x) + Q_i(g)] \]
\[ = \sigma^2 B_i(g) \quad \text{(say)} \]

where

\[ B_i(g) = \kappa(g) S_i^2(x) + Q_i(g) \]

and

\[ Q_i(g) = \frac{N_i}{N_{i-1}} q_i(g) = \frac{N_i}{N_{i-1}} \left\{ \frac{k_{i-1}}{N_i} \bar{X}_i^2 - \frac{2}{\overline{X}} \sigma_i(x, x') \right\} \]
\[ = \bar{X}_i^2 - \frac{2}{\overline{X}} \sigma_i(x, x') \]

and \( \kappa(g) \) is defined by (7.2.14).

Model-based Allocations under the Model \( \Delta_g \)

From (7.3.6), it follows that

\[ \mathcal{E}_{\bar{x}}[S_i^2(v) | X] \propto B_i(g). \]

Thus \( S_i^2(v) \) is expected under the model \( \Delta_g \) to be proportional to \( B_i(g) \).

Therefore the TN-optimum allocation under the Model-\( \Delta_g \) is obtained by
substituting $B_i(g)$ in place of the unknown $S^2_i(v)$ in the TNOA i.e.,

$$n_i(g) \propto N_i \sqrt{B_i(g)}.$$  \hspace{1cm} (7.3.9)

**Definition 7.3.1.** Allocation (7.3.9) is called $\Delta_g$-Model Optimum Allocation ($\Delta_g$-MOA) for the strategy (St.SRSWOR: $\hat{Y}_{CR}$).

Again from (7.3.5)

$$E_{\delta_i}(S^2_i(v)|X) \propto S^2_i(x)$$

or

$$\propto Q_i(g)$$

provided $S^2_i(x) \propto Q_i(g)$.

Thus two more allocations can be obtained under the condition that $S^2_i(x) \propto Q_i(g)$ as follows:

1.

$$n_i \propto N_i S_i(x) \hspace{1cm} (7.3.10)$$

2.

$$n_i(g) \propto N_i \sqrt{Q_i(g)} \hspace{1cm} (7.3.11)$$
provided

\[ \theta_i(g) = \frac{S_i(x)}{\sqrt{Q_i(g)}} \]  

(7.3.12)

are equal in all strata.

**Definition 7.3.2.** Allocations given by (7.3.10) and (7.3.11) under the condition (7.3.12) are called \( \Delta_g \)-Conditional Model Auxiliary Variable Optimum Allocation (\( \Delta_g \)-CMAVOA) and \( \Delta_g \)-Conditional Model Optimum Allocation (\( \Delta_g \)-CMOA) respectively under the superpopulation model \( \Delta_g \) for the Strategy (St.SRSWOR: \( \hat{Y}_{CR} \)).

### 7.3.3 Approximate Allocations

**Approximations to \( \Delta_g \)-MOA and \( \Delta_g \)-CMOA**

We shall now obtain some quick approximations to \( \Delta_g \)-MOA and \( \Delta_g \)-CMOA in this subsection. The second term on the right hand side of the (7.3.8) viz., \( \frac{2}{X} S_i(x, x^g) \) being the ratio of twice the covariance between \( x \) and \( x^g \) divided by comparatively very large quantity the population total \( X \) is expected to be very negligible. Therefore, \( Q_i(g) \) may be approximated by

\[ Q_i'(g) = \frac{1}{N_i} \sum_j X_{ij}^g \equiv X_i^g. \]  

(7.3.13)

The RNT relative to \( Q_i(g) \) in this approximation is given by
\[
\frac{2}{\lambda} \frac{S_i(x, x^p)}{Q_i(g)} = \frac{2}{\lambda} \frac{\sigma_i(x, x^p)}{\eta_i(g)} = t_i'(g) = \frac{Q_i(g) - Q_i(g)}{Q_i(g)} = Q_i'(g) - 1. \tag{7.3.14}
\]

Consequently \( B_i(g) \) may be approximated by

\[
B_i'(g) = \kappa(g) S_i^2(x) + Q_i'(g). \tag{7.3.15}
\]

The RNT for the approximation will be

\[
\frac{2}{\lambda} \frac{S_i(x, x^p)}{B_i(g)} = \frac{2}{\lambda} \frac{\sigma_i(x, x^p)}{b_i(g)} = m_i'(g) = \frac{B_i'(g) - B_i(g)}{B_i(g)} = \frac{B_i'(g)}{B_i(g)} - 1.
\]

We note that

\[
\kappa(g) = \sum_i \sum_i x_{ij}^g > 0 \text{ and } S_i^2(x) \geq 0,
\]

\[
\Rightarrow \kappa(g) S_i^2(x) \geq 0
\]

and hence

\[
B_i(g) \geq Q_i(g).
\]
Similarly

\[ B'_i(g) \geq Q'_i(g) \]  

\[ B'_i(g) - B_i(g) = \frac{2}{x} S_i(x, x^o) = Q'_i(g) - Q_i(g) \]

\[ m'_i(g) = \frac{2}{x} S_i(x, x^o) = \frac{B'_i(g) - B_i(g)}{B_i(g)} \]

\[ = \frac{Q'_i(g) - Q_i(g)}{B_i(g)} \leq \frac{Q_i(g) - Q_i(g)}{Q_i(g)} = t'_i(g) \]

\[ \Rightarrow m'_i(g) \leq t'_i(g). \]  

Thus if the RNT \( m'_i(g) \) is such that

\[ m'_i(g) \approx 0 \ll 1,\]

then the \( \Delta_g \)-MOA (7.3.9) may be approximated by the following approximate \( \Delta_g \)-MOA:

\[ n_i(g) \propto N_i \sqrt{B'_i(g)}. \]  

**Definition 7.3.3.** Allocation given by (7.3.18) is called the \( \Delta_g \)-Approximate Model Optimum Allocation (\( \Delta_g \)-AMOA) under the model \( \Delta_g \) for the strategy \( \text{St.SRSWOR: } \hat{Y}_{CR} \).

Similarly if the RNT \( t'_i(g) \) is such that \( t'_i(g) \approx 0 \ll 1 \) then the \( \Delta_g \)-CMOA given by (7.3.11) can be approximated by following approximate allocation:

\[ n_i(g) \propto N_i \sqrt{Q'_i(g)} \]  

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provided

\[ \theta'_i(g) = \frac{S_i(x)}{\sqrt{Q'_i(g)}} \]  

(7.3.20)

are equal in all strata.

**Definition 7.3.4.** Allocation given by (7.3.20) under the large population total approximation is called the \( \Delta_g \)-Approximate Conditional Model Optimum Allocation (\( \Delta_g \)-ACMOA) under the model \( \Delta_g \) for the strategy (St.SRSWOR: \( \hat{Y}_{CR} \)).

\( \Delta_g \)-MOAs under the ESCV-Assumption

The quantity \( Q'_i(g) \) in (7.3.13) may be written as

\[ Q'_i(g) = \overline{X^g_i} = \frac{1}{N_i} \sum_j X^g_{ij} \]

\[ = \sigma_i^2(x^{g/2}) + \left( \overline{X^g_i} \right)^2 \]  

(7.3.21)

\[ = \left( \overline{X^g_i} \right)^2 \left\{ C_i^2(x^{g/2}) + 1 \right\}. \]  

(7.3.22)

Therefore, using (7.3.22) in (7.3.7) and (7.3.6), we have

\[ \mathcal{E}_g \{ S^2_i(v) \} = \sigma^2 B'_i(g) \]

\[ = \sigma^2 \left[ \kappa(g) S^2_i(x) + \left( \overline{X^g_i} \right)^2 \left\{ C_i^2(x^{g/2}) + 1 \right\} \right]. \]
Now if $C_i(x^{g/2}) \propto \text{constant}$, then

$$\mathbb{E}[S_i^2(u)|X] \propto S_i^2(x) \propto \left(\frac{X_i^{g/2}}{s_i}\right)^2$$

if $S_i^2(x) \propto \left(\frac{X_i^{g/2}}{s_i}\right)^2$.

Therefore,

if \( (i) \ t'_i(g) \approx 0 < C 1 \)

and \( (ii) C_i(x^{g/2}) \) are equal,

then we obtain $\Delta_g$-ACMOA based on the mean under the assumption of equal SCV($x^{g/2}$):

$$n_i(g) \propto N_i \left(\frac{X_i^{g/2}}{s_i}\right) = \sum_j X_{ij}^{g/2} \quad (7.3.23)$$

provided

$$\eta_i(g) \propto \frac{S_i(x)}{X_i^{g/2}}$$

are equal in all strata.

**Definition 7.3.5.** Allocation given by (7.3.23) obtained from ACMOA under the assumption of equal SCV($x^{g/2}$) based on mean of GAV-$x^{g/2}$ is called the $\Delta_g$-Mean-based ESCVACMOA ($\Delta_g$-MESCVACMOA) or the Generalized
Auxiliary Variable Proportional Allocation (GAVPA) under the model $\Delta_g$ for the strategy (St.SRSWOE: $\hat{Y}_{CR}$).

Again, (7.3.21) may alternatively be written as

$$Q'_1(g) = \sigma^2(x^{g/2}) \left\{ 1 + \frac{1}{C'_1(x^{g/2})} \right\}.$$  

(7.3.24)

Therefore, on substituting (7.3.24) in (7.3.7) and (7.3.6), we have

$$\mathcal{E}_{\delta_y}[S^2_i(v)|X] = \sigma^2 B'_1(g)$$

$$= \sigma^2 \left[ \kappa(g) S^2_i(x) + \sigma^2(x^{g/2}) \left\{ 1 + \frac{1}{C'_1(x^{g/2})} \right\} \right].$$

Under the assumption $C_i(x^{g/2}) \propto \text{constant},$

$$\mathcal{E}_{\delta_y}[(S^2_i(v)|X)] \propto S^2_i(x)$$

or

$$\propto \sigma^2(x^{g/2}),$$

provided $S^2_i(x) \propto \sigma^2(x^{g/2}).$

Therefore if

(i) $t_i(g) \approx 0 \ll 1$ and

(ii) $C_i(x^{g/2})$'s are equal,

then $\Delta_g$-ACMOA based on standard deviation

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\( \sigma_i(x^{g/2}) \) under the assumption of \( \text{ESCV}(x^{g/2}) \) reduces to

\[
n_i(g) \propto N_i \sigma_i(x^{g/2}) \tag{7.3.25}
\]

provided

\[
\zeta_i(g) = \frac{S_i(x)}{\sigma_i(x^{g/2})} = \frac{S_i(x)}{\left(\frac{x^{g/2}}{\sigma_i(x^{g/2})}\right)}
= \frac{\eta_i(g)}{C_i(x^{g/2})} \propto \eta_i(g)
\]

are equal in all strata.

7.3.4 Summary of the Results of Subsections 7.3.2 and 7.3.3

In this subsection, we summarise the results obtained in subsections 7.3.2 and 7.3.3 in three theorems. Table 7.3.4.1 gives the summary of different types of allocations.

**Theorem 7.3.6.** The TNOA under the superpopulation model \( \Delta_g \) defined by (7.2.4) for the strategy (St.SRSWOR: \( \hat{Y}_{CR} \)) reduces to the following \( \Delta_g \)-model optimum allocation:

\[
n_i(g) \propto N_i \sqrt{B_i(g)} ,
\]

where

\[
B_i(g) = \kappa(g) S_i^2(x) + Q_i(g),
\]

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with

\[ K(g) = \frac{\sum X_{ij}^g}{X^2} \]

and

\[ Q_i(g) = \frac{1}{N_i} \sum_j X_{ij}^g - \frac{2}{X} S_i(x, x^g). \]

**Corollary 7.3.7.** If \( S_i(x, x^g) \) can be considered negligible relative to \( B_i(g) \)
i.e., if

\[ m_i'(g) = \frac{\frac{2}{X} S_i(x, x^g)}{B_i(g)} \approx 0 \ll 1, \]

then the \( \Delta_y\text{-MOA} \) of the theorem 7.3.6 may be approximated by \( \Delta_y\text{-AMOA}: \)

\[ n_i(g) \propto N_i \sqrt{B_i'(g)} \]

where

\[ B_i'(g) = \kappa(g) S_i^2(x) + Q_i'(g) \]

and

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Theorem 7.3.8. The Tschuprow-Neyman Optimum Allocation for the strategy (St.SRSWOR: YEN) under the model $\Delta_g$ and under the condition that

$$\theta_i(g) = \frac{S_i(x)}{\sqrt{Q_i(g)}}$$

are equal in all strata, reduces either to $\Delta_g$-Conditional Model Auxiliary Variable Optimum Allocation ($\Delta_g$-CMAVOA):

$$n_i \propto N_i S_i(x)$$

or to $\Delta_g$-Conditional Model Optimum Allocation ($\Delta_g$-CMOA):

$$n_i(g) \propto N_i \sqrt{Q_i(g)}.$$

Corollary 7.3.9. If $\frac{2}{X} S_i(x, x^g)$ is negligible relative to $Q_i(g)$ i.e.,

if $t_i(g) = \frac{2}{X} S_i(x, x^g) \approx 0 \ll 1,$

then $Q_i(g)$ may be replaced by

$$Q_i'(g) = \frac{1}{N_i} \sum_j X_{ij}^g.$$

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in the $\Delta_g$-CMOA of the theorem 7.3.8 i.e., $\Delta_g$-CMOA may be approximated by $\Delta_g$-Approximate CMOA ($\Delta_g$-ACMOA) given by

$$n_i(g) \propto N_i \sqrt{Q'_i(g)}$$

provided

$$\theta'_i(g) = \frac{S_i(x)}{\sqrt{Q'_i(g)}}$$

are equal in all strata.

**Theorem 7.3.10.** If (i) $\frac{2}{X} S_i(x, x^9)$ is negligible compared to

$$Q_i(g) = \bar{X}_i^g = \frac{1}{N_i} \sum_j X_{ij}^g$$

i.e., $\theta'_i(g) \approx 0 \ll 1$

(ii) $C_i(x^{9/2}) \propto$ constant

i.e., the $\text{CV}(x^{9/2})$ is equal in all strata

then TNOA under model $\Delta_g$, for the strategy (St.SRSWOR: $\bar{Y}_{CR}$) under the assumption of $\text{ESCV}(x^{9/2})$ reduces to the following Mean-based and SD-based $\text{ESCVACMOAs}$ viz., $\Delta_g$-MESCVA$\text{MOA}$ and $\Delta_g$-SDESCVACMOA as given below:
1. \( \Delta_g\text{-MESCVACMOA} \)

\[
n_i(g) \propto \sum_j X_{ij}^{g/2}
\]

provided

\[
\eta_i(g) = \frac{S_i(x)}{X_i^{g/2}}, \quad i = 1, 2, \ldots, k
\]

are equal in all strata.

2. \( \Delta_g\text{-SDESCVACMOA} \)

\[
n_i(g) \propto N_i S_i(x)
\]

provided

\[
\zeta_i(g) = \frac{S_i(x)}{\sigma_i(x^{g/2})}, \quad i = 1, 2, \ldots, k
\]

are equal in all strata.

A summary of various model-based allocations obtained in 7.3.2 and 7.3.3 is given in the following table no 7.3.4.1
### Table - 7.3.4.1

Summary of Allocations for CRE under St. SRSWOR

<table>
<thead>
<tr>
<th>Condition →</th>
<th>Without any Assump.</th>
<th>Conditional Allocation</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓Approx.</td>
<td>↓Assump.</td>
<td></td>
</tr>
<tr>
<td>Without any approx.</td>
<td>MOA</td>
<td>CMOA</td>
</tr>
<tr>
<td>MOA</td>
<td>$n_i(g)$</td>
<td>Allocation</td>
</tr>
<tr>
<td>Ni</td>
<td>$N_i \sqrt{B_i(g)}$</td>
<td>$N_i \sqrt{Q_i(g)}$</td>
</tr>
<tr>
<td>Condition</td>
<td>Condition</td>
<td>Condition</td>
</tr>
<tr>
<td></td>
<td>$\theta_i(g) = \frac{S_i(x)}{\sqrt{Q_i(g)}}$</td>
<td>$\theta_i(g) = \frac{S_i(x)}{\sqrt{Q_i(g)}}$</td>
</tr>
<tr>
<td>$\frac{2}{x}S_i(x^{\sigma/2})$</td>
<td>AMOA</td>
<td>ACMOA</td>
</tr>
<tr>
<td>ESCV Assump.</td>
<td>$\alpha N_i \sqrt{B_i(g)}$</td>
<td>$\alpha N_i \sqrt{Q_i(g)}$</td>
</tr>
<tr>
<td>RNT.</td>
<td>$\eta_i'(g) \approx 0$</td>
<td>$\theta_i'(g) \approx 0$</td>
</tr>
<tr>
<td>Under ESCV assump.:</td>
<td>MESCVCAMOA</td>
<td>Condition:</td>
</tr>
<tr>
<td>$C_i(x^{\sigma/2})$ equal</td>
<td>Allocation:</td>
<td>$\eta_i(g) = \frac{S_i(x)}{(x^{\sigma/2})}$</td>
</tr>
<tr>
<td></td>
<td>$\alpha \sum_j x_{ij}^{\sigma/2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SDESCVCAMOA</td>
<td>Condition:</td>
</tr>
<tr>
<td></td>
<td>Allocation</td>
<td>$\xi_i(g) = \frac{S_i(x)}{S_i(x^{\sigma/2})}$</td>
</tr>
</tbody>
</table>
Particular Cases

The particular cases of various model-based allocations derived in sub-sections 7.3.2 and 7.3.3, which can be deduced for any value of \( g \), are given for three values of \( g \) viz., \( g = 0, 1 \) and \( 2 \) of the model \( \Delta_g \) in the following table 7.3.4.2.

Table - 7.3.4.2
\( \Delta_g \)-MOA and AMOA for Different \( g \) for CRE under St. SRSWOR

<table>
<thead>
<tr>
<th>Allocation</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOA</td>
<td>( N_1 \sqrt{\frac{N}{n_1} S_1^2(x) + 1} )</td>
<td>( N_1 \sqrt{\bar{X}_1 - \frac{S_i^2(x)}{\chi}} )</td>
<td>( N_1 \sqrt{\kappa(2) S_1^2(x) + q_i(2)} )</td>
</tr>
<tr>
<td>Approx.:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RNT: ( m_i(g) )</td>
<td>Not required</td>
<td>( m_i(1) = \frac{S_i^2(x)}{B_i(1)} \approx 0 \ll 1 )</td>
<td>( m_i(2) = \frac{S_i^2(x)}{B_i(2)} \approx 0 \ll 1 )</td>
</tr>
<tr>
<td>AMOA</td>
<td>( N_1 \sqrt{\frac{N}{n_1} S_1^2(x) + 1} )</td>
<td>( N_1 \sqrt{\frac{1}{2} S_1^2(x) + \bar{X}_1} )</td>
<td>( N_1 \sqrt{\kappa(2) S_1^2(x) + q_i(2)} )</td>
</tr>
</tbody>
</table>

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**Table - 7.3.4.3**

$\Delta_g$-CMOA and $\Delta_g$-ESCVACMOAs and Their Approximates for Different $g$

for CRE under St. SRSWOR

<table>
<thead>
<tr>
<th>Allocation</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>CMOA</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Allocn.: $n_i(g) \propto N_i$</td>
<td>$N_i$</td>
<td>$N_i$</td>
<td>$N_i$</td>
</tr>
<tr>
<td>Cond.: $\theta_i(g)$</td>
<td>$\sqrt{\frac{X_i^2 - S_i^2(x)}{S_i^2(x)}}$</td>
<td>$\sqrt{\frac{1}{N_i} \sum X_{ij}^2 - \frac{1}{X_i} S_i(x, x^2)}$</td>
<td>$\sqrt{\frac{1}{N_i} \sum X_{ij}^2 - \frac{1}{X_i} S_i(x, x^2)}$</td>
</tr>
<tr>
<td>Eq.of $\frac{S_i(x)}{\sqrt{Q_i(x)}}$</td>
<td>$= S_i(x)$</td>
<td>$= S_i(x)$</td>
<td>$= S_i(x)$</td>
</tr>
<tr>
<td>Approximation: $\frac{\sqrt{S_i(x, x^2)}}{X_i} \approx 0$</td>
<td>Not required</td>
<td>$\frac{\sqrt{S_i^2(x)}}{X_i} \approx 0$</td>
<td>$\frac{\sqrt{S_i(x, x^2)}}{X_i} \approx 0$</td>
</tr>
<tr>
<td>ACMOA</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Allocn.: $n_i(g) \propto N_i$</td>
<td>$N_i$</td>
<td>$N_i\sqrt{X_i}$</td>
<td>$\sqrt{N_i \sum X_{ij}^2}$</td>
</tr>
<tr>
<td>Cond.: $\theta_i(g)$</td>
<td>$\theta_i^0(0) = S_i(x)$</td>
<td>$\theta_i^1(1) = \frac{S_i(x)}{\sqrt{X_i}}$</td>
<td>$\theta_i^2(2) = \frac{S_i(x)}{\sqrt{X_i} \sum X_{ij}^2}$</td>
</tr>
<tr>
<td>$\frac{S_i(x)}{\sqrt{Q_i(x)}}$</td>
<td>$= S_i(x)$</td>
<td>$= S_i(x)$</td>
<td>$= S_i(x)$</td>
</tr>
<tr>
<td>ESCV Assump.:</td>
<td>Not equal required</td>
<td>$C_i(\sqrt{X_i})$ equal</td>
<td>$C_i(x)$ equal</td>
</tr>
<tr>
<td>$C_i(x^{2/3})$ equal</td>
<td>$C_i(\sqrt{X_i})$ equal</td>
<td>$C_i(\sqrt{X_i})$ equal</td>
<td></td>
</tr>
<tr>
<td>MESCVCMAOA</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Allocn.: $n_i(g) \propto N_i$</td>
<td>$N_i$</td>
<td>$\sum_j \sqrt{X_{ij}}$</td>
<td>$X_i$</td>
</tr>
<tr>
<td>Cond.: $\eta_i(g)$</td>
<td>$S_i(x)$</td>
<td>$\eta_i^0(1) = \frac{S_i(x)}{X_i^{1/2}}$</td>
<td>$\eta_i^0(2) = C_i(x)$</td>
</tr>
<tr>
<td>equal</td>
<td>equal</td>
<td>equal</td>
<td></td>
</tr>
<tr>
<td>SDESCVCMAOA</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Allocn.: $n_i(g) \propto N_i S_i(x^{2/3})$</td>
<td>Does not equal</td>
<td>$N_i S_i(\sqrt{X_i})$</td>
<td>$N_i S_i(x)$</td>
</tr>
<tr>
<td>Cond.: $\xi_i(g)$ equal</td>
<td>$\xi_i^0(1) = \frac{S_i(x)}{S_i(\sqrt{X_i})}$</td>
<td>$\xi_i^0(2) = \frac{S_i(x)}{S_i(\sqrt{X_i})} = 1$ (satisfied)</td>
<td></td>
</tr>
</tbody>
</table>

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7.4 Empirical Illustration

The MOA based on TNOA for combined ratio estimator (CRE) under the superpopulation $\Delta_g$ for stratified SRSWOR design derived in section 7.3.2 and with its approximations are worked out for the two populations. In the illustration, the same five STs A, B, C, D and E equalizing $\sum_j X_{ij}^g$ for $g = 0, 0.5, 0.1, 1.5$ and $2.0$ respectively used in preceding chapters are also used in the current chapter.

The variances of CRE for stratified SRSWOR design in all the STs related to Pop.-I and II for sample sizes 5 and 4 are presented in the tables 7.4.1(I) and 7.4.1(II) respectively. In these tables col. 3 gives the relative efficiencies (REs) of different STs w.r.t. ST-A, col.- 4 and 5 show the REs of CRE and SRE w.r.t. the unbiased expansion estimator (UEE).

The tables 7.4.2(I-A-E) and 7.4.2(IIA-E) given in appendix D provide for all STs for the Pop.-I and II respectively, the REs of MOA, AMOA under the approximation $m_i^g(g) \approx 0$, CMOA under the condition of equal $\theta_i(g)$ along with condition CV, the ACMOA under the approximation $RNT$ $t_i^g(g) \approx 0$ and under the condition of equal $\theta_i'(g)$ along with $t_i'(g)$ values and the condition CV and ESCV allocations - MESCVA-CMOA and SDESCVA-CMOA along with assumption CV and the conditions $\eta_i(g)$ and $\zeta_i(g)$ respectively for different $g$. 

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Table - 7.4.1(I)

Variances of CRE for 5 STs and their REs w.r.t. ST-A and REs of CRE and SRE w.r.t. UEE for Pop.-I

| ST | $V(\hat{Y}_{CR})$ | $\text{RE}(|A)$ | $V(\hat{Y}_{SR})$ | $\text{RE}(\hat{Y}_{CR}|\hat{Y})$ | $\text{RE}(\hat{Y}_{SR}|\hat{Y})$ |
|----|-----------------|----------------|-----------------|----------------|----------------|
| A  | 2421x10^6       | 1              | 2496x10^6       | 0.4434         | 0.4301         |
| B  | 2189x10^6       | 1.1056         | 2785x10^6       | 0.6035         | 0.4426         |
| C  | 2743x10^6       | 1.8825         | 12440x10^6      | 0.7206         | 0.1589         |
| D  | 3348x10^6       | 0.7229         | 6724x10^6       | 0.6637         | 0.3305         |
| E  | 5343x10^6       | 0.4530         | 8188x10^6       | 0.5041         | 0.3289         |

Table - 7.4.1(II)

Variances of CRE for 5 STs and their REs w.r.t. ST-A and REs of CRE and SRE w.r.t. UEE for Pop.-II

| ST | $V(\hat{Y}_{CR})$ | $\text{RE}(|A)$ | $V(\hat{Y}_{SR})$ | $\text{RE}(\hat{Y}_{CR}|\hat{Y})$ | $\text{RE}(\hat{Y}_{SR}|\hat{Y})$ |
|----|-----------------|----------------|-----------------|----------------|----------------|
| A  | 4671510         | 1              | 4669155         | 4.1757         | 4.1778         |
| B  | 3946803         | 1.1836         | 3939080         | 3.6917         | 3.6989         |
| C  | 3307417         | 1.4124         | 3308048         | 3.6376         | 3.6369         |
| D  | 2796982         | 1.6702         | 2792290         | 4.9305         | 4.9388         |
| E  | 2319163         | 2.0143         | 2320214         | 7.9866         | 7.9829         |

7.4.1 Comparison among STs and Comparison between SRE and CRE under TNOA

It can be observed from the table 7.4.1(I) that ST-B is the best followed by STs A and C respectively for Pop.-I whereas for Pop.-II from table 7.4.1(II) ST-E is the best followed by STs D and C respectively as in the
case of Separate Ratio Estimator (SRE). Thus we see that almost same STs are the best for both the ratio estimators. Further for populations with low $g$ like Pop.-I, low STs A or B i.e., equalizing $N_i$ or $\sum_j \sqrt{X_{ij}}$ respectively are the best whereas for populations with high $g$ like Pop.-II, the high ST equalizing $\sum_j X_{ij}^2$ will be most suitable.

In case of Pop.-I, the combined ratio estimator (CRE) is more efficient than separate ratio estimator (SRE) for all the STs, the difference in the RE is particularly striking for ST-C equalizing stratum total $X_i$. On the other hand for Pop.-II, CRE is slightly more efficient for ST-C and E.

### 7.4.2 Behavior of Efficiency of Allocations

The REs of most of the allocations for Pop.-I are decreasing with increasing $g$ except for allocations MOA, AMOA and CMOA for STs A, B and for low values of $g$ for middle ST-C and the allocation ACMOA for ST-B. For Pop.-I, the efficiencies of all the allocations are increasing with increasing $g$ for all the STs except ST-A for SDESCVAMOA.

### 7.4.3 Effect of Approximation

The highest values of RNTs $m_i'(g)$ and $t_i'(g)$ for the two strata are 0.0082 and 0.0016 for $g = 2$ for ST-B and 0.0258 and 0.0022 for $g = 2$ for ST-E in case of Pop-I. Similarly for Pop.-II, the highest RNTs for the two strata are
0.00099 and 0.0230 for $g = 2$ for ST-A and 0.0154 and 0.0031 for $g = 2$ for ST-E. Thus we see that the RNTs are too little to have any substantial impact on the REs of the corresponding allocations.

The AMOA is more efficient than MOA for all the STs for Pop.-I. Similarly under the condition of equal $\theta_i(g)$, ACMOA is better than the conditional allocation without approximation viz., CMOA. Thus for Pop.-I using approximate allocation increases the RE of the allocation without approximation. In case of Pop.-II, the RE of allocations becomes higher for STs A and C only. The REs of AMOA and ACMOA although come down in comparison to those of MOA and CMOA respectively, yet the reduction is very nominal. Therefore, approximate allocations AMOA and ACMOA may be used in preference to the corresponding without approximate allocations MOA and CMOA respectively without much loss of efficiency.

7.4.4 Effect of Condition

For Pop.-I, conditional allocations $\Delta_g$-CMOA and its approximation $\Delta_g$-ACMOA are less efficient than the corresponding unconditional ones-$\Delta_g$-MOA and $\Delta_g$-AMOA respectively for most of the STs except for only a few $g$ values for STs B and C. However, in case of Pop.-II, the conditional allocations are better than unconditional ones for higher values of $g$ in higher STs. Thus the unconditional allocations are suitable for both the populations and for all STs except for higher values $g$ in the middle STs for populations with low $g$ like Pop.-I and for higher values of $g$ in the middle and higher STs for
populations with high $g$ like Pop.-II where conditional allocations are preferable.

The condition CV is increasing and the efficiency is decreasing for all STs except ST-B in Pop.-I, ST-E and SDESCVACMOA in Pop.-II. This is clear from tables 7.4.3.(I) and (II) that the correlation coefficient between condition CV and efficiency is high negative value for all allocations in all STs except for CMOA and ACMOA in ST-C for Pop.-I and SDESCVACMOA in ST-A and CMOA, ACMOA and MESCVACMOA in ST-D for Pop.-II in which case, the correlation coefficient is either poor negative or high positive value. Thus in general, better the condition is satisfied, the higher is the efficiency of the allocation. Therefore the condition CV is helpful in the choice of the allocation in that the allocation with the lowest condition CV is expected to be the best.

Table - 7.4.3(I)

Correlation Coefficient between Condition CV and R.E. of Allocation for Pop.-I

<table>
<thead>
<tr>
<th>Allocation</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
</tr>
</thead>
<tbody>
<tr>
<td>CMOA</td>
<td>-0.9622</td>
<td>-0.8019</td>
<td>-0.3813</td>
<td>-0.9941</td>
<td>-0.9650</td>
</tr>
<tr>
<td>ACMOA</td>
<td>-0.9615</td>
<td>-0.8006</td>
<td>-0.3715</td>
<td>-0.9943</td>
<td>-0.9655</td>
</tr>
<tr>
<td>MESCVACMOA</td>
<td>-0.9682</td>
<td>-0.7196</td>
<td>-0.6544</td>
<td>-0.9869</td>
<td>-0.9540</td>
</tr>
<tr>
<td>SDESCVACMOA</td>
<td>-0.7969</td>
<td>-0.9952</td>
<td>-0.9905</td>
<td>-0.9928</td>
<td>-0.9706</td>
</tr>
</tbody>
</table>
Table - 7.4.3(II)
Correlation Coefficient between Condition CV and 
R.E. of Allocation for Pop.-II

<table>
<thead>
<tr>
<th>Allocation</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
</tr>
</thead>
<tbody>
<tr>
<td>CMOA</td>
<td>-0.9306</td>
<td>-0.9793</td>
<td>-0.9587</td>
<td>0.3747</td>
<td>0.9946</td>
</tr>
<tr>
<td>ACMOA</td>
<td>-0.9282</td>
<td>-0.9794</td>
<td>-0.9572</td>
<td>0.3726</td>
<td>0.9946</td>
</tr>
<tr>
<td>MESCVACMOA</td>
<td>-0.9462</td>
<td>-0.9813</td>
<td>-0.9448</td>
<td>0.4709</td>
<td>0.9940</td>
</tr>
<tr>
<td>SDESCVACMOA</td>
<td>0.9889</td>
<td>-0.9365</td>
<td>-0.9985</td>
<td>-0.9996</td>
<td>-0.9976</td>
</tr>
</tbody>
</table>

7.4.5 Effect of Assumption

There are two allocations under the assumption of equal SCV($x^{g/2}$) viz.,

1. MESCVACMOA \( \propto \sum_j X_{ij}^{g/2} \)

and 2. SDESCVACMOA \( \propto N_i S_i(x^{g/2}) \)

derived from ACMOA under the assumption of equal SCV($x^{g/2}$). On comparison, we find that mostly MESCVACMOA is inferior to ACMOA in almost all STs for Pop.-I except for a few values of $g$ in some of the STs viz., for middle values of $g$ in ST-B and for $g = 0.5$ in ST-C. However, SDESCVACMOA is better than ACMOA for most of the STs except STs B and C, the highest value of $g = 2$ in the lowest ST-A and for lowest value of $g = 0.5$ in the highest ST-E. Thus we see that ACMOA is better than MESCVACMOA whereas SDESCVACMOA is more efficient than ACMOA barring a few exceptions for Pop.-I.
Similarly ACMOA is more efficient than the first allocation MESCVACMOA in Pop.-II for lower STs A and B only whereas the allocation is more efficient than its parent allocation ACMOA for the middle and higher STs C, D and E. On the other hand, SDESCVACMOA is superior to ACMOA for lower values of $g$ for STs A and B whereas ACMOA is better than SDESCVACMOA for the middle and higher STs C, D and E. Thus we see that the performance of the two ESCV-allocations is opposite in Pop.-II i.e., if one is better for lower STs than ACMOA then the other is better than ACMOA for higher STs. On the whole, SDESCVACMOA appears to be the best for populations with low $g$ like Pop.-I whereas for populations with high $g$ like Pop.-II, the allocation is the best for lower STs A and B only but MESCVACMOA is preferable in the middle and high STs C, D and E.

Table - 7.4.4(I)
Correlation Coefficient between Assumption CV and RE of Allocation for Pop.-I

<table>
<thead>
<tr>
<th>Allocation</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
</tr>
</thead>
<tbody>
<tr>
<td>MESCVACMOA</td>
<td>-0.9118</td>
<td>0.9518</td>
<td>-0.8253</td>
<td>-0.9715</td>
<td>-0.6099</td>
</tr>
<tr>
<td>SDESCVACMOA</td>
<td>-0.6322</td>
<td>0.9979</td>
<td>0.9984</td>
<td>0.9957</td>
<td>0.8206</td>
</tr>
</tbody>
</table>

Table - 7.4.4(II)
Correlation Coefficient between Assumption CV and RE of Allocation for Pop.-II

<table>
<thead>
<tr>
<th>Allocation</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
</tr>
</thead>
<tbody>
<tr>
<td>MESCVACMOA</td>
<td>0.9951</td>
<td>-0.9814</td>
<td>0.9979</td>
<td>0.9999</td>
<td>0.9990</td>
</tr>
<tr>
<td>SDESCVACMOA</td>
<td>-0.9533</td>
<td>-0.9230</td>
<td>0.9997</td>
<td>0.9980</td>
<td>0.9928</td>
</tr>
</tbody>
</table>
The assumption of equal SCV($x^{r/2}$) is improving in ST-A but deteriorating in all other STs in Pop.-I but the efficiency of the ESCV-allocations is improving for some STs whereas it is reverse in other STs. Similarly the assumption and RE of the ESCV-allocations are not going along in Pop.-II also. This is evident from tables 7.4.4(I) and 7.4.4(II) respectively for Pop.-I and II showing high positive correlation between the assumption CV and the REs of the two allocations for most of the STs in both the populations. Therefore the level of satisfaction of the assumption is not helpful in choosing an allocation with higher efficiency.

7.4.6 Overall Comparison

Various allocations are arranged in decending order of efficiency according to the majority of $g$ values for each ST. The ST ranking of the allocations are presented in tables 7.4.5(I) and (II) for Pop.-I and II respectively. Overall ordering of allocations on the basis of majority of STs is also given in the tables.
Table - 7.4.5(I)

Correlation Coefficient between Assumption CV and RE of Allocation for Pop.-I

<table>
<thead>
<tr>
<th>Rank</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
<th>Overall Ranking</th>
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<td>ME</td>
<td>CM</td>
<td>SDE</td>
<td>AM</td>
<td>(0-0.5) SDE</td>
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<tr>
<td></td>
<td></td>
<td>(0.5-1)</td>
<td></td>
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<tr>
<td></td>
<td>ACM</td>
<td>(1.5-2.0)</td>
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<tr>
<td>3</td>
<td>AM</td>
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<td>M</td>
<td>AM</td>
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</tr>
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<td>ME</td>
<td>SDE</td>
<td>SD</td>
<td>ME</td>
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We have seen in earlier sub-sections that for Pop.-I the approximate allocations are better than those without approximation and that unconditional allocations are more efficient than the conditional ones. Further, the SD-based ESCV-allocation SDESCVACMOA is better than the one without assumption ACMOA as well as the mean-based ESCV-allocation MESCVACMOA. We can see from the table 7.4.5(I) that the allocation CMAVOA $\propto N_i S_i(x)$ is the best followed by

$$SDESCVACMOA \propto N_i S_i(x^0)^{1/2}$$ and AMOA $\propto N_i \sqrt{E_i'(g)}$

respectively.
Here we note that

\[ \text{MESCVACMOA} \propto \sum_j X_{ij}^{g/2} \]

under the assumption of ESCV is the worst with the lowest RE. Thus if \( g \) is known generalized AVOA \( \propto N_i S_i(x^{g/2}) \), which is SDESCVACMOA for \( \text{CRE} \), is the best for the populations with low \( g \) like Pop.-I. But if \( g \) is not known, the CMAVOA \( \propto N_i S_i(x) \) may be used.

On the other hand in Pop.-II, the behavior of allocations is totally different rather opposite for lower and higher STs. The allocation which is the best for lower ST is the worst for the higher ones and vice versa. For instance CMAVOA and SDESCVACMOA are the best allocations for low STs but are the worst for higher STs. Similarly, MESCVACMOA, which is the best for higher STs, is the worst for lower STs.

It was observed in subsection 7.4.3 that the approximate allocations - AMOA and ACMOA are preferable to MOA and CMOA due to their simplicity at the cost of slight loss of efficiency. Further as noted in subsection 7.4.4 that the conditional allocations are better than unconditional ones for higher STs whereas for lower STs the opposite is true. Therefore AMOA is preferable for lower STs and ACMOA is appropriate for higher STs in populations with high \( g \) like Pop.-I.
We have also seen in that either of the two ESCV-allocations is better than their parent ACMOA. For low STs SDESCVACMOA is more efficient than both AMOA and ACMOA and in fact is the best of all except CMAVOA. On the other hand MESCVACMOA is better than both ACMOA and AMOA and is the best of all for higher STs. From table 7.4.5(II) of ranking, we note that for lower STs

\[ \text{CMAVOA} \propto N_i S_i(x) \]

is the best allocation followed by

\[ \text{SDESCVACMOA} \propto N_i S_i(x^{g/2}) \]

and

\[ \text{AMOA} \propto N_i \sqrt{B'_i(g)} \]

respectively, whereas for higher STs

\[ \text{MESCVACMOA} \propto \sum_j X_{ij}^{g/2} \]

is the best followed by CMOA and MOA respectively.
Approximate allocations are preferable to those without approximation as the former are more efficient than the latter. In most of the situations wherever the approximate allocations are less efficient than those without approximations for higher STs in populations with high $g$ like Pop.-II the loss of efficiency is negligible. Although the unconditional allocations are more efficient than the conditional ones, yet the ESCV-allocations derived from approximate conditional allocation under the assumption of equal SCV are more efficient than their parent allocation. The mean-based ESCV-allocation

$$MESCVCAMOA \propto \sum_{j} X_{ij}^{g/2}$$

is the best allocation for high STs in populations with high $g$ like Pop.-II. Except this, for all STs in all populations the SD-based ESCV-allocation - SDESCVCAMOA $\propto N_i S_i (x^{g/2})$ is expected to be the best allocation.

The RNTs in large strata approximation are negligible for the two populations under illustration and in practice where population and hence strata will be large, the RNTs in actual survey populations can safely be considered negligible. The large strata approximation does not affect the efficiency of allocations and in some cases even increases the efficiency of allocations. The level of satisfaction of the condition in conditional allocations plays an important role in the efficiency of allocation: The better the condition is satisfied, the higher is the efficiency of the conditional allocation. Therefore the conditions of different allocations may be checked to determine
the best allocation at the planning stage of the survey. Even when $g$ is not known, optimum choice of $g$ can be made to give the highest efficiency. The allocations under the assumption of equal SCV are robust w.r.t the satisfaction of the assumption of equal SCV($x^{g/2}$). These allocations are the best depending on the type of population and the stratification used.