Chapter 6

Allocation for Separate Ratio Estimator

Summary

Cochran (1977) [8] noted, "In the planning of a sample, the allocation with a ratio estimate may appear a little perplexing . . ." and suggested two rules - one making allocation proportional to product of stratum size and the square root of mean and the other providing allocation proportional to the stratum total of the auxiliary variable, but without any justification.

In this chapter, the problem of allocation of total sample size to different strata is studied for separate ratio estimator under superpopulation model approach. The Model Optimum Allocation (MOA) based on TNOA under the model is derived. Various approximate allocations have been obtained under different approximations. Several particular cases of these allocations have also been deduced under particular cases of the model for certain values
of the superpopulation model parameter $g$. The two allocations suggested by Cochran (1977) [8] are shown to follow as particular cases of the MOA and thereby providing their justification. The above results have also been obtained for SRSWOR design within each stratum. The results are illustrated with the two live populations for both stratified SRSWOR design.

6.1 Introduction

Consider a finite population $U = \{U_1, U_2, \ldots, U_N\}$ of $N$ units divided into $k$ strata of sizes $N_i$, $i = 1, 2, \ldots, k$. Suppose we are interested in estimating parametric functions such as population total, population mean etc., of the study variate $Y$. In many practical situations, auxiliary information on a character closely related to $Y$ is available. It is well known (see Sec.2.5.1 of chapter 2) that the use of ratio estimation in stratified sampling combines the advantages of both the techniques of stratified sampling as well as ratio method of estimation. In the current and following chapters we will make use of the information on the same auxiliary variable $X$ both at planning stage for allocation of sample size and at estimation stage by using the ratio estimators in stratified sampling. Thus the problem of allocation of sample size to strata is studied for ratio estimators in stratified sampling using superpopulation model approach. As already discussed in chapter 2, there are two types of ratio estimators-separate ratio (SR) estimator (SRE) and combined ratio (CR) estimator (CRE). In this chapter, the problem of allocation for SRE will be investigated for stratified SRSWR and SRSWOR designs.
6.2 Stratified SRSWR Design

6.2.1 SRE, TNOA and Superpopulation Model

Separate Ratio Estimator

Suppose simple random sampling with replacement (SRSWR) is used within each stratum. The Separate Ratio (SR) Estimator (SRE) for the population total

\[ Y = \sum_{i=1}^{k} \sum_{j=1}^{N_i} Y_{ij} \]

is given by

\[ \hat{Y}_{SR} = \sum_{i=1}^{k} \frac{({\hat{Y}}_{i})_{st}}{({\hat{X}}_{i})_{st}} \cdot X_i = \sum_{i=1}^{k} \left( \frac{\bar{y}_i}{\bar{x}_i} X_i \right) \]  \hspace{1cm} (6.2.1)

where

\[ \bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} \]

and

\[ \bar{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij} \]

are the sample means of \( Y \) and \( X \) respectively and \( X_i \) is the stratum total of \( X \)-character in the \( i^{th} \) stratum.
Assume the regression of \(Y\) on \(X\) is linear passing through the origin in each stratum. It is also assumed that the sample sizes are large enough in each stratum so that the assumption

\[
\left| \frac{\bar{x}_i}{\bar{X}_i} - 1 \right| < 1
\]  

holds \(\forall i\) and consequently

\[
\left( 1 - \frac{\bar{x}_i}{\bar{X}_i} \right)^{-1}
\]

can be expanded.

Then the bias of \(\hat{Y}_{SR}\), \(B(\hat{Y}_{SR})\) to the second order of approximation, which is given by

\[
B(\hat{Y}_{SR}) = \sum_{i=1}^{k} \left( \frac{N_i^2}{n_i} \right) \frac{1}{X_i} \left\{ R_i \left( \sigma_i^2(x) - \sigma_i(x, y) \right) \right\}
\]

where

\[
\sigma_i(x, y) = \frac{1}{N_i - 1} \sum_j (X_{ij} - \bar{X}_i)(Y_{ij} - \bar{Y}_i)
\]

is the covariance between \(X\) and \(Y\) in the \(i^{th}\) stratum, becomes negligible. The approximate variance which is the same as \(\text{MSE}(\hat{Y}_{SR})\) under large sample approximation is given by

184
\[ V(\hat{Y}_{SR})_{WR} = \sum_{i} \frac{N_i^2}{n_i} \sigma_i^2(u), \]  \hspace{1cm} (6.2.3)

where

\[ \sigma_i^2(u) = \frac{1}{N_i} \sum_{j=1}^{N_i} U_{ij}^2. \]  \hspace{1cm} (6.2.4)

wherein \( U_{ij} = Y_{ij} - R_i X_{ij} \) is the transformed variable for all \( j = 1, 2, \ldots, N_i \) and \( i = 1, 2, \ldots, k \).

Thus

\[ \sigma_i^2(u) = \frac{1}{N_i} \sum_{j} \left( Y_{ij}^2 + R_i^2 X_{ij}^2 - 2R_i X_{ij}Y_{ij} \right). \]  \hspace{1cm} (6.2.5)

**Tschuprow-Neyman Optimum Allocation**

The Tschuprow-Neyman optimum allocation (TNOA) of fixed total sample size \( n \) which minimises the variance in (6.2.3) is given by

\[ n_i \propto N_i \sigma_i(u). \]  \hspace{1cm} (6.2.6)

In case of the usual unbiased expansion estimator (UEE), the unknown WSSD in the TNOA could be replaced by the known WSSD of the auxiliary variable. But the things are not so straightforward for ratio estimator for which we need some estimate of \( \sigma_i^2(u) \), where

\[ \sigma_i^2(u) = \sigma_i^2(y) + R_i^2 \sigma_i^2(x) - 2R_i \sigma_i(y, x) \]  \hspace{1cm} (6.2.7)
All the quantities in the expression of $a_f(u)$ except $a(x)$ are unknown for which no intuitive estimates are available as in the case of UEE. Therefore, it is not easy to use the above formula for computing workable allocations as it appears because the quantity $a_i^2(u)$ is too complex to speculate. Cochran (1977) [8] expresses the difficulty and suggested two rules as follows: In the planning of a sample, the allocation with a ratio estimate may appear a little perplexing, because it seems difficult to speculate about the likely values of $a_i(u)$. Two rules are helpful. With a population in which the ratio estimate is a best linear unbiased estimate, $a_i(u)$ will be roughly proportional to $\sqrt{X_i}$. In this case the $n_i$ should be proportional to $N_i \sqrt{X_i}$. Sometimes the variance of $U_{ij}$ may be more nearly proportional to $X_{ij}^2$. This leads to the allocation of $n_i$ proportional to $N_i X_i = X_i$, that is, to the stratum total of $X$.

Superpopulation Model for Separate Ratio Estimator

The difficulty in using TNOA in practice has been pointed out in the last paragraph. Cochran (1977) [8] has given two rules for allocation with a ratio estimate in practice, however, no justification was provided for the rules suggested. Therefore it is desirable to model the relationship between the study variable $Y$ and the auxiliary variable $X$, in order to explore their relationship not only for the precise estimation but also for the allocation of sample size to strata. For the stratification setup using separate ratio estimator, we explicitly define the model $\Delta_g$ as the class of all prior distributions.
\( \delta_g \) for which we have

\[
\begin{align*}
\text{i) } & \quad E_{\delta_g}(Y_{ij}|X_{ij}) = \beta_i X_{ij} \\
\text{ii) } & \quad \mathcal{V}_{\delta_g}(Y_{ij}|X_{ij}) = \sigma^2 X_{ij}^g \\
\text{iii) } & \quad C_{\delta_g}(Y_{ij}, Y_{ij'}|X_{ij}, X_{ij'}) = 0 \\
\text{iv) } & \quad C_{\delta_g}(Y_{ij}, Y_{ij'}|X_{ij}, X_{ij'}) = 0
\end{align*}
\]

where \( \beta, \sigma^2 \) and \( g \) are superpopulation parameters with \( \sigma^2 > 0 \) and \( g \geq 0 \).

The script letters \( E_{\delta_g}, \mathcal{V}_{\delta_g} \) and \( C_{\delta_g} \) denote the conditional expectation, variance and covariance given \( X \)'s respectively under superpopulation model \( \delta_g \).

In practice, it is found that \( g \) lies between 0 and 2 and more often close to 2.

It is well known that the ratio estimator is the best linear unbiased estimator (BLUE) for the populations with \( g = 1 \), Royall (1970) [60], Cochran (1977) [8], yet we formulate the model with general \( g \) in order to take care of any departure from \( g = 1 \) for real life populations.

### 6.2.2 TNOA under Superpopulation Model

We now derive \( E_{\delta_g}\{\sigma_i^2(u)|X\} \), i.e., the conditional expectation of \( \sigma_i^2(u) \) given \( X \) under the above superpopulation model \( \Delta_g \). The TN optimum allocation under the model \( \Delta_g \) may be obtained by substituting \( E_{\delta_g}\{\sigma_i^2(u)|X\} \) in place of the unknown \( \sigma_i^2(u) \) in the TNOA in (6.2.6).

Thus
\[ N_i \delta \{ \sigma_i^2(u) \mid X \} = \delta \left[ \left\{ \sum_j \left( Y_{ij}^2 + R_i^2 X_{ij}^2 - 2R_i Y_{ij} X_{ij} \right) \right\} \mid X \right]. \]

\[ = \left\{ \delta \left( \sum_j Y_{ij}^2 \right) \mid X \right\} + \delta \left\{ \left( \sum_j R_i^2 X_{ij}^2 \right) \mid X \right\} - 2\delta \left\{ \sum_j R_i Y_{ij} X_{ij} \mid X \right\}. \]

(6.2.8)

Now

\[ \delta \left\{ \left( \sum_j Y_{ij}^2 \right) \mid X \right\} = \sum_j \delta \{ Y_{ij}^2 \mid X_{ij} \} \]

\[ = \sum_j \left[ \delta \{ Y_{ij} \mid X_{ij} \} + \{ \delta \{ Y_{ij} \mid X_{ij} \} \}^2 \right] \]

\[ = \sum_j \left( X_{ij}^g + \beta_i^2 X_{ij}^2 \right) \]

(6.2.9)

Next,

\[ \delta \left\{ \left( \sum_j R_i^2 X_{ij}^2 \right) \mid X \right\} = \delta \left\{ \sum_j \frac{Y_{ij}^2}{X_{ij}^2} X_{ij}^2 \mid X \right\} \]

\[ = \left[ \left( \frac{\sum_j X_{ij}^2}{X_i^2} \right) \right] \delta \{ Y_i^2 \mid X \}. \]

188
\[
\begin{align*}
\sum_{i} \left( \frac{X_{ij}^2}{x_i^2} \right) \left\{ \varepsilon_{\delta_{g}} \left( \sum_{j} Y_{ij}^2 | X_{ij} \right) + \varepsilon_{\delta_{g}} \left( \sum_{j \neq j'} Y_{ij} Y_{ij'} | X_{ij}, X_{ij'} \right) \right\} \\
= \left( \sum_{i} \frac{X_{ij}^2}{x_i^2} \right) \left( \sigma^2 \sum_{i} X_{ij}^2 + \beta_i^2 \sum_{j} X_{ij}^2 \right) + \sum_{j \neq j'} \sum \varepsilon_{\delta_{g}} (Y_{ij} Y_{ij'} | X_{ij}, X_{ij'}) \\
= \left( \sum_{i} \frac{X_{ij}^2}{x_i^2} \right) \left[ \sigma^2 \sum_{j} X_{ij}^2 + \beta_i^2 \sum_{j} X_{ij}^2 \right] \\
+ \sum_{j \neq j'} \sum \varepsilon_{\delta_{g}} (Y_{ij} Y_{ij'} | X_{ij}, X_{ij'}) + \varepsilon_{\delta_{g}} (Y_{ij} | X_{ij}) \varepsilon_{\delta_{g}} (Y_{ij'} | X_{ij'}) \\
= \sum_{i} \frac{X_{ij}^2}{x_i^2} \left[ \sigma^2 \sum_{j} X_{ij}^2 + \left\{ \beta_i^2 \sum_{j} X_{ij}^2 + \beta_i^2 \sum_{j \neq j'} \sum X_{ij} X_{ij'} \right\} \right] \\
= \sigma^2 \left( \sum_{j} \frac{X_{ij}^2}{x_i^2} \right) \sum_{j} X_{ij}^2 + \beta_i^2 \left( \sum_{j} \frac{X_{ij}^2}{x_i^2} \right) X_{ij}^2 \\
\varepsilon_{\delta_{g}} \left\{ \left( \sum_{j} R_{ij}^2 X_{ij}^2 \right) | X \right\} = \sigma^2 \sum_{j} \frac{X_{ij}^2}{x_i^2} \sum_{j} X_{ij}^2 + \beta_i^2 \sum_{j} X_{ij}^2 \\
\varepsilon_{\delta_{g}} \left\{ \sum_{j} (X_{ij}^2) R_{ij}^2 | X \right\} = \sigma^2 \kappa_i(x) \sum_{j} X_{ij}^2 + \beta_i^2 \sum_{j} X_{ij}^2 \\
(6.2.10)
\end{align*}
\]
where
\[ K_i(x) = \frac{\sum X_{ij}^2}{X_i^2} = \frac{1 + C_i^2(x)}{N_i} \]  
(6.2.11)
and \( C_i(x) = \frac{\sigma_i(x)}{X_i} \) is the coefficient of variation of \( X \).

Finally,
\[ \mathcal{E}_{\delta_y} \left[ \left( R_i \sum_j X_{ij} Y_{ij} \right) \mid X \right] = \frac{1}{X_i} \mathcal{E}_{\delta_y} \left[ \left( \sum_j X_{ij} Y_{ij} \right) \left( \sum_j Y_{ij} \right) \mid X \right] \\
= \frac{1}{X_i} \mathcal{E}_{\delta_y} \left[ \sum_j X_{ij} Y_{ij}^2 + \sum_j X_{ij} \sum_{j' \neq j} Y_{ij} Y_{ij'} X \right] \\
= \frac{1}{X_i} \mathcal{E}_{\delta_y} \left[ \sum_j X_{ij} \mathcal{E}_{\delta_y} (Y_{ij}^2 \mid X_{ij}) + \sum_j X_{ij} \sum_{j' \neq j} \mathcal{E}_{\delta_y} (Y_{ij} Y_{ij'} \mid X_{ij}, X_{ij'}) \right] \\
= \frac{1}{X_i} \left[ \sigma^2 \sum_j X_{ij}^g + \beta_i^2 \sum_j X_{ij}^2 (X_{ij} + \sum_{j' \neq j} X_{ij'}) \right] \\
= \frac{1}{X_i} \left[ \sigma^2 \sum_j X_{ij}^g + \beta_i^2 \sum_j X_{ij}^2 \left( \sum_j X_{ij} \right) \right]. \\
\mathcal{E}_{\delta_y} \left[ \left( R_i \sum_j X_{ij} Y_{ij} \right) \mid X \right] = \sigma^2 \left( \frac{\sum_j X_{ij}^g + 1}{X_i} \right) + \beta_i^2 \sum_j X_{ij}^2. \)  
(6.2.12)

Now substituting from (6.2.9), (6.2.10), (6.2.12) in (6.2.8), we get
\[ E_{\delta_y} (\sigma_i^2(u) | X) = \frac{1}{N_i} \left[ \left( \sigma^2 \sum_j X_{ij}^g + \beta_i^2 \sum_j X_{ij}^2 \right) + \left\{ \sigma^2 \kappa_i(x) \sum_j X_{ij}^g \right\} \\
+ \beta_i^2 \sum_j X_{ij}^2 \right] - 2 \left\{ \frac{\sum_j X_{ij}^{g+1}}{X_i} + \beta_i^2 \sum_j X_{ij}^2 \right\} \]

\[ = \sigma^2 \left[ \{1 + \kappa_i(x)\} \frac{1}{N_i} \sum_j X_{ij}^g - \frac{2}{X_i} \frac{1}{N_i} \sum_j X_{ij}^{g+1} \right] \]

where

\[ E_{\delta_y} \{ \sigma_i^2(u) | X \} = \sigma^2 a_i(g) \] (6.2.13)

with

\[ a_i(g) = \{1 + \kappa_i(x)\} \bar{X}_i^g - \frac{2}{X_i} \bar{X}_i^{g+1} \] (6.2.14)

is the mean of \( h^{th} \) power of \( X \) in the \( i^{th} \) stratum.

From (6.2.13), we have

\[ E_{\delta_y} \{ \sigma_i^2(u) | X \} \propto a_i(g). \]
Thus $\sigma_i^2(u)$ is expected under the model $\Delta_g$ to be proportional to $a_i(g)$. Hence the TN optimum allocation under the model $\Delta_g$ is obtained by substituting $a_i(g)$ in place of unknown $\sigma_i^2(u)$ in the TNOA and is given below:

$$n_i(g) \propto N_i \sqrt{a_i(g)} \quad (6.2.15)$$

**Definition 6.2.1.** The allocation (6.2.15) is called $\Delta_g$-Model Optimum Allocation ($\Delta_g$-MOA) for the sampling strategy (St.SRSWR: $Y_{SR}$).

$$n_i(g) \propto N_i \sqrt{a_i(g)} .$$

When $g$ is known, the $\Delta_g$-MOA can readily be computed. However, if $g$ is not known, a pilot survey may be planned to collect data for estimating the superpopulation parameters. If that is also not feasible, the main survey may be conducted in phased manner, estimating the superpopulation parameters from the data collected in the first phase and utilising these estimates for better planning of the final phase.

We have proved the following theorem:

**Theorem 6.2.2.** The Tschuprow-Neyman optimum allocation under the superpopulation model $\Delta_g$ defined by (6.2.1) for ratio estimator under stratified sampling with simple random sampling strategy (St.SRSWR: $\hat{Y}_{SR}$) reduces to
the $\Delta_g$-Model Optimum Allocation ($\Delta_g$-MOA):

\[ n_i(g) \propto N_i \sqrt{\left\{1 + \kappa_i(x)\right\}} \frac{1}{N_i} \sum_j X_{ij}^g - \frac{2}{X_i} \frac{1}{N_i} \sum_j X_{ij}^{g+1} \]

where

\[ \kappa_i(x) = \frac{1 + C_i^2(x)}{N_i} \]

and $C_i(x) = \frac{\sigma_i(x)}{X_i}$ is the coefficient of variation of the auxiliary variable $X$.

### 6.2.3 Approximation to $\Delta_g$-MOA

We now obtain some quick approximations to the $\Delta_g$-MOA which may be useful in practice. If the stratum total is large compared to the mean of $(g + 1)^{th}$ power of the auxiliary variable $X$ in each stratum then the second term of $a_i(g)$ in (6.2.14) viz.,

\[ \frac{2}{X_i} \left(X_i^{g+1}\right) \]

is expected to be small quantity and more so specially compared to $a_i(g)$ i.e.,

\[ \frac{2}{X_i} X_i^{g+1} \ll a_i(g). \]
Therefore the relative neglected term (RNT) $l'_i(g)$ is given by
\[
l'_i(g) = \frac{2}{X_i} \left( \frac{X_i^{g+1}}{a_i(g)} \right) \approx 0 \ll 1.
\] (6.2.16)

This approximation may be called Large Stratum Totals Approximation (LSTA). Under the LSTA, $a_i(g)$ may be approximated by
\[
a'_i(g) = \{1 + \kappa_i(x)\} \left( \frac{X_i^g}{X_i} \right). \tag{6.2.17}
\]

Therefore the first approximate $\Delta_g$-MOA ($\Delta_g$-AMOA) based on $\left( \frac{X_i^g}{X_i} \right)$ is given by
\[
n_i(g) \propto N_i \sqrt{a'_i(g)} = N_i \sqrt{\{1 + \kappa_i(x)\} \left( \frac{X_i^g}{X_i} \right)}. \tag{6.2.18}
\]

Further if $\kappa_i(x)$ can be neglected relative to unity i.e.,
\[
\kappa_i(x) = \frac{1 + C_i^2(x)}{N_i} \approx 0 \ll 1
\]
holds, then $a_i(g)$ may be approximated by
\[
a''_i(g) = \frac{1}{N_i} \sum_j X_i^g = \left( \frac{X_i^g}{X_i} \right) \tag{6.2.19}
\]
and the RNT is given by
\[ l''_i(g) = \frac{\kappa_i(x) \left( \bar{X}_i^g \right) - \frac{2}{N_i} \sum_j X_{ij}^{g+1}}{a_i(g)} \]

\[ = \frac{a''_i(g)}{a_i(g)} - 1. \tag{6.20} \]

The corresponding approximate allocation from \( \Delta_g \)-MOA is given as

\[ n_i(g) \propto N_i \sqrt{a''_i(g)} = N_i \sqrt{\left( \bar{X}_i^g \right)} = \sqrt{N_i \sum_j X_{ij}^g}. \tag{6.21} \]

Now

\[ \frac{1}{N_i} \sum_j X_{ij}^{g+1} = \sigma_i(x, x^g) + \bar{X}_i \left( \bar{X}_i^g \right) \tag{6.22} \]

where

\[ \sigma_i(x, x^g) = \frac{1}{N_i} \sum_j (X_{ij} - \bar{X}_i) \left\{ X_{ij}^g - \left( \bar{X}_i^g \right) \right\} = \text{Cov}_i(x, x^g) \]

is the covariance between \( x \) and \( x^g \) in the \( i^{th} \) stratum.
\[
\frac{2}{X_i} \left( X_i^{g+1} \right) = \frac{2}{X_i} \left\{ \sigma_i(x, x^g) + \bar{X}_i \left( X_i^g \right) \right\}
\]

\[
= \frac{2}{N_i} \rho_i(x, x^g) C_i(x) \sigma_i(x^g) + \frac{2}{N_i} \left( \bar{X}_i^g \right)
\]

assuming

\[
\frac{2}{N_i} \rho_i(x, x^g) C_i(x) \sigma_i(x^g) \approx 0.
\]

Therefore when \( \frac{2}{X_i} \left( X_i^{g+1} \right) \) is not negligible compared to \( a_i(g) \) i.e., \( l'_i(g) \) is not negligible relative to unity, \( \frac{2}{X_i} \left( X_i^{g+1} \right) \) can be approximated by \( \frac{2}{N_i} \left( \bar{X}_i^g \right) \) and consequently \( a_i(g) \) can be approximated by

\[
a_i'''(g) = \left( 1 + \kappa_i(x) - \frac{2}{N_i} \right) \left( \bar{X}_i^g \right). \tag{6.2.23}
\]

If the RNT for this approximation given by

\[
l'''_i(g) = \frac{2}{X_i} \sigma_i(x, x^g) = \frac{a_i'''(g)}{a_i(g)} - 1 \quad \tag{6.2.24}
\]

is such that

\[
l'''_i(g) \approx 0 \ll 1 \quad \tag{6.2.25}
\]

then the corresponding approximate \( \Delta_g \)-MOA is given by

\[
n_i(g) \propto N_i \sqrt{a_i'''(g)} = N_i \left( 1 + \kappa_i(x) - \frac{2}{N_i} \right) \left( \bar{X}_i^g \right). \tag{6.2.26}
\]
Definition 6.2.3. The three allocations given by (6.2.18), (6.2.21) and (6.2.26) will be called \( \Delta_g \)-Approximate MOAs, (\( \Delta_g \) -AMOAs) viz., \( \Delta_g \)-AMOA1, \( \Delta_g \)-AMOA2 and \( \Delta_g \)-AMOA3 respectively for the sampling strategy (St.SRSWR: \( \hat{Y}_{SR} \)).

Thus we have proved the following theorem:

**Theorem 6.2.4.** If \( \frac{2}{X_i} \bar{X}_i^{g+1} \) can be neglected relative to \( a_i(g) \) i.e.,

\[
\ell_i' = \frac{\frac{2}{X_i} \bar{X}_i^{g+1}}{a_i(g)} \approx 0 \ll 1
\]

then the \( \Delta_g \)-MOA for the strategy (St.SRSWR: \( \hat{Y}_{SR} \)) may be approximated by \( \Delta_g \)-Approximate MOA1 (\( \Delta_g \)-AMOA1) given by

\[
n_i(g) \propto N_i \sqrt{1 + \kappa_i(x)} \bar{X}_i^g
\]

where

\[
\bar{X}_i^g = \frac{1}{N_i} \sum_{j=1}^{N_i} X_{ij}^g
\]

and

\[
\kappa_i(x) = \frac{1 + C_i^g(x)}{N_i}.
\]

Further, if \( \kappa_i(x) \) is also negligible relative to unity, besides \( \ell_i' \), then \( \Delta_g \)-MOA
may be replaced by the following approximate \( \Delta_g\text{-MOA}_2 \) viz., \( \Delta_g\text{-AMOA}_2 \):

\[
n_i(g) \propto N_i \sqrt{X_i^g} = \sqrt{N_i \sum X_{ij}^g}.
\]

However if \( \frac{2}{X_i} X_i^{g+1} \) is not negligible relative to \( a_i(g) \) but it can be replaced by \( \frac{2}{N_i} X_i^g \) i.e., if

\[
\frac{2}{X_i} X_i^{g+1} \approx \frac{2}{N_i} X_i^g
\]

then \( \Delta_g\text{-MOA} \) can be approximated by \( \Delta_g\text{-AMOA}_3 \) given by

\[
n_i(g) \propto N_i \sqrt{\left\{ 1 + \kappa_i(x) - \frac{2}{N_i} \right\} X_i^g}.
\]

6.2.4 \( \Delta_g\text{-AMOAs under the ESCV-Assumption} \)

Now we obtain some \( \Delta_g\text{-AMOAs under the assumption that the coefficients of variation of } x^g/2 \text{ are equal in all strata. Now}

\[
\overline{X_i^g} = \frac{1}{N_i} \sum_j X_{ij}^g = \sigma_i^2(x^{g/2}) + \left( \frac{\overline{x}_i^{g/2}}{2} \right)^2.
\]

(6.2.27)
Now substituting the value of $X_i^g$ from (6.2.28) in $a'_i(g)$ in (6.2.17), we have
\[
a'_i(g) = \{1 + \kappa_i(x)\} \left( X_i^{g/2} \right)^2 \left\{ C_i^2(x^{g/2}) + 1 \right\}.
\] (6.2.29)

Now if $C_i^2(x^{g/2}) \propto \text{const.}$ then
\[
a'_i(g) \propto \{1 + \kappa_i(x)\} \left( X_i^{g/2} \right)^2
\]
and therefore $\Delta_g$-AMOA under the assumption of $\text{ESC}V(x^{g/2})$ is given below:
\[
n_i(g) \propto \sum_j X_{ij}^{g/2} \sqrt{(1 + \kappa_i(x))}.
\] (6.2.30)

Similarly substituting $X_i^g$ from (6.2.28) in $a''_i(g)$ and $a'''_i(g)$ in (6.2.19) and (6.2.23) respectively, we have
\[
a''_i(g) = \left( X_i^{g/2} \right)^2 \left( C_i^2(x) + 1 \right)
\] (6.2.31)
and
\[
a'''_i(g) = \left( 1 + \kappa_i(x) - \frac{2}{N_i} \right) \left( X_i^{g/2} \right)^2 \left( C_i^2(x) + 1 \right).
\] (6.2.32)

Consequently $\Delta_g$-AMOAs corresponding to $a''_i(g)$ and $a'''_i(g)$ under the assumption of equal $\text{SC}V(x^{g/2})$ are respectively given as follows:
\[ n_t(g) \propto \sum_j X_{ij}^{g/2} \tag{6.2.33} \]

and

\[ n_t(g) \propto \sqrt{\left(1 + \kappa_i(x) - \frac{2}{N_i}\right) \sum_j X_{ij}^{g/2}}. \tag{6.2.34} \]

**Definition 6.2.5.** Three allocations given by (6.2.30), (6.2.33) and (6.2.34) based on \( X_{i}^{g/2} \) under the assumption of equal SCV \( (x^g)^2 \) for all strata are called \( \Delta_g \)-Mean-based ESCVAMOs (\( \Delta_g^{M}-\text{ESCVAMOAs} \)) viz., \( \Delta_g^{M}-\text{ESCVAMO1} \), \( \Delta_g^{M}-\text{ESCVAMO2} \) and \( \Delta_g^{M}-\text{ESCVAMO3} \) respectively for the strategy \( \hat{Y}_{SR} \).

Next we alternatively write the value of \( X_f \) from (6.2.27) as follows:

\[ X_f = \frac{\sigma^2_i(x^g/2)}{2} + \left( X_{i}^{g/2} \right)^2 \tag{6.2.35} \]

Now substituting this value of \( X_f \) from (6.2.35) in \( a'_i(g) \), \( a''_i(g) \) and \( a''''_i(g) \), given by (6.2.17) (6.2.19) and (6.2.23) respectively, we get

\[ a'_i(g) = \{1 + \kappa_i(x)\} \sigma_i^2(x^g/2) \left(1 + \frac{1}{C_i^2(x^g/2)}\right), \tag{6.2.36} \]

\[ a''_i(g) = \sigma_i^2(x^g/2) \left(1 + \frac{1}{C_i^2(x^g/2)}\right), \tag{6.2.37} \]

200
Again if it is assumed that the SCV\(x^{g/2}\) are equal in all strata, then the \(\Delta_g-AMOAs\) based on above \(a^i_1(g), a^i_2(g)\) and \(a^i_3(g)\) given by (6.2.36), (6.2.37) and (6.2.38) are given below:

\[
n_i(g) \propto N_i \sigma_i(x^{g/2}) \sqrt{1 + \kappa_i(x)} , \quad (6.2.39)
\]

\[
n_i(g) \propto N_i \sigma_i(x^{g/2}) \quad (6.2.40)
\]

and

\[
n_i(g) \propto N_i \sigma_i(x^{g/2}) \sqrt{1 + \kappa_i(x) - \frac{2}{N_i}} . \quad (6.2.41)
\]

**Definition 6.2.6.** Three allocations given by (6.2.39), (6.2.40) and (6.2.41) based on \(\sigma_i(x^{g/2})\) under the assumption of ESCV\(x^{g/2}\) for all strata are called \(\Delta_g\)-Standard Deviation (SD)-based \(\Delta_g\)-ESCVAMOAs (\(\Delta_g\)-SDESCVAMOAs) viz., \(\Delta_g\)-SDESCVAMOA1, \(\Delta_g\)-SDESCVAMOA2 and \(\Delta_g\)-SDESCVAMOA3 respectively for the strategy (St.SRSWR: \(\hat{Y}_{SR}\)).

**Theorem 6.2.7.** If \(C_i(x^{g/2})\), the stratum coefficients of variation of \(x^{g/2}\) are equal in all strata, then each of \(\Delta_g\)-AMOA1, \(\Delta_g\)-AMOA2 and \(\Delta_g\)-AMOA3 for the strategy (St.SRSWR: \(\hat{Y}_{SR}\)) give rise to 2 types of \(\Delta_g\)-ESCVAMOAs viz., \(\Delta_g\)-MESCVAMOAs based on mean \(X_i^{g/2}\) and \(\Delta_g\)-SDESCVAMOAs based on standard deviation \(\sigma_i(x^{g/2})\) of \(x^{g/2}\) in the \(i\)th stratum:
I. Based on $\Delta_g$-AMOA1

1. $\Delta_g - MCVAMOA1 : n_i(g) \propto \sum_j X_{ij}^{g/2} \sqrt{1 + \kappa_i(x)}$

and

2. $\Delta_g - SDESCVAMOA1 : n_i(g) \propto N_i \sigma_i(x_i^{g/2}) \sqrt{1 + \kappa_i(x)}$.

II. Based on $\Delta_g$-AMOA2

1. $\Delta_g - MESCVAMOA2 : n_i(g) \propto \sum_j X_{ij}^{g/2}$

and

2. $\Delta_g - SDESCVAMOA2 : n_i(g) \propto N_i \sigma_i(x_i^{g/2})$.

III. Based on $\Delta_g$-AMOA3

1. $\Delta_g - MESCVAMOA3 : n_i(g) \propto \sum_j X_{ij}^{g/2} \sqrt{1 + \kappa_i(x) - \frac{2}{N_i}}$

and

2. $\Delta_g - SDESCVAMOA3 : n_i(g) \propto N_i \sigma_i(x_i^{g/2}) \sqrt{1 + \kappa_i(x) - \frac{3}{N_i}}$.

All the $\Delta_g$-AMOAs and $\Delta_g$-ESCVAMOAs obtained above are presented in the table 6.2.2.1 below:
Table - 6.2.2.1

Summary of Approximate Allocations
for SRE for St.SRSWR

<table>
<thead>
<tr>
<th>Approx. (R.N.T.)</th>
<th>( l'_1(g) \approx 0 )</th>
<th>( l''_1(g) \approx 0 )</th>
<th>( l'''_1(g) \approx 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplier</td>
<td>( \sqrt{1 + \kappa_1(x)} )</td>
<td>1</td>
<td>( \sqrt{1 + \kappa_1(x) - \frac{2}{N_1}} )</td>
</tr>
<tr>
<td>Basic Quantity</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. \( N_1 \sqrt{X_i^2} \)

- AMOA1: \( \propto N_1 \sqrt{a'_1(g)} \)
- AMOA2: \( \propto N_1 \sqrt{a''_1(g)} \)
- AMOA3: \( \propto N_1 \sqrt{a'''_1(g)} \)

Assump.: \( C_1(x^{9/2}) \)

2. \( \sum_{j} X_{ij}^{9/2} \)

- MESCVA MOA1: \( \propto \sum_{j} X_{ij}^{9/2} \)
- MESCVA MOA2: \( \propto \sum_{j} X_{ij}^{9/2} \)
- MESCVA MOA3: \( \propto \sum_{j} X_{ij}^{9/2} \)

3. \( N_1 \sigma_1(x^{9/2}) \)

- SDESCVA MOA1: \( \propto N_1 \sigma_1(x^{9/2}) \)
- SDESCVA MOA2: \( \propto N_1 \sigma_1(x^{9/2}) \)
- SDESCVA MOA3: \( \propto N_1 \sigma_1(x^{9/2}) \)

\[ \sqrt{1 + \kappa_1(x) - \frac{2}{N_1}} \]

6.2.5 Particular Cases of \( \Delta_g \)-Model-based Allocations

Some particular cases of the \( \Delta_g \)-MOA, \( \Delta_g \)-AMOAs and \( \Delta_g \)-ESCVAMOAs obtained in subsections 6.2.2, 6.2.3 and 6.2.4 are deduced for certain values.
of the superpopulation parameter $g$ of the model $\Delta_g$ in this subsection.

Case I: Model $\Delta_0$

Substituting $g = 0$ in $a_i(g)$ in (6.2.14), we have,

$$a_i(0) = \{1 + \kappa_i(x)\} - \frac{2}{\bar{X}_i} \bar{X}_i$$

$$= 1 + \kappa_i(x) - \frac{2}{N_i}$$

$$a_i(0) = 1 - \left(1 - \frac{C_i^2(x)}{N_i}\right).$$

Therefore $\Delta_0$-MOA is given by

$$n_i(0) \propto N_i \sqrt{1 + \kappa_i(x) - \frac{2}{N_i}}$$

$$\propto N_i \sqrt{1 - \frac{1 - C_i^2(x)}{N_i}}.$$

Case II: Model $\Delta_1$

Similarly substituting $g = 1$ in $a_i(g)$ in (6.2.14) we have

$$a_i(1) = \{1 + \kappa_i(x)\} \bar{X}_i - \frac{2}{\bar{X}_i} \bar{X}_i^2$$

$$= \{1 + \kappa_i(x)\} \bar{X}_i - \frac{2}{\bar{X}_i} \left\{\sigma_i^2(x) + \bar{X}_i^2\right\}$$

$$= \bar{X}_i \left\{1 + \kappa_i(x)\right\} - \frac{2}{N_i} \left\{1 + C_i^2(x)\right\}$$

$$= \bar{X}_i \{1 - \kappa_i(x)\}.$$

Therefore $\Delta_1$-MOA becomes
Finally putting $g = 2$ in $a_i(g)$ in (6.2.14), we have

$$a_i(2) = \{1 + \kappa_i(x)\} \frac{X_i^2}{X_i^3} - \frac{2}{X_i} \left( X_i^3 \right)$$

$$= \{1 + \kappa_i(x)\} \left( \sigma_i^2(x) + \bar{X}_i^2 \right) - \frac{2}{X_i} \frac{1}{N_i} \sum_j X_{ij}^3$$

$$= \bar{X}_i^2 \left[ N_i \kappa_i(x) \left\{1 + \kappa_i(x)\right\} - \frac{2}{N_i} \frac{X_i^3}{X_i^3} \right]$$

$$= \bar{X}_i^2 \left[ N_i \kappa_i(x) \left\{1 + \kappa_i(x)\right\} - \frac{2}{N_i} \frac{\mu'_3(x)}{\mu'_3(x)} \right]$$

where

$$\mu'_r(x) = \frac{1}{N_i} \sum_j X_{ij}^r,$$

is the $r$th raw moment of the auxiliary variable $X$.

Hence $A_2$-MOA reduces to

$$n_i(2) \propto X_i \sqrt{N_i \kappa_i(x) \left\{1 + \kappa_i(x)\right\} - \frac{2}{N_i} \frac{\mu'_3(x)}{\mu'_3(x)}}.$$
1 and 2 and the same are presented in tables 6.2.3.1 and 6.2.3.2 below:

**Table - 6.2.3.1**

RNT and Multiplier for AMOAs and ESCVAMOAs for St.SRSWR.

<table>
<thead>
<tr>
<th>Suffix of the Allocation</th>
<th>RNT (Approx.)</th>
<th>Multiplier</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$l'_1(g) = \frac{\frac{\sigma_i(x^g)}{a_i(g)}}{x^g} \approx 0$</td>
<td>$\sqrt{1 + \kappa_i(x)}$</td>
</tr>
<tr>
<td>2</td>
<td>$l''_1(g) = \frac{\kappa_i(x) x^g - \frac{3}{2} x^g + 1}{a_i(g)} \approx 0$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$l''_1(g) = \frac{\frac{\sigma_i(x^g)}{a_i(g)}}{x^g} \approx 0$</td>
<td>$\sqrt{1 + \kappa_i(x)} - \frac{2}{N_i}$</td>
</tr>
</tbody>
</table>

**Table - 6.2.3.2**

**Expression of AMOAs and ESCVAMOAs for different $g$ for St.SRSWR.**

<table>
<thead>
<tr>
<th>Allocation</th>
<th>Type</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMOAs</td>
<td>1</td>
<td>$N_i$</td>
</tr>
<tr>
<td>Assump.: $C_i(x^{g/2})$ equal</td>
<td>Not required</td>
<td>$C_i(\sqrt{x})$ equal</td>
</tr>
<tr>
<td>ESCVAMOAs</td>
<td>2</td>
<td>$N_i \sqrt{x_i} \sum \sqrt{X_{ij}}$</td>
</tr>
<tr>
<td>SDESCVAMOAs</td>
<td>3</td>
<td>does not exist</td>
</tr>
</tbody>
</table>

* The name of the allocation is obtained by suffixing 1, 2 or 3 in the allocation type from table 6.2.3.2.

** The formula for the allocation is obtained by multiplying the expression from table 6.2.3.2 by the corresponding multiplier from table 6.2.3.1.
6.3 Stratified SRSWOR Design

6.3.1 TNOA under Superpopulation Model

Suppose simple random sampling without replacement is used within each stratum, the separate ratio(SR) estimator (SRE) for the population total is a biased estimator. Under the assumption (6.2.2) mentioned in the subsection 6.2.1, the bias of \( \hat{Y}_{SR} \) is given by

\[
B(\hat{Y}_{SR}) = \sum_{i=1}^{k} N_i^2 \left( \frac{1 - f_i}{n_i} \right) \frac{1}{X_i} \left\{ R_i S^2_i(x) - S_i(x, y) \right\}
\]

where

\[
S_i(x, y) = \frac{1}{N_i - 1} \sum_j (X_{ij} - \bar{X}_i)(Y_{ij} - \bar{Y}_i)
\]

is the covariance between \( X \) and \( Y \) in the \( i^{th} \) stratum.

The bias to the second order of approximation is negligible. The approximate variance under large sample approximations is given by

\[
V(\hat{Y}_{SR})_{WOR} = \sum_i N_i^2 \left( \frac{1 - f_i}{n_i} \right) S^2_i(u), \tag{6.3.1}
\]

where
\[ S_i^2(u) = \frac{N_i}{N_i - 1} \sigma_i^2(u) = \frac{1}{N_i - 1} \sum_{j=1}^{N_i} U_{ij}^2 \]  \hspace{2cm} (6.3.2) 

wherein \( U_{ij} = Y_{ij} - R_i X_{ij} \) is the transformed variable for all \( j = 1, 2, \ldots, N_i \) and \( i = 1, 2, \ldots, k \) and \( f_i = n_i/N_i \) is the sampling fraction in the \( i^{th} \) stratum.

Thus

\[ S_i^2(u) = \frac{1}{N_i - 1} \sum_j \left( Y_{ij}^2 + R_i^2 X_{ij}^2 - 2 R_i X_{ij} Y_{ij} \right) \]  \hspace{2cm} (6.3.3) 

\[ = S_i^2(y) + R_i^2 S_i^2(x) - 2 R_i S_i(x, y). \]

The TNOA of fixed total sample size \( n \) which minimises the variance in (6.3.1) is given by

\[ n_i \propto N_i S_i(u). \]  \hspace{2cm} (6.3.4) 

In \( S_i^2(u) \), all the quantities except \( S_i(x) \) are unknown. In order to compute TNOA, some estimates of \( S_i^2(u) \) are needed. As in case of SRSWR, the expectation of \( S_i^2(u) \) is obtained under the model \( \Lambda_g \) in (6.2.1) in the following subsection so that the same could be substituted in place of the unknown \( S_i^2(u) \) in the TNOA in (6.3.4).
6.3.2 TNOA under the Model $\Delta_g$

We write the expression for expectation of $S^2_i(u)$ given $X$ under the model $\Delta_g$ from that for $\sigma^2_i(u)$ from (6.2.13), which would be substituted for the unknown $S_i(u)$ in the TNOA.

\[
E_{\delta_g} \{S^2_i(u)|X\} = \left( \frac{N_i}{N_i-1} \right) E_{\delta_g} \{\sigma^2_i(u)|X\} = \left( \frac{N_i}{N_i-1} \right) \sigma^2 a_i(g)
\]

\[
= \sigma^2 \left[ \left(1 + \kappa_i(x)\right) \frac{1}{N_i-1} \sum_j X_{ij}^g - \frac{2}{N_iN_i-1} \frac{1}{X_i} \sum_j X_{ij}^{g+1} \right]
\]

\[
= \sigma^2 A_i(g), \quad \text{(6.3.5)}
\]

where

\[
A_i(g) = \left\{1 + \kappa_i(x)\right\} M_i(g) - \frac{2}{X_i} M_i(g + 1) \quad \text{(6.3.6)}
\]

with

\[
M_i(h) = \frac{1}{N_i-1} \sum_j X_{ij}^h
\]

is the modified mean of $h^{th}$ power of $X$ in the $i^{th}$ stratum.

From (6.3.5), we have

\[
E_{\delta_g} \{S^2_i(u)|X\} \propto A_i(g)
\]

209
Thus $S^2_i(u)$ is expected under the model $\Delta_g$ to be proportional to $A_i(g)$. Hence the TN optimum allocation under the model $\Delta_g$ is obtained by substituting $A_i(g)$ in place of unknown $S^2_i(u)$ in the TNOA and is given below:

$$n_i(g) \propto N_i \sqrt{A_i(g)}$$

(6.3.7)

**Definition 6.3.1.** The allocation (6.3.7) is called $\Delta_g$-Model Optimum Allocation ($\Delta_g$-MOA) for the sampling strategy (St.SRSWOR: $\hat{Y}_{SR}$).

The result is stated in the following theorem:

**Theorem 6.3.2.** The Tschuprow-Neyman optimum allocation under the superpopulation model $\Delta_g$ defined in (6.2.1) for Separate Ratio Estimator under stratified sampling strategy (St.SRSWOR: $\hat{Y}_{SR}$) reduces to the $\Delta_g$-model optimum allocation ($\Delta_g$-MOA).

$$n_i(g) \propto N_i \sqrt{\left\{1 + \kappa_i(x)\right\}} \frac{1}{N_i - 1} \sum_j X^g_{ij} - \frac{1}{N_i} \sum_j X^g_{ij+1},$$

where

$$\kappa_i(x) = \frac{1 + C_i^2(x)}{N_i}$$

and

$$C_i(x) = \frac{\sigma_i(x)}{\bar{X}_i}$$
is the coefficient of variation of $X$.

### 6.3.3 Approximations to $\Delta_g$-MOA

We shall now find some useful approximations to the $\Delta_g$-MOA. The second term of $A_i(g)$ in (6.3.6) viz., \( \frac{2}{X_i} M_i(g+1) \) is expected to be negligible quantity relative to $A_i(g)$ i.e.,

\[
\frac{2}{X_i} M_i(g+1) \ll A_i(g).
\]

Thus, if the relative neglected term (RNT) is such that

\[
l'_i(g) = \frac{\frac{2}{X_i} M_i(g+1)}{A_i(g)} \approx 0 \ll 1,
\]

then $A_i(g)$ may be approximated under LSTA by

\[
A'_i(g) = (1 + \kappa_i(x)) M_i(g) = \frac{N_i}{N_i - 1} a'_i(g) .
\]

(6.3.8)

The RNT may be expressed as

\[
l'_i(g) = \frac{\frac{2}{X_i} M_i(g+1)}{A_i(g)} = \frac{A'_i(g) - A_i(g)}{A_i(g)} = \frac{A'_i(g)}{A_i(g)} - 1 .
\]

(6.3.9)
Hence the first approximation to $\Delta_g$-MOA allocation based on $M_i(g)$ is given by

$$n_i(g) \propto N_i \sqrt{A_i'(g)} = N_i \sqrt{1 + \kappa_i(x)} M_i(g). \quad (6.3.10)$$

Further if $\kappa_i(x)$ may be assumed negligible relative to unity i.e.,

$$\kappa_i(x) = \frac{1 + C_i^2(x)}{N_i} \approx 0 \ll 1 \quad (6.3.11)$$

then $A_i(g)$ may be approximated by

$$A_i'(g) = M_i(g) = \frac{1}{N_i - 1} \sum_j X_{ij}^g \quad (6.3.12)$$

and the RNT in this case may be expressed as

$$l_i''(g) = \frac{\kappa_i(x)}{M_i(g)} - \frac{2}{X_i} M_i(g + 1) A_i(g) = \frac{A_i''(g)}{A_i(g)} - 1. \quad (6.3.13)$$

The second approximation to the $\Delta_g$-MOA in this case is given by

$$n_i(g) \propto N_i \sqrt{A_i''(g)} = N_i \sqrt{M_i(g)}. \quad (6.3.14)$$

Now,

$$M_i(g + 1) = \frac{1}{N_i - 1} \sum_j X_{ij}^{g+1} = S_i(x, x^g) + \frac{N_i}{N_i - 1} \bar{X}_i \bar{X}_i^g$$

where

$$S_i(x, x^g) = \frac{1}{N_i - 1} \sum_i (X_{ij} - \bar{X}_i)(X_{ij}^g - \bar{X}_i^g)$$

212
\[
\frac{1}{N_i-1} \left[ \sum_i X_{ij}^{g+1} - N_i \bar{X}_i \bar{X}_i^g \right]
\]

is the covariance between \(x\) and \(x^g\)-values of the auxiliary variable \(X\) in the \(i^{th}\) stratum.

Hence

\[
\frac{2}{X_i} M_i(g + 1) = \frac{2}{X_i} \left[ S_i(x, x^g) + \frac{N_i}{N_i-1} \bar{X}_i \bar{X}_i^g \right]
\]

\[
= \frac{2}{N_i} \rho_i(x, x^g) C_i'(x) S_i(X^g) + \frac{2}{N_i} M_i(g)
\]

\[
\approx \frac{2}{N_i} M_i(g).
\]

Since the first term being the product of 3 quantities, of which two viz., \(\rho_i(x, x^g)\) and \(C_i(x)\) are very small, when divided by relatively large stratum size \(N_i\) can be safely assumed negligible. Then we have

\[
\frac{2}{X_i} M_i(g + 1) \approx \frac{2}{N_i} M_i(g).
\]

Thus even if \(l_i'(g)\) may not be negligible i.e., if the term \(\frac{2}{X_i} M_i(g + 1)\) may not be completely dropped, the same may be approximated by \(\frac{2}{N_i} M_i(g)\).

Therefore, \(A_i(g)\) may be approximated by

\[
A_i''(g) = \left\{ 1 + \kappa_i(x) - \frac{2}{N_i} \right\} M_i(g). \quad (6.3.15)
\]
The RNT in this approximation is given by

$$i''_i(g) = \frac{2}{\hat{X}_i} S_i(x, x^g) = A''_i(g) - 1.$$  \hspace{1cm} (6.3.16)

If

$$i''_i(g) \approx 0 < 1,$$

the third approximate $\Delta_g$-MOA based on $M_i(g)$ is given as follows:

$$n_i(g) \propto N_i \sqrt{A''_i(g)} = N_i \sqrt{(1 + \kappa_i(x) - \frac{2}{N_i})M_i(g)}.$$  \hspace{1cm} (6.3.17)

**Definition 6.3.3.** The three allocations given by (6.3.10), (6.3.14) and (6.3.17) are called $\Delta_g$-approximate MOAs ($\Delta_g$-AMOAs) viz., $\Delta_g$-AMOA1, $\Delta_g$-AMOA2 and $\Delta_g$-AMOA3 respectively for the sampling strategy (St.SRSWOR: $\tilde{Y}_{SR}$).

### 6.3.4 Large Strata Approximations

If the strata are so large that the term in $\frac{1}{N_i}$ can be ignored in all strata, then

$$M_i(g) = \frac{1}{N_i - 1} \sum_j X_{ij}^g \approx \frac{1}{N_i} \sum_j X_{ij}^g = \bar{X}_i^g$$  \hspace{1cm} (6.3.18)

where $\bar{X}_i^g$ is the mean of $g^{th}$ power of $X$ in the $i^{th}$ stratum, $i=1, 2 \ldots k$.

Consequently the second set of approximate $\Delta_g$-MOA may be obtained by
replacing $M_i(g)$ by $\overline{X_i^g}$ in all the three $\Delta_g$-AMOAs based on $M_i(g)$ as follows:

The $A'_i(g)$ in (6.3.8) may further be approximated by

$$a'_i(g) = \{1 + \kappa_i(x)\} \overline{X_i^g}, \quad (6.3.19)$$

which amounts to neglecting

$$\frac{a'_i(g) - A_i(g)}{A_i(g)} = \frac{a'_i(g)}{A_i(g)} - 1$$

in comparison to $A_i(g)$.

The first large strata $\Delta_g$-AMOA from $\Delta_g$-AMOA1, in (6.3.10) using $a'_i(g)$ in (6.3.19) based on $\overline{X_i^g}$ is given by

$$n_i(g) \propto N_i \sqrt{a'_i(g)} = N_i \sqrt{\{1 + \kappa_i(x)\} \overline{X_i^g}} \quad (6.3.20)$$

i.e.,

$$n_i(g) \propto \sqrt{\{1 + \kappa_i(x)\} N_i \left( \sum_j X_{ij}^g \right)}$$

Similarly the large strata approximations of the quantities $A''_i(g)$ and $A'''_i(g)$ in (6.3.12) and (6.3.15) are given by

$$a''_i(g) = \overline{X_i^g} \quad (6.3.21)$$

and

$$a'''_i(g) = \left( 1 + \kappa_i(x) - \frac{2}{N_i} \right) \overline{X_i^g} \quad (6.3.22)$$
respectively.

Consequently, the large strata $\Delta_g$-AMOAs from $\Delta_g$-AMOA2 and $\Delta_g$-AMOA3 in (6.3.14) and (6.3.17) respectively i.e., $\Delta_g$-AMOAs corresponding to $a''_i(g)$ and $a''_i(g)$ of (6.3.21) and (6.3.22) respectively based on $\overline{X_i^g}$ are given by

\[ n_i(g) \propto N_i \sqrt{a''_i(g)} = N_i \sqrt{\overline{X_i^g}} = \sqrt{N_i \sum_j X_{ij}^g} , \quad (6.3.23) \]

and

\[ n_i(g) \propto N_i \sqrt{a''_i(g)} = N_i \sqrt{\left\{ 1 + \kappa_i(x) - \frac{2}{N_i} \right\} \overline{X_i^g}} \]

\[ = \sqrt{\left\{ 1 + \kappa_i(x) - \frac{2}{N_i} \right\} N_i \sum_j X_{ij}^g} . \quad (6.3.24) \]

**Definition 6.3.4.** The three $\Delta_g$-AMOAs viz., $\Delta_g$-AMOA1, $\Delta_g$-AMOA2 and $\Delta_g$-AMOA3 given by (6.3.10), (6.3.14) and (6.3.17) are called $\Delta_g$-large strata $\Delta_g$-AMOAs ($\Delta_g$-LSAMOAs) viz., $\Delta_g$-LSAMOA1, $\Delta_g$-LSAMOA2 and $\Delta_g$-LSAMOA3 respectively for the sampling strategy (St.SRSW: $\hat{Y}_{SR}$).

$\Delta_g$-AMOAs under ESCV-Assumption

Now

\[ M_i(g) = \frac{1}{N_i - 1} \sum_j \left( X_{ij}^{g/2} \right)^2 \]

\[ M_i(g) = S_i^2(x^{g/2}) + N_i \left( \overline{X_i^{g/2}} \right)^2 \]

(6.3.25)
\[ N_j = \frac{N_i}{N_i - 1} \left( \frac{\overline{x}_i^{g/2}}{N} \right)^2 \left\{ \frac{S^2(x^{g/2})}{\overline{N}_i} \right\} + 1 \]

where

\[ C_i(x^{g/2}) = \frac{\sigma_i(x^{g/2})}{\overline{x}_i^{g/2}} \]

is the coefficient of variation of \( x^{g/2} \).

Now substituting the value of \( M_i(g) \) from (6.3.26) in \( A_i''(g) \) of (6.3.8), we have,

\[ A_i'(g) = \left\{ 1 + \kappa_i(x) \right\} \left\{ \frac{N_i}{N_i - 1} \right\} \left\{ \left( \frac{X_i^{g/2}}{N_i} \right)^2 \right\} \left( 1 + C_i^2(x^{g/2}) \right) \].

(6.3.27)

If it is assumed that \( C_i(x^{g/2}) \), the stratum coefficients of variation of \( x^{g/2} \) are equal in all strata, the \( \Delta_g \)-AMOA1 under this assumption, is given as follows:

\[ n_i(g) \propto \sum_j X_{ij}^{g/2} \sqrt{\left( \frac{N_i}{N_i - 1} \right) \{1 + \kappa_i(x)\}}. \]

(6.3.28)

Similarly substituting \( M_i(g) \) from (6.3.26) in \( A_i'''(g) \) and \( A_i''''(g) \) given by (6.3.12) and (6.3.15) respectively, we get
\[
A''_i(g) = \left( \frac{N_i}{N_i - 1} \right) \left( \frac{X_{i}^{9/2}}{N_i} \right)^2 \left\{ 1 + C_i^2(x_{i}^{9/2}) \right\}
\]  
(6.3.29)

and

\[
A'''_i(g) = \left\{ 1 + \kappa_i(x) \frac{2}{N_i} \right\} \left( \frac{N_i}{N_i - 1} \right) \left( \frac{X_{i}^{9/2}}{N_i} \right)^2 \left\{ 1 + C_i^2(x_{i}^{9/2}) \right\}.
\]  
(6.3.30)

Consequently \( \Delta_g\)-AMOAs under the assumption of ESCV(\(x_{i}^{9/2}\)) based on mean obtained from \(A''_i(g)\) and \(A'''_i(g)\) given by (6.3.29) and (6.3.30) respectively are given below:

\[
n_i(g) \propto \sum_j X_{ij}^{9/2} \sqrt{\frac{N_i}{N_i - 1}}
\]  
(6.3.31)

and

\[
n_i(g) \propto \sum_j X_{ij}^{9/2} \sqrt{\left\{ 1 + \kappa_i(x) \frac{2}{N_i} \right\} \frac{N_i}{N_i - 1}}
\]  
(6.3.32)

respectively.

**Definition 6.3.5.** The three allocations based on \(X_{i}^{9/2}\), the mean of generalized auxiliary variable (GAV) \(x_{i}^{9/2}\) under the assumption of ESCV(\(x_{i}^{9/2}\)), the coefficient of variation of GAV for all the strata given by (6.3.28), (6.3.31) and (6.3.32) are called \(\Delta_g\)-Mean-based ESCVAMOAs (\(\Delta_g\)-MESCVMOA1, \(\Delta_g\)-MESCVMOA2 and \(\Delta_g\)-MESCVMOA3 respectively for the strategy (St.SRSWR: \(Y_{SR}\)).
Large Strata Approximation

If the strata is so large that \( \frac{1}{N_i-1} \approx 0 \) or equivalently \( \frac{N_i}{N_i-1} \approx 1 \) then \( A'_i(g) \), \( A''_i(g) \) and \( A'''_i(g) \) in (6.3.27), (6.3.29) and (6.3.30) respectively may be approximated by

\[
a'_i(g) = \left(1 + \kappa_i(x)\right) \left(\frac{x_i^{g/2}}{x_i}\right)^2 \left\{1 + C_i^2(x^{g/2})\right\} ,
\]

(6.3.33)

\[
a''_i(g) = \left(\frac{x_i^{g/2}}{x_i}\right)^2 \left\{1 + C_i^2(x^{g/2})\right\} ,
\]

(6.3.34)

and

\[
a'''_i(g) = \left\{1 + \kappa_i(x) - \frac{2}{N_i}\right\} \left(\frac{x_i^{g/2}}{x_i}\right)^2 \left\{1 + C_i^2(x^{g/2})\right\} \]

(6.3.35)

respectively.

Now if \( C_i(x^{g/2}) \), the coefficient of variation of \( x_i^{g/2} \) is equal in all strata then under the large strata approximation, the \( \Delta_g \)-MESCVAMOA1, \( \Delta_g \)-MESCVAMOA2 and \( \Delta_g \)-MESCVAMOA3 based on \( a'_i(g) \), \( a''_i(g) \) and \( a'''_i(g) \) in (6.3.33)–(6.3.35) respectively may be approximated by the following three allocations:

\[
n_i(g) \propto \sqrt{\left\{1 + \kappa_i(x)\right\}} \sum_j X_{ij}^{g/2} ,
\]

(6.3.36)

\[
n_i(g) \propto \sum_j X_{ij}^{g/2} ,
\]

(6.3.37)

\[
n_i(g) \propto \sum_j X_{ij}^{g/2} \sqrt{\left(1 + \kappa_i(x) - \frac{2}{N_i}\right)}
\]

(6.3.38)
Definition 6.3.6. The three allocations given by (6.3.36), (6.3.37) and (6.3.38) based on $X_i^{g/2}$ under the assumption of ESCV($x^{g/2}$) for all strata are called $(\Delta_g$-large strata-MESCVOAMOs $(\Delta_g$-LSMESCVOAMOs) viz., $(\Delta_g$-LSMESCVOAMO1, $(\Delta_g$-LSMESCVOAMO2 and $(\Delta_g$-LSMESCVOAMO3 respectively for the strategy (St.SRSWR:$\hat{Y}_{SR}$).

Alternatively (6.3.25) may be written as

$$M_i(g) = S_i^2(x^{g/2}) \left\{ 1 + \frac{(X_i^{g/2})^2}{\sigma_i^2(x^{g/2})} \right\}$$

$$= S_i^2(x^{g/2}) \left\{ 1 + \frac{1}{C_i^2(x^{g/2})} \right\}$$

(6.3.39)

Now substituting the value of $M_i(g)$ in $A_i'(g)$, $A_i''(g)$ and $A_i'''(g)$ given by (6.3.8), (6.3.12) and (6.3.15) respectively, we get

$$A_i'(g) = \{1 + \kappa_i(x)\} S_i^2(x^{g/2}) \left\{ 1 + \frac{1}{C_i^2(x^{g/2})} \right\}$$

(6.3.40)

$$A_i''(g) = S_i^2(x^{g/2}) \left\{ 1 + \frac{1}{C_i^2(x^{g/2})} \right\}$$

(6.3.41)

and

$$A_i'''(g) = \left\{ 1 + \kappa_i(x) - \frac{2}{N_i} \right\} S_i^2(x^{g/2}) \left\{ 1 + \frac{1}{C_i^2(x^{g/2})} \right\}$$

(6.3.42)
respectively.

Now if the assumption of \( \text{ESCV}(x^{g/2}) \) holds, then \( \Delta_g \)-AMOAs' under this assumption based on the quantities \( A'_i(g) \), \( A''_i(g) \) and \( A''''_i(g) \) given by (6.3.40), (6.3.41) and (6.3.42) respectively given below:

\[
n_i(g) \propto N_i S_i(x^{g/2}) \sqrt{1 + \kappa_i(x)} \quad (6.3.43)
\]

\[
n_i(g) \propto N_i S_i(x^{g/2}) \quad (6.3.44)
\]

and

\[
n_i(g) \propto N_i S_i(x^{g/2}) \sqrt{1 + \kappa_i(x) - \frac{2}{N_i}} \quad (6.3.45)
\]

respectively.

**Definition 6.3.7.** The three allocations given by (6.3.43), (6.3.44) and (6.3.45) based on \( S_i(x^{g/2}) \) under the assumption of equal \( \text{SCV}(x^{g/2}) \) for all strata are called \( \Delta_g \)-Standard Deviation based ESCVAMOAs (\( \Delta_g \)-SDESCVAMOAs) viz., \( \Delta_g \)-SDESCVAMOA1, \( \Delta_g \)-SDESCVAMOA2 and \( \Delta_g \)-SDESCVAMOA3 respectively for the strategy (St.SRSWR: \( \hat{Y}_{SR} \)).

All the \( \Delta_g \)-AMOAs and \( \Delta_g \)-ESCVAMOAs obtained above are presented in the table 6.3.2.1 below:
### Table - 6.3.2.1
Summary of Approximate Allocations
for SRE under Stratified SRSWOR

<table>
<thead>
<tr>
<th>Approx. (R.N.T.)</th>
<th>( l_i(g) \approx 0 )</th>
<th>( l_i''(g) \approx 0 )</th>
<th>( l_i'''(g) \approx 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplier</td>
<td>( \sqrt{1 + \kappa_i(x)} )</td>
<td>1</td>
<td>( \sqrt{1 + \kappa_i(x) - \frac{2}{N_i}} )</td>
</tr>
<tr>
<td>Basic Quantity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( N_i \sqrt{M_i(g)} )</td>
<td>AMOA1</td>
<td>AMOA2</td>
<td>AMOA3</td>
</tr>
<tr>
<td>( \propto N_i \sqrt{A_i'(g)} )</td>
<td>( \propto N_i \sqrt{A_i'(g)} )</td>
<td>( \propto N_i \sqrt{A_i'(g)} )</td>
<td></td>
</tr>
<tr>
<td>Under Approx:</td>
<td>LSAMOA1</td>
<td>LSAMOA2</td>
<td>LSAMOA3</td>
</tr>
<tr>
<td>( \frac{N_i}{N_i - 1} \approx 1 )</td>
<td>( \propto N_i \sqrt{a_i'(g)} )</td>
<td>( \propto N_i \sqrt{a_i'(g)} )</td>
<td>( \propto N_i \sqrt{a_i''(g)} )</td>
</tr>
<tr>
<td>Assump: ( C_i(x^{g/2}) = 1 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Under Approx:</td>
<td>MESCVA1</td>
<td>MESCVA2</td>
<td>MESCVA3</td>
</tr>
<tr>
<td>( \frac{N_i}{N_i - 1} \approx 1 )</td>
<td>( \sum_j X_{ij}^{g/2} \sqrt{\frac{N_i}{N_i - 1}} )</td>
<td>( \sum_j X_{ij}^{g/2} \sqrt{\frac{N_i}{N_i - 1}} )</td>
<td>( \sum_j X_{ij}^{g/2} \sqrt{\frac{N_i}{N_i - 1}} )</td>
</tr>
<tr>
<td>( \frac{N_i}{N_i - 1} \approx 1 )</td>
<td>( \sum_j X_{ij}^{g/2} \sqrt{\frac{N_i}{N_i - 1}} )</td>
<td>( \sum_j X_{ij}^{g/2} \sqrt{\frac{N_i}{N_i - 1}} )</td>
<td>( \sum_j X_{ij}^{g/2} \sqrt{\frac{N_i}{N_i - 1}} )</td>
</tr>
<tr>
<td>Under Approx:</td>
<td>MESCVA1</td>
<td>MESCVA2</td>
<td>MESCVA3</td>
</tr>
<tr>
<td>( N_i S_i(x^{g/2}) )</td>
<td>( N_i S_i(x^{g/2}) )</td>
<td>( N_i S_i(x^{g/2}) )</td>
<td>( N_i S_i(x^{g/2}) )</td>
</tr>
</tbody>
</table>
6.3.5 Particular Cases of $\Delta_g$-Model-based Allocations

In this subsection, we deduce some particular cases of the $\Delta_g$-MOA, $\Delta_g$-AMOAs and $\Delta_g$-ESCVAMOAs for certain values of interest of the superpopulation parameter $g$ under the model $\Delta_g$.

Case I: Model $\Delta_0$

Substituting $g = 0$ in $A_i(g)$ in (6.3.6), we have

$$A_i(0) = (1 + \kappa_i(x)) \frac{N_i}{N_i - 1} - \frac{2}{X_i} \frac{1}{N_i - 1} X_i$$

$$= 1 + \frac{C_i^2(x)}{N_i - 1}$$

therefore, from (6.3.7), we get $\Delta_0$-MOA

$$n_i(0) \propto N_i \sqrt{1 + \frac{C_i^2(x)}{N_i - 1}.} \quad (6.3.46)$$

Case II: Model $\Delta_1$

Similarly for $g = 1$,

$$A_i(1) = \{1 + \kappa_i(x)\} \frac{X_i}{N_i - 1} - \frac{2}{X_i} \frac{1}{N_i - 1} \sum_j X_{ij}^2$$

$$= \{1 + \kappa_i(x)\} \frac{N_i}{N_i - 1} X_i - \frac{2}{(N_i - 1)X_i} \left(\sigma_i^2(x) + \bar{X}_i^2\right)$$

$$= \bar{X}_i \left[\{1 + \kappa_i(x)\} \left(\frac{N_i}{N_i - 1}\right) - 2 \left\{\frac{1 + C_i^2(x)}{N_i - 1}\right\}\right]$$

Hence $\Delta_1$-MOA is given by
\[ n_i(1) \propto N_i \sqrt{X_i} \left\{ 1 + \kappa_i(x) \right\} \left( \frac{N_i}{N_i - 1} \right) - 2 \left\{ \frac{1 + C_i^2(x)}{N_i - 1} \right\} . \] (6.3.47)

**Case-III: Model \( \Delta_2 \)**

Similarly for \( g = 2 \),

\[
A_i(2) = \left\{ 1 + \kappa_i(x) \right\} \frac{1}{X_i} \sum_j X_{ij}^2 - \frac{2}{X_i N_i - 1} \sum_j X_{ij}^3
\]

\[
= \left\{ 1 + \kappa_i(x) \right\} \left\{ S_i^2(x) + \frac{N_i}{N_i - 1} X_i \right\} - \frac{2}{X_i N_i - 1} \sum_j X_{ij}^3
\]

Therefore \( \Delta_2\)-MOA reduces to

\[ n_i(2) \propto N_i \sqrt{\left\{ 1 + \kappa_i(x) \right\} \sum_j X_{ij}^2 - \frac{2}{X_i N_i - 1} \sum_j X_{ij}^3} . \] (6.3.48)

Similarly we can deduce \( \Delta_g\)-AMOAs and \( \Delta_g\)-ESCVAMOAs for these values of \( g \) and the same are presented in tables 6.3.3.1 and 6.3.3.2 below:

**Table - 6.3.3.1**

RNT and Multiplier for AMOAs and ESCVAMOAs for St.SRSWOR

<table>
<thead>
<tr>
<th>Suffix of the Allocation</th>
<th>RNT (Approx.)</th>
<th>Multiplier</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( L_i(g) = \frac{1}{X_i} M_i g ) ( A_i(g) \approx 0 )</td>
<td>( \sqrt{1 + \kappa_i(x)} )</td>
</tr>
<tr>
<td>2</td>
<td>( L_i''(g) = \frac{1}{X_i} M_i g ) ( A_i(g) \approx 0 )</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>( L_i'''(g) = \frac{1}{X_i} S_i ) ( A_i(g) \approx 0 )</td>
<td>( \sqrt{1 + \kappa_i(x) - \frac{2}{N_i}} )</td>
</tr>
</tbody>
</table>
Table - 6.3.3.2

Expression** of AMOAs and ESCVAMOAs under $\Delta_g$ for different $g$ for St.SRSWOR

<table>
<thead>
<tr>
<th>Allocation Type*</th>
<th>Approx. Assump.</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1. AMOA</td>
<td>$N_i \sqrt{\frac{N_i}{N_i-1}}$</td>
<td>$N_i \sqrt{X_i}$</td>
</tr>
<tr>
<td>2. LSAMOA</td>
<td>Approx: $\frac{N_i}{N_i-1} \approx 1$</td>
<td>$N_i$</td>
</tr>
<tr>
<td>Assumption</td>
<td>$C_i(x^{3/2})$'s are equal</td>
<td>Not Required</td>
</tr>
<tr>
<td>3. MESCVAMOA</td>
<td>$N_i \sqrt{\frac{N_i}{N_i-1}} \sum_j \sqrt{X_{ij}}$</td>
<td>$X_i \sqrt{\frac{N_i}{N_i-1}}$</td>
</tr>
<tr>
<td>4. MESCVLSAMOA</td>
<td>Approx: $\frac{N_i}{N_i-1} \approx 1$</td>
<td>$N_i$</td>
</tr>
<tr>
<td>5. SDESCVAMOA</td>
<td>Does not exist</td>
<td>$N_i S_i(\sqrt{x})$</td>
</tr>
</tbody>
</table>

* The name of the allocation is obtained by suffixing 1, 2 or 3 in the allocation type from table 6.3.3.2.
** The formula for the allocation is obtained by multiplying the expression from table 6.3.3.2 by the corresponding multiplier from table 6.3.3.1.
6.4 Empirical Illustration

The $\Delta_g$-MOA and its approximate allocations for stratified SRSWOR designs derived in section 6.3 are illustrated with two live populations: Pop.-I and Pop.-II. The same five stratification types (STs) A, B, C, D and E equalizing $\sum_j X_{ij}^g$ for $g=0, 0.5, 1.0, 1.5$ and $2.0$ respectively considered in the chapter 4 have been used for the illustration.

The variances of the separate ratio estimator (SRE) in the stratified SRSWOR design for estimating population total under TNOA for the STs for Pop.-I and II for samples of size 5 and 4 are presented in tables 6.4.1(I) and 6.4.1(II) respectively. In these tables col.3 gives the relative efficiency of various STs w.r.t. ST-A for SRE while col. 5 shows the relative efficiency (RE) of SRE w.r.t. unbiased expansion estimator (UEE) for the five STs. The $\Delta_g$-MOA and its approximations for stratified SRSWOR design are calculated for 5 values of $g$ viz., 0, 0.5, 1.0, 1.5 and 2.0 for the five STs A, B, C, D and E and their REs w.r.t. TNOA are given in tables 6.4.2(IA to IE) and 6.4.2(IIA to IIE) of Appendix C for Pop.-I and II respectively alongwith their assumption CV and the three approximations corresponding to RNTs $l'$, $l''$ and $l'''$. 
Table - 6.4.1.1(I)

Variances of SRE and UEE under TNOA REs for 5 STs w.r.t. ST-A and RE of SRE w.r.t. UEE for Pop-I.

| Strat. | \( V(\hat{Y}_{SR}) \) (1) | R.E.(/A) (2) | \( V(\hat{Y}) \) (3) | R.E.(\( \hat{Y}_{SR}|\hat{Y} \)) (5) |
|--------|-----------------|-------------|-----------------|-----------------|
| A      | 2496x10^6       | 1           | 1073x10^6       | 0.4301          |
| B      | 2985x10^6       | 0.8361      | 1321x10^6       | 0.4426          |
| C      | 12440x10^6      | 0.2006      | 1976x10^6       | 0.1589          |
| D      | 6724x10^6       | 0.3712      | 2222x10^6       | 0.3305          |
| E      | 8188x10^6       | 0.3048      | 2693x10^6       | 0.3289          |

Table - 6.4.1.1(II)

Variances of SRE and UEE under TNOA REs for 5 STs w.r.t. ST-A and RE of SRE w.r.t. UEE for Pop-II.

| Strat.Type | \( V(\hat{Y}_{SR}) \) | R.E.(\( \hat{Y}_{SR}|A \)) | \( V(\hat{Y}) \) | R.E.(\( \hat{Y}_{SR}|\hat{Y} \)) |
|------------|-----------------|-----------------|-----------------|-----------------|
| A          | 4.7x10^6        | 1               | 2.0x10^6        | 4.1778          |
| B          | 3.9x10^6        | 1.1893          | 1.5x10^6        | 3.6989          |
| C          | 3.3x10^6        | 1.4115          | 1.2x10^6        | 3.6369          |
| D          | 2.8x10^6        | 1.6722          | 1.4x10^6        | 4.9388          |
| E          | 2.3x10^6        | 2.0124          | 1.9x10^6        | 7.9829          |

6.4.1 Comparison of STs and REs of SRE under TNOA

It can be noted from the table 6.4.1(I) that for Pop.-I, ST-A has the least variance for SRE followed by ST-B with its RE w.r.t. ST-A being 0.8361. It is surprising, however, that ST-C equalizing stratum total \( X_i \) is the worst
with the lowest efficiency of just 0.2006 only. The ST-A is the best followed by B for UEE $\hat{Y}_{st, UE}$. However, the gain of $\hat{Y}_{SR}$ over $\hat{Y}_{st, UE}$ is slightly higher at 0.44 for ST-B than that for ST-A at 0.43. The ST-E equalizing $\sum_{j} X_{ij}^2$ for UEE is the least efficient but the least gain by using SRE over UEE is for ST-C with RE 0.1589. For Pop.-II we see from table 6.4.1.1(ii) that ST-E equalizing $\sum_{j} X_{ij}^2$ is the best followed by ST-D with RE of 1.67 as against 2.01 for ST-E compared to ST-A. The ST-C equalizing stratum total $X_i$ is the best for UEE, however, the gain of SRE over UEE is maximum for ST-E. Thus we observe that even for the same population, different STs are suitable for different estimators.

6.4.2 Behaviour of Efficiency of Allocation

The REs of most of the allocations for Pop.-I are decreasing with increasing $g$ with a few exceptions. The REs for ST-A are increasing with $g$ except the allocations $\Delta g$-SDESCVAMOA between higher values of $g$. The REs of all allocations are decreasing with increasing $g$ for all STs except allocations $\Delta g$-SDESCVAMOAs for STs B and D for lower values of $g$. Thus the RE of $\Delta g$-SDESCVAMOA is increasing for lower values of $g$ for STs A, B and D only.

In Pop.-II the REs of the allocations are increasing with $g$ for all the STs except $\Delta g$-SDESCVAMOAs for ST-A only. The REs of most of the allocations are the highest for $g = 2$ for Pop.-II and for $g = 0$ for Pop.-I. As noted earlier the value of superpopulation parameter $g$ seems to be close to zero for Pop.-I and close to 2 for Pop.-II.
6.4.3 Effect of Approximations

(a) $\Delta_g$-MOA vs AMOAs:

We observe from tables 6.4.2(IA-IE) for Pop.-I that $\Delta_g$-MOA is slightly better than its approximate allocations AMOA1 and AMOA2 under $l'$ and $l''$-approximations for all STs except ST-A. On the other hand AMOA3 under $l''$-approximation is more efficient than MOA for all the STs. In Pop.-II it can be noted from tables 6.4.2 (IIA-IIIE) that the approximate allocations under all the three approximations are superior to MOA for almost all $g$ for most of the STs with a few exceptions. Thus we observe from the above that the effect of approximations is different in different types of populations i.e., for the populations of the type Pop.-I MOA is better whereas for the type Pop.-II, the approximate allocations are more efficient than the MOA. The trend is clear for the middle and higher STs C, D and E.

Short names replacing abbreviated names of allocations in the following tables are given below:

<table>
<thead>
<tr>
<th></th>
<th>MOA</th>
<th>AMOA1</th>
<th>AMOA2</th>
<th>AMOA3</th>
<th>LSAMOA1</th>
<th>LSAMOA2</th>
<th>LSAMOA3</th>
<th>MESCVAMOA1</th>
<th>MESCVAMOA2</th>
<th>MESCVAMOA3</th>
<th>MESCVLSAMOA1</th>
<th>MESCVLSAMOA2</th>
<th>MESCVLSAMOA3</th>
<th>SDESCVAMOA1</th>
<th>SDESCVAMOA2</th>
<th>SDESCVAMOA3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>MOA</td>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>MESCVAMOA2</td>
<td>M2</td>
<td>M3</td>
<td>M1'</td>
<td>M2'</td>
<td>M3'</td>
<td>SD1</td>
<td>SD2</td>
<td>SD3</td>
</tr>
<tr>
<td>2</td>
<td>AMOA1</td>
<td>AM1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>MESCVAMOA3</td>
<td>M3</td>
<td>M3</td>
<td>M1'</td>
<td>M2'</td>
<td>M3'</td>
<td>SD1</td>
<td>SD2</td>
<td>SD3</td>
</tr>
<tr>
<td>3</td>
<td>AMOA2</td>
<td>AM2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>MESCVLSAMOA1</td>
<td>M1'</td>
<td>M3</td>
<td>M1'</td>
<td>M2'</td>
<td>M3'</td>
<td>SD1</td>
<td>SD2</td>
<td>SD3</td>
</tr>
<tr>
<td>4</td>
<td>AMOA3</td>
<td>AM3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>MESCVLSAMOA2</td>
<td>M2'</td>
<td>M3</td>
<td>M1'</td>
<td>M2'</td>
<td>M3'</td>
<td>SD1</td>
<td>SD2</td>
<td>SD3</td>
</tr>
<tr>
<td>5</td>
<td>LSAMOA1</td>
<td>AM1'</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>MESCVLSAMOA3</td>
<td>M3'</td>
<td>M3</td>
<td>M1'</td>
<td>M2'</td>
<td>M3'</td>
<td>SD1</td>
<td>SD2</td>
<td>SD3</td>
</tr>
<tr>
<td>6</td>
<td>LSAMOA2</td>
<td>AM2'</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>SDESCVAMOA1</td>
<td>SD1</td>
<td>M3</td>
<td>M1'</td>
<td>M2'</td>
<td>M3'</td>
<td>SD1</td>
<td>SD2</td>
<td>SD3</td>
</tr>
<tr>
<td>7</td>
<td>LSAMOA3</td>
<td>AM3'</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>SDESCVAMOA2</td>
<td>SD2</td>
<td>M3</td>
<td>M1'</td>
<td>M2'</td>
<td>M3'</td>
<td>SD1</td>
<td>SD2</td>
<td>SD3</td>
</tr>
<tr>
<td>8</td>
<td>MESCVAMOA1</td>
<td>M1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>SDESCVAMOA3</td>
<td>SD3</td>
<td>M3</td>
<td>M1'</td>
<td>M2'</td>
<td>M3'</td>
<td>SD1</td>
<td>SD2</td>
<td>SD3</td>
</tr>
</tbody>
</table>

229
Table - 6.4.3(I)

Stratification Ranking of Allocations
w.r.t. RE for Pop.-I

<table>
<thead>
<tr>
<th>Rank</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>SD3</td>
<td>SD3</td>
<td>SD3</td>
<td>SD1,3</td>
<td>SD3</td>
<td>SD3</td>
</tr>
<tr>
<td>2</td>
<td>SD1</td>
<td>SD2</td>
<td>SD2</td>
<td>SD2</td>
<td>SD2</td>
<td>SD2</td>
</tr>
<tr>
<td>3</td>
<td>SD2</td>
<td>SD1</td>
<td>SD1</td>
<td>SD1</td>
<td>SD1</td>
<td>SD1</td>
</tr>
<tr>
<td>4</td>
<td>AM1,3</td>
<td>AM1',3'</td>
<td>AM3'</td>
<td>AM3'</td>
<td>AM3'</td>
<td>AM3'</td>
</tr>
<tr>
<td>5</td>
<td>AM1,3</td>
<td>AM1',3'</td>
<td>AM3'</td>
<td>AM3'</td>
<td>AM3'</td>
<td>AM3'</td>
</tr>
<tr>
<td>6</td>
<td>AM1,3</td>
<td>AM1',3'</td>
<td>AM2'</td>
<td>AM2'</td>
<td>AM2'</td>
<td>AM2'</td>
</tr>
<tr>
<td>7</td>
<td>AM1,3</td>
<td>AM1',3'</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>M</td>
</tr>
<tr>
<td>8</td>
<td>AM2,2'</td>
<td>AM1'</td>
<td>AM1'</td>
<td>M3'</td>
<td>AM1'</td>
<td>AM1'</td>
</tr>
<tr>
<td>9</td>
<td>AM2,2'</td>
<td>AM2</td>
<td>AM2</td>
<td>AM1'</td>
<td>AM2</td>
<td>AM2</td>
</tr>
<tr>
<td>10</td>
<td>M</td>
<td>AM1</td>
<td>AM1</td>
<td>AM2</td>
<td>AM1</td>
<td>AM1</td>
</tr>
<tr>
<td>11</td>
<td>M1,1'M3,3'</td>
<td>M3'</td>
<td>M3'</td>
<td>M3</td>
<td>M3'</td>
<td>M3'</td>
</tr>
<tr>
<td>12</td>
<td>M1,1'M3,3'</td>
<td>M3</td>
<td>M3</td>
<td>M2'</td>
<td>M3</td>
<td>M3'</td>
</tr>
<tr>
<td>13</td>
<td>M1,1'M3,3'</td>
<td>M2'</td>
<td>M2'</td>
<td>M1</td>
<td>M2'</td>
<td>M2'</td>
</tr>
<tr>
<td>14</td>
<td>M1,1'M3,3'</td>
<td>M1</td>
<td>M1</td>
<td>M1'</td>
<td>M1</td>
<td>M1'</td>
</tr>
<tr>
<td>15</td>
<td>M2,2'</td>
<td>M2</td>
<td>M2</td>
<td>M2</td>
<td>M2</td>
<td>M2</td>
</tr>
<tr>
<td>16</td>
<td>M2,2'</td>
<td>M1</td>
<td>M1</td>
<td>M1</td>
<td>M1</td>
<td>M1</td>
</tr>
<tr>
<td>Rank</td>
<td>ST-A</td>
<td>ST-B</td>
<td>ST-C</td>
<td>ST-D</td>
<td>ST-E</td>
<td>Overall</td>
</tr>
<tr>
<td>------</td>
<td>---------------</td>
<td>---------------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>---------</td>
</tr>
<tr>
<td>1</td>
<td>SD2</td>
<td>SD1</td>
<td>M1</td>
<td>M1</td>
<td>M1</td>
<td>M1</td>
</tr>
<tr>
<td>2</td>
<td>SD1,3(.5-1)</td>
<td>SD2</td>
<td>AM1</td>
<td>AM1</td>
<td>AM1</td>
<td>AM1</td>
</tr>
<tr>
<td></td>
<td>AM3,3'(1.5-2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>SD1,3(.5-1)</td>
<td>SD3</td>
<td>M2</td>
<td>M2</td>
<td>M2</td>
<td>M2</td>
</tr>
<tr>
<td></td>
<td>AM1,1'(1.5-2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>M(0-.5)</td>
<td>AM1</td>
<td>M1'</td>
<td>M1'</td>
<td>AM2</td>
<td>M1'</td>
</tr>
<tr>
<td></td>
<td>AM1,1'(1.5-2)</td>
<td></td>
<td></td>
<td></td>
<td>AM1'</td>
<td></td>
</tr>
<tr>
<td></td>
<td>AM3,3'(0,1, 2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>AM2,2'(0-.5,2)</td>
<td>M1(0-.1)</td>
<td>AM2</td>
<td>AM2</td>
<td>M1'</td>
<td>AM2</td>
</tr>
<tr>
<td></td>
<td>AM3,3'</td>
<td>AM1'</td>
<td></td>
<td></td>
<td>AM2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0,1, 2)</td>
<td>(1.5-1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>AM2,2'</td>
<td>AM1'(0-.5,2)</td>
<td>AM1'</td>
<td>AM1'</td>
<td>AM1'</td>
<td>AM1'</td>
</tr>
<tr>
<td></td>
<td>(1.5-2)</td>
<td>(1.5-1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>AM3,3'(0-.5,2)</td>
<td>M1(0-.1)</td>
<td>M3</td>
<td>M3</td>
<td>M2'</td>
<td>M3</td>
</tr>
<tr>
<td></td>
<td>(.5-1, 2)</td>
<td>AM1,1'</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>AM3,3'(0-.5)</td>
<td>M1'(0-.1)</td>
<td>M2</td>
<td>M2'</td>
<td>M3</td>
<td>M2'</td>
</tr>
<tr>
<td></td>
<td>M3,3'(0,1,5-2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>AM1,1'</td>
<td>M2</td>
<td>AM3</td>
<td>AM3</td>
<td>M</td>
<td>AM3</td>
</tr>
<tr>
<td></td>
<td>(0-.5)M3,3'</td>
<td>(0-.5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.5-2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table - 6.4.3(II)

ST Ranking of Allocations w.r.t. E.E for Pop.-II

<table>
<thead>
<tr>
<th>Rank</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>AM1,1'</td>
<td>M2</td>
<td>M</td>
<td>AM2'</td>
<td>AM2'</td>
<td>AM2'</td>
</tr>
<tr>
<td></td>
<td>(0-.5)</td>
<td>(0,1)</td>
<td>(0,.5,1.5)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>M1,1'</td>
<td></td>
<td>M</td>
<td>AM2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0,1.5-2)</td>
<td></td>
<td>(1.5-2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>M1,1'</td>
<td>M3</td>
<td>AM2'</td>
<td>M</td>
<td>AM3</td>
<td>M</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>M2,2'(1.5-2)</td>
<td>M(,5-1)</td>
<td>M3'</td>
<td>M3'</td>
<td>M3'</td>
<td>M3'</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>M1,1'(.5-1,2)</td>
<td>AM(0-.5)</td>
<td>AM3'</td>
<td>AM3'</td>
<td>AM3'</td>
<td>AM3'</td>
</tr>
<tr>
<td></td>
<td>M2,2'(0,1.5-2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>M3,3'(.5-1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>M1,1'(.5-1)</td>
<td>M2'(0-1)</td>
<td>SD1</td>
<td>SD1</td>
<td>SD1</td>
<td>SD1</td>
</tr>
<tr>
<td></td>
<td>M3,3'(.5-1)</td>
<td>M3(1.5-2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>SD2(1.5-2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>M2,2'(0-1)</td>
<td>AM3'(0-1)</td>
<td>SD2</td>
<td>SD2</td>
<td>SD2</td>
<td>SD2</td>
</tr>
<tr>
<td></td>
<td>SD1(1.5-2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>M2,2'(0-1)</td>
<td>M3'</td>
<td>SD3</td>
<td>SD3</td>
<td>SD3</td>
<td>SD3</td>
</tr>
</tbody>
</table>

(b) Comparison Among 1-approximations with Different Multipliers

From table 6.4.3(I) for Pop.-I we find that the allocations under 1''-approximation are the best followed by those under 1' and 1''-approximations.
respectively for all 5 types of allocations. However, in case of SDESCVA-MOA second allocation with multiplier unity is the best.

Next we note from table 6.4.3(II) for Pop.-II, that for all the 5 major allocations, the allocations (1) under $\ell'$-approximation with multiplier $\sqrt{1 + \kappa_i(x)}$ are the most efficient followed by (2) under $\ell''$ with multiplier unity and (3) under $\ell''$'-approximations with multiplier $\sqrt{1 + \kappa_i(x) - \frac{2}{N}}$ respectively for all STs except ST-A.

We note from the above that all the five major allocations behave in similar way under the three $\ell$-approximations for both the populations for all the STs except for the lower ST-A. As observed above the allocations with multiplier $\sqrt{1 + \kappa_i - \frac{2}{N}}$ and $\sqrt{1 + \kappa_i}$ are the best allocations for Pop.I and II respectively. However, there is not much difference in efficiencies of these allocations and therefore, either may be used. Further the allocations with factor unity under $\ell''$-approximation is the most simple and easy for interpretation, which come at the cost of sacrifice of little efficiency.

(c) Effect of Large Strata Approximation

It may be noted again from 6.4.2(I) that approximate allocations under the approximation of large stratum sizes viz., LSAMOA 1, 2 and 3 are better than their counterparts without approximation-AMOA 1, 2 and 3 respectively for all STs of Pop.-I. Similarly the large strata approximate allocations under the assumption of ESCV-MESCVLSAMOA 1, 2 and 3 are more efficient than the corresponding allocation without approximation for
Further from table 6.4.2(II) we observe that all approximate allocations including those under the assumption of ESCV are more efficient than the corresponding allocations under approximation of large strata for all the STs for Pop.-II. Thus we see that the performance of allocations under large strata approximation differs from stratum to stratum and population to population. The differences in REs of large strata approximate allocations become higher for higher values of $g$ and for higher STs in Pop.-II. However in actual surveys, the loss of efficiency due to LSA is expected to be marginal only. Therefore on account of simplicity and easy interpretation, large strata approximate allocation may be preferred to those without LSA even at the cost of little efficiency.

6.4.4 Role and Effect of ESCV-Assumption

(a) Role of Assumption

The CV of assumption SCV($x^{g/2}$) of the ESCV-allocations is increasing with increasing $g$ i.e., the assumption is deteriorating for most of the STs except ST-A for higher values of $g$ and the REs of all the allocations based on mean viz., $\Delta_g$ MESCVA MOA 1, 2 and 3 and MESCVA LSAMOA1, 2 and 3 are decreasing for all the STs except ST-A. However, the REs of the SD-based allocations -SDESCVA MOA 1, 2 and 3 is decreasing for STs C, E and for higher values of $g$ for STs A, B and D. The relationship of the CV of assumption SCV($x^{g/2}$) and the efficiency of an allocation may also be exam-
ined from the table 6.4.4(I) giving correlation coefficient between assumption SCV and efficiency. The direct relation is signified by high negative correlation and inverse relation by poor negative or positive correlation coefficient e.g., for allocations SDESCVAMOA 1, 2 and 3 in STs A and D except B.

### Table - 6.4.4(I)

**Correlation between Assumption SCV and REs of Allocations for Pop.-I**

<table>
<thead>
<tr>
<th>Strat. Type</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓All.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M1</td>
<td>-0.9081</td>
<td>-0.9960</td>
<td>-0.9830</td>
<td>-0.9746</td>
<td>-0.6366</td>
</tr>
<tr>
<td>M1'</td>
<td>-0.9081</td>
<td>-0.9957</td>
<td>-0.9826</td>
<td>-0.9743</td>
<td>-0.6337</td>
</tr>
<tr>
<td>SD1</td>
<td>-0.5643</td>
<td>-0.9194</td>
<td>-0.9520</td>
<td>-0.1730</td>
<td>-0.4903</td>
</tr>
<tr>
<td>↑′-app.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M2</td>
<td>-0.9084</td>
<td>-0.9957</td>
<td>-0.9827</td>
<td>-0.9744</td>
<td>-0.6343</td>
</tr>
<tr>
<td>M2'</td>
<td>-0.9084</td>
<td>-0.9954</td>
<td>-0.9822</td>
<td>-0.9740</td>
<td>-0.6315</td>
</tr>
<tr>
<td>SD2</td>
<td>-0.5701</td>
<td>-0.9143</td>
<td>-0.9509</td>
<td>+0.6448</td>
<td>-0.4743</td>
</tr>
<tr>
<td>↑′-app.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M3</td>
<td>-0.9081</td>
<td>-0.9953</td>
<td>-0.9822</td>
<td>-0.9740</td>
<td>-0.6313</td>
</tr>
<tr>
<td>M3'</td>
<td>-0.9081</td>
<td>-0.9950</td>
<td>-0.9818</td>
<td>-0.9737</td>
<td>-0.6286</td>
</tr>
<tr>
<td>SD3</td>
<td>-0.5640</td>
<td>-0.9057</td>
<td>-0.9493</td>
<td>+0.9012</td>
<td>-0.4474</td>
</tr>
</tbody>
</table>

235
The CV of assumption SCV($x_i^{a/2}$) is increasing with $g$ i.e., the assumption is deteriorating for higher $g$ for all STs except ST-B. We note that the efficiency and the assumption CV are moving in the same direction for ST-B and SDESCVAMOA 1, 2 and 3 only and in opposite direction for all other STs. The fact is confirmed by the table 6.4.4(II) giving the correlation coefficient between them. In this population except ST-B and allocations SDESCVAMOA 1, 2 and 3 in ST-B, all other allocations in ST-A have high positive correlation. On the basis of the above, it cannot be established that
there exist any relation between the two.

Thus it cannot be concluded on the basis of these two populations that the assumptions CV could be a guide to choose the most efficient allocation with suitable $g$. Therefore the level of satisfaction of the assumption is not indicative of the efficiency of an allocation.

(b) Effect of Assumption

We observe for Pop.-I from table 6.4.2(IA to IE) that the first approximate allocations under the ESCV assumption - MESCVAO1 and its large strata approximation - MESCVAO2 and MESCVAO3 are slightly less efficient compared to their respective allocations without assumption viz., AMOA, LSAMOA for all STs. However, the SD-based allocations under assumption of ESCV ($\frac{x^6}{z^2}$) - SDESCVAMA1 and SDESCVAMO2 are much more efficient than their counterparts without assumption - AMOA 1, 2 and 3 respectively for almost all STs.

Next we see from table 6.4.3(II) that the first allocation under ESCV-assumption - MESCVEA1 and MESCVEA2 are better than the corresponding allocation without assumption - AMOA and LSAMOA for higher STs C, D and E under $l'$-approximation but less efficient for lower STs A and B. On the other hand the opposite is true with the second allocation under ESCV-assumption SDESCVAMA1 which is more efficient for
low STs A and B but less efficient than AMOA1 for high STs C, D and E. Similar is the case under $l''$ and $l'''$-approximations. However, the differences in efficiency between an ESCV allocation and corresponding AMOA is only marginal. Therefore on account of simplicity, the allocation under ESCV-assumption is better to use in practice compared to the approximate allocation without assumption i.e., AMOA.

(c) Comparison between the Two ESCV Allocations

In Pop.-I the second allocation SDESCVAMOA is much more efficient than the first MESCVAMOA1 as well as its LSA-MESCVLSAMOA except for $g = 2$ of ST-A. On the other hand for Pop.-II SDESCVAMOA is better than MESCVAMOA and MESCVLSAMOA1 for ST-B and low-values of $g$ in ST-A whereas for the middle and higher STs C, D and E and higher values of $g$ in ST-A MESCVAMOA and MESCVLSAMOA are more efficient. On the basis of these two populations one may guess that for populations with low $g$ SDESCVAMOA is suitable and for populations with high $g$ MESCVAMOA or MESCVLSAMOA may be appropriate for atleast middle and higher STs. More investigations may be needed to establish this fact. We have noted that large strata (LS) approximate allocations are better than those without it for both the populations. However, the differences in the efficiencies of these allocations is not substantial. Therefore, one may use MESCVLSAMOA based on $\sum_{j} \chi_{ij}^2/2$ on account of its simplicity and interpretation, though at the cost of little loss of efficiency for some populations.
6.4.5 Overall Comparison of Allocations and Discussion

For the Pop.-I from table 6.4.3(I) we observe the four broad groups of allocations viz., SDESCVAMOAs, AMOAs, MOA and MESCVAMOAs appearing in almost the same order in all the STs. The SD-based allocations - SDESCVAMOAs are the most efficient followed by some of the AMOAs viz., AMOA3 & AMOA2 then AMOA as well as the remaining AMOAs and finally, MESCVAMOAs for all STs except A in which MOA follows all AMOAs. Thus for populations like Pop.-I with low $g$, either SD-based allocations or AMOAs /MOA may be used in preference to the mean-based allocations. The MOA/AMOAs are in between the two ESCV allocations.

We have noted earlier and now also from the table 6.4.3(I) that the sequence of allocations in order of efficiency with $1$-approximations is 3, 2 and 1 i.e., the allocations with multiplier $\sqrt{1 + \kappa_i - \frac{2}{N_i}}$ being the best, with $\sqrt{1 + \kappa_i}$ the second and with multiplier unity the least efficient. Further, the allocations under LSA are better than those without it. Therefore, more specifically, for populations with low $g$, the allocation SDESCVAMOA3 proportional to

$$N_i \, S_i(x_\theta^2) \sqrt{1 + \kappa_i - \frac{2}{N_i}}$$

or LSAMOA3 proportional to

$$N_i \sqrt{\left(1 + \kappa_i - \frac{2}{N_i}\right) X_i \theta^2}$$
are expected to be the most efficient allocations.

In case of Pop.-II, we observe that the behavior of allocations differs in lower STs A and B from that in the middle and high STs C, D and E. Because of this reason it has not been possible to do the overall ranking of allocations together for all the STs. However, overall ranking has been given for the middle and higher STs-C, D and E in the last column of the table 6.4.3(II). The allocations SDESCVAMOAs are the best for lower values of $g$ upto 1 for ST-A followed by MOA and AMOAs and MESCVAMOAs respectively, whereas for higher $g$, MOA/AMOAs are the best followed by MESCVAMOAs and SDESCVAMOAs respectively.

Thus the ST-A is not stable because the allocations SDESCVAMOAs which are the best for some values of $g$ become the worst for other values. In ST-B, SDESCVAMOAs are the most efficient followed by AMOAs and MESCVAMOAs respectively. In all higher STs C, D and E the allocations MESCVAMOAs are the best followed by AMOAs and MOA but SDESCVAMOAs are the least efficient allocations. Most of the approximate allocations are superior to MOA in all STs. Thus for populations with high $g$ like Pop.-II, MESCVAMOAs may be used for middle and higher STs whereas for lower STs AMOAs or SDESCVAMOAs may be preferred to mean-based allocations.
We recall that the order of various allocations in respect of efficiency is 1, 2 and 3 i.e., the allocations with multiplier $\sqrt{1 + \kappa_i}$ are the best, with multiplier unity the second and with $\sqrt{1 + \kappa_i - \frac{2}{N_i}}$ the least efficient. Further, the LSA-allocations are more efficient than the corresponding without LSA-allocation. Therefore for the populations with higher $g$ like Pop.-II MESC VSAMOA1 proportional to

$$\sum_j x_{ij} g^2 \sqrt{\left\{1 + \kappa_i(x)\right\} \frac{N_i - 1}{N_i}}$$

for higher STs and SDESC VSAMOA1 proportional to

$$N_i S_i(x^{g/2}) \sqrt{1 + \kappa_i}$$

or SDESC VSAMOA2 proportional to $N_i S_i(x^{g/2})$ for lower STs under the assumption of equal SCV ($x^{g/2}$) are expected to be the most efficient.

The two populations considered here for illustration are different in nature and hence the appropriate allocation for use in practice differs not only for the two populations but from one ST to another for the same population. Even for some STs viz., the lower STs A and B, the performance of allocations varies from lower $g$ to higher $g$. In this respect, the low $g$ STs are not stable and hence not suitable for use in practice.

The allocations with multipliers $\sqrt{1 + \kappa_i(x) - \frac{2}{N_i}}$ and $\sqrt{1 + \kappa_i(x)}$ are the best for Pop.-I and II respectively. But their efficiencies are not sub-

241
stantially different from the much simpler allocations with multiplier unity. Therefore these multipliers may be ignored. Further, the large strata approximation i.e., neglecting the terms in $1/N_i$ enhances the efficiency for certain types of populations like Pop.-I and on the otherhand reduces it for others like Pop.-II. But the loss of efficiency on account of LSA for actual survey, when strata are large, can be expected to be only marginal. Therefore, in practice, allocations with LSA neglecting the terms in $1/N_i$ may be used in practice. In this way we end up with SDESCVAMOA2 proportional to

$$N_i S_i(x^{g/2})$$

or LSAMOA2 proportional to

$$N_i \sqrt{\bar{X}^g} = \sqrt{N_i \sum_j X_{ij}^g}$$

for populations with low $g$ like Pop.-I. For the populations with high $g$ like Pop.-II, the allocation MESCVLSAMOA2 proportional to $\sum_j X_{ij}^{g/2}$ for higher STs and the allocation SDESCVAMOA2 proportional to $N_i S_i(x^{g/2})$ for low STs would be suitable.

6.4.6 Conclusion

The Model Optimum Allocation (MOA) for separate ratio estimator under different approximations and assumptions gives rise to several quick
approximations, many of which are more efficient than MOA itself. The two approximate allocations Generalized Auxiliary Variable Optimum Allocation (GAVOA) and Generalized Auxiliary Variable Proportional Variation (GAVPA) proportional to

\[ N_i S_i(x_i^{g/2}) \quad \text{and} \quad \sum_j X_{ij}^{g/2} \]

respectively, are quite close to TNOA and are more efficient than MOA itself for certain populations and stratification types.

If \( g \) is known for survey population, one may choose either GAVOA or GAVPA depending upon the nature of the population and the ST used. For populations with low \( g \) and for populations with high \( g \) but with lower ST equalizing \( \sum_j X_{ij}^g, g < 1 \), the GAVOA appears to be the most appropriate. On the other hand, for populations with higher \( g \) and with higher STs equalizing \( \sum_j X_{ij}^g, 1 \leq g \leq 2 \), the GAVPA seems to be adequate. The efficiency of allocations is not very sensitive to variations in \( g \). Therefore, even if the precise value of \( g \) is not known and if rough estimate can be guessed, then the same may be used for choosing and computing the appropriate model-based allocation.

The performance of allocations are very sensitive to the STs and in this respect, the lower STs equalizing \( \sum_j X_{ij}^g, g < 1 \) are very much unsta-
ble for populations with higher $g$. Therefore the higher STs are preferable to the lower for such populations. The behavior of allocations and hence the criteria for the choice of suitable approximate allocation for SRSWR design are expected to be similar as for SRSWOR design.