Chapter 4

Allocation for Unbiased Expansion Estimator

Summary

In this chapter, we initiate the study of the problem of allocation of sample size to strata in stratified SRSWR sampling design under superpopulation model. The model-based allocations are obtained from TN optimum allocation under a suitable condition. Further, two generalized allocations are also derived under the assumption of equal stratum coefficients of variation (ESCV) of a generalized form of auxiliary variable. Furthermore, approximate allocations of all the model-based allocations are obtained under the large strata approximation. The model-based allocations are also deduced under certain particular cases of the superpopulation model. Similarly the study is carried out for stratified SRSWOR design. Next we explore applications and implications of the condition and ESCV assumption on the performance of the allocations and suitable criteria are evolved for the choice of
the best possible allocation. Finally, the results for SRSWOR are illustrated with two live populations. The effects of condition, assumption of ESCV and large strata approximation are studied on the efficiency of the corresponding allocations and their efficiencies are compared empirically. Usefulness of the condition is also empirically demonstrated for choosing the most efficient allocation for use in practice.

4.1 Introduction

Consider a finite population of size $N$ divided into $k$ strata of sizes $N_i$, $i = 1, 2, \ldots, k$ i.e.,

$$
U = \{U_{11}, \ldots, U_{1N_1}, U_{21}, \ldots, U_{2N_2}, \ldots, U_{k1}, \ldots, U_{kN_k}\}.
$$

Let $Y$ be the study variate taking values $\{Y_{ij}\}$ for $j^{th}$ unit of the $i^{th}$ stratum, $j = 1, 2, \ldots, N_i$, and $i = 1, 2, \ldots, k$ i.e.,

$$
Y = \{Y_{11}, \ldots, Y_{1N_1}, Y_{21}, \ldots, Y_{2N_2}, \ldots, Y_{k1}, \ldots, Y_{kN_k}\}.
$$

Most often our interest lies in estimating certain parametric functions of $\{Y_{ij}\}$ such as
population total : $Y = \sum_i \sum_j Y_{ij}$

population mean : $\bar{Y} = Y/N$ or

population variance : $\sigma^2_y = \sigma^2(y) = \frac{1}{N} \sum_i \sum_j (Y_{ij} - \bar{Y})^2$; etc.

Suppose we are interested in the estimation of population total, which is the primary object of estimation in sample surveys. As we have already mentioned, supplementary or auxiliary information on a related variable, which is closely related to the study variable, is usually available beforehand in most of the sample survey situations. In stratified sampling whenever prior information on an auxiliary variable $X$ taking value $\{X_{ij}\}$ on the unit $\{U_{ij}\}$, where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, N_i$ is available, there are several methods by which precision of the estimates can be increased. The available auxiliary information in stratified sampling can be utilized broadly at either of the two stages viz., design or planning stage and estimation stage or both stages together. Thus, the main uses of auxiliary information in stratified sampling at the two stages may be mentioned as follows:

I: At Design Stage

The auxiliary information is used at design stage

(a) in stratifying the population that results in the homogeneous strata,

(b) in choice of sampling design within each stratum such as SRS, PPS or \(\pi\)PS depending upon the type of auxiliary information available in different
strata,
(c) In the allocation of sample size to strata,
(d) In the selection of sample by varying probability sampling by using auxiliary information as size measure etc.

II: At Estimation Stage

The auxiliary information can also be used at estimation stage for instance
(a) classical ratio and regression estimators in conjunction with the design SRS,
(b) Hansen and Hurwitz estimator in the PPSWR design,
(c) Horvitz-Thompson estimator in conjunction with PPSWOR design based on auxiliary information etc.

Obviously the available auxiliary information may be utilized in combinations of one or more of the above. In this chapter, the auxiliary information will be utilized for (Ic) viz., the allocation of sample size to strata at the design stage under the superpopulation model involving the auxiliary variable in the situation of (a) of the stage I above.

In stratified sampling if Simple Random Sampling With Replacement (SRSWR) is adopted within each stratum, the Tschuprow - Neyman optimum allocation (TNOA) of total sample size n to strata is given by
which involves unknown parameters \( \{ \sigma_i \} \). In order to compute TNOA given by equation (4.1.1), we need at least proportionate values of \( \{ \sigma_i^2 \} \). We have already discussed some methods of obtaining working estimates of \( \{ \sigma_i^2 \} \) in order to use the allocation.

In practice sometimes past data are not satisfactory and sufficient time might not be available for conducting a pilot survey or to conduct the main survey itself in phased manner \{ Sukhatme (1935) [71], Mahalanobis (1942) [33] \}. Therefore there is need to have some estimates of \( \{ \sigma_i^2 \} \) based on the prior information on the related \( \mathcal{X} \)-values which can be substituted for \( \{ \sigma_i^2 \} \) in order to compute the optimum allocation. Generally the known \( \{ \sigma_i(x) \} \), the Within Stratum Standard Deviations (WSSDs) of the auxiliary variable \( \mathcal{X} \) are used as estimates of the unknown \( \{ \sigma_i \} \), the WSSDs of the study variable \( \mathcal{Y} \). When information on an auxiliary variable \( \mathcal{X} \) which is highly correlated with the study variable \( \mathcal{Y} \) is available, it was demonstrated by Cochran (1946) [6] that superpopulation model could be constructed using this auxiliary information. This is achieved for the stratification set up by considering

\[
Y = (Y_{11}, \ldots, Y_{1N_1}, Y_{21}, \ldots, Y_{2N_2}, \ldots, Y_{k1}, \ldots, Y_{kN_k})
\]

as a realization of an N-dimensional random vector.
\[ y = (y_{11}, \ldots, y_{1N_1}, y_{21}, \ldots, y_{2N_2}, \ldots, y_{k1}, \ldots, y_{kN_k}) \]

the joint distribution of elements of which depends upon

\[ x = (x_{11}, \ldots, x_{1N_1}, x_{21}, \ldots, x_{2N_2}, \ldots, x_{k1}, \ldots, x_{kN_k}) \]

and some unknown parameters. This formulation is termed as Superpopulation Model. The idea of superpopulation, in its most pure form, may be explained as the finite population is actually drawn from a bigger universe.

In many situations it is natural to let a model summarize and formalise our prior knowledge about the population, whether this be based on long range experience or on personal subjective belief. Superpopulation models need not be Bayesian in the sense of expressing personal subjective belief (Cassel et al. 1977)[3]. They can be as objective as some of the models used in classical statistical theory (Royall, 1971) [61].

Superpopulation models have a long history in the literature of survey sampling. Cochran (1939,46) [4], [6] was one of the foremost users of the superpopulation models which were used for comparison of the efficiencies of stratified sampling and systematic sampling. Hanurav (1965) [23] and Rao (1968) [49] were the first to use the superpopulation model for the allocation of sample size to the strata in stratified sampling.

A model is survey sampler’s conceptualization of superpopulation and essentially defines a class of distributions \( \delta_\theta \). Let us consider one such superpopulation model. The joint distribution of the random variables
\( \mathcal{Y} = (Y_{11}, \ldots, Y_{1N_1}, Y_{21}, \ldots, Y_{2N_2}, \ldots, Y_{k1}, \ldots, Y_{kN_k}) \)

corresponding to stratified population

\( \mathcal{U} = (U_{11}, \ldots, U_{1N_1}, U_{21}, \ldots, U_{2N_2}, \ldots, U_{k1}, \ldots, U_{kN_k}) \)

is specified, though not completely, by the first two moments. Throughout the thesis we will use the notation \( Y_{ij} \) for both as a random variable and as the value it takes on the \( j^{th} \) unit of the \( i^{th} \) stratum given a finite population of size \( N_i; i=1,2,\ldots,k; j=1,2,\ldots,N_i \).

Let \( \mathcal{E}_{\delta_g}, \mathcal{V}_{\delta_g}, \) and \( \mathcal{C}_{\delta_g} \) denote the conditional expectation, variance and covariance given \( X \)'s respectively w.r.t. the model \( \delta_g \)

\[
\begin{align*}
&i) \quad \mathcal{E}_{\delta_g}(Y_{ij}|X_{ij}) = \alpha + \beta \ X_{ij} \\
&ii) \quad \mathcal{V}_{\delta_g}(Y_{ij}|X_{ij}) = \sigma^2 \ X_{ij}^g \\
&iii) \quad \mathcal{C}_{\delta_g}(Y_{ij}, Y_{ij'}|X_{ij}, X_{ij'}) = 0 \\
&iv) \quad \mathcal{C}_{\delta_g}(Y_{ij}, Y_{ij'}|X_{ij}, X_{ij'}) = 0
\end{align*}
\]

(4.1.2)

where \( i = 1, 2, \ldots, k \) and \( j, j' = 1, 2, \ldots, N_i, (j \neq j') \).

In the model (4.1.2), \( \alpha, \beta, \sigma^2 \) and \( g \) are superpopulation parameters, with \( \sigma^2 \geq 0, \alpha, \beta \) are unknown parameters of the prior distribution \( \delta_g \) whereas \( g \) is found to lie between 0 and 2 and more often close to 2.
As mentioned earlier, some estimate of the unknown $\{\sigma_i^2\}$ is needed in order to compute TNOA. Usually $\{\sigma_i(x)\}$, the corresponding WSSD of the auxiliary variable is substituted in place of the unknown $\{\sigma_i\}$ in the TNOA.

Earlier, Mahalanobis (1952) [36] proposed equalization of the stratum totals together with equal allocation as an approximation to TN-optimum allocation while Kitagawa (1956) [30] provided the justification for equipartition. Hansen, Hurwitz and Madow (1953) [21] gave several illustrations to show that allocation proportional to stratum totals is a "near-optimum" allocation.

**Definition 4.1.1.** The allocation, $n_i \propto N_i \sigma_i(x)$ is called Auxiliary Variable Optimum Allocation (AVOA) and is given by

$$n_i = n \frac{N_i \sigma_i(x)}{\sum_i N_i \sigma_i(x)}.$$  \hspace{1cm} (4.1.3)

The justification of the substitution on the assumption that the estimates $\{\sigma_i^2(x)\}$-values are not substantially different from the proportionate values of the unknown $\{\sigma_i^2\}$ was appraised in the light of the prior distributions specified by equation (4.1.2) using superpopulation model approach first by Hanurav (1965) [23] and then extensively by Rao (1968,77) [49], [50].

We now give a theorem due to T.J. Rao (1968) [49].

**Theorem 4.1.2.** Under the model specified by equations (4.1.2), the Tschuprow-Neyman Optimum Allocation (TNOA) reduces to
\[ n_i(g) \propto \sqrt{X_i^2 - \gamma_i(g)} \quad (4.1.4) \]

\[ n_i(g) \propto X_i \sqrt{1 - \frac{\gamma_i(g)}{X_i^2}} \]

provided CCV

\[ \theta_i(g) = \frac{\sigma_i(x)}{\sqrt{X_i^2 - \frac{\gamma_i(g)}{N_i^2}}} \quad (4.1.5) \]

are equal in all strata,

where

\[ \gamma_i(g) = N_i \left( \sum_j X_{ij}^2 - \sum_j X_{ij}^2 \right) \]

for \( \gamma_i(g) > 0 \).

**Corollary 4.1.3.** For a particular case of \( g = 2 \) of the model (4.1.2), the
TNOA reduces to allocation proportional to stratum totals of the auxiliary
variable if its coefficient of variation is equal in all strata i.e., if \( C_i(x) \propto \)
constant., then \( n_i(2) \propto X_i \).

**Definition 4.1.4.** The allocation \( n_i(g) \propto X_i \sqrt{1 - \frac{\gamma_i(g)}{X_i^2}} \) given by (4.1.4)
is called \( \Delta_g \)-Rao’s allocation proportional to corrected stratum total of
auxiliary variable or \( \Delta_g \)-Rao’s Corrected Auxiliary Variable -Proportional
Allocation (\( \Delta_g \)-RCAVPA).
Definition 4.1.5. Under the model (4.1.2) with $g = 2$ the $\Delta_2$-RCAVPA reduces to the allocation proportional to stratum total of the auxiliary variable and is called the $\Delta_2$-Rao’s Auxiliary Variable-Proportional Allocation ($\Delta_2$-RAVPA).

Definition 4.1.6. For the superpopulation model $\Delta_g$ we define $x^{g/2}$, the $g/2^{th}$ power of the auxiliary variable $X$ as $\Delta_g$-Generalized Auxiliary Variable (\(\Delta_g\)-GAV).

Definition 4.1.7. For stratified SRSWR or SRSWOR design, allocation proportional to stratum total of the $\Delta_g$-GAV i.e,

$$n_i(g) \propto \sum_j X_{ij}^{g/2}, \quad i = 1,2,\ldots, k$$

is called $\Delta_g$-GAV Proportional Allocation ($\Delta_g$-GAVPA).

Definition 4.1.8. For stratified SRSWR design, allocation proportional to the product of stratum size and $\sigma_i(x^{g/2})$, the standard deviation of the $\Delta_g$-GAV i.e.,

$$n_i(g) \propto N_i \sigma_i(x^{g/2}), \quad i = 1,2,\ldots, k$$

is called $\Delta_g$-GAV Optimum Allocation ($\Delta_g$-GAVOA), where

$$\sigma_i^2(x^{g/2}) = \frac{1}{N_i} \sum_j \left(X_{ij}^{g/2} - \overline{X}_i^{g/2}\right)^2$$
is the variance of $GAV_{x^{g/2}}$.

**Definition 4.1.9.** For stratified SRSWOR design allocation proportional to the product of stratum size $N_i$ and $S_i(x^{g/2})$, the standard deviation of the GAV, viz.,

$$n_i(g) \propto N_i S_i(x^{g/2}), \quad i = 1, 2, \ldots, k$$

is called $\Delta_g$-GAV Optimum Allocation ($\Delta_g$-GAVOA), where

$$S_i^2(x^{g/2}) = \frac{N_i}{N_i - 1} \sigma_i^2(x^{g/2}).$$

In this chapter we further investigate the problem of allocation of sample size to strata under the superpopulation model $\Delta_g$ for the stratified SRSWR design in section 4.2 and for stratified SRSWOR design in section 4.3 and give some quick approximate allocations and study the effect of the condition (4.1.5) on the efficiency of the allocations.

4.2 Stratified SRSWR Design

4.2.1 TNOA under Superpopulation Model

We now derive the expectation of $\{\sigma_i^2\}$ under the model $\Delta_g$ specified by equations (4.1.2) above, when simple random sampling design with replacement (SRSWR) is used within each stratum.

Now, let us write
\[ \sigma_i^2 = \frac{1}{N_i} \sum_j (Y_{ij} - \bar{Y}_i)^2 \]

as

\[ \sigma_i^2 = \frac{1}{N_i} \left[ \left(1 - \frac{1}{N_i}\right) \sum_j Y_{ij}^2 - \frac{1}{N_i} \sum_{j \neq j'} Y_{ij} Y_{ij'} \right]. \]

Now, under the prior distributions \( \Delta_g \) specified by equations (4.1.2), we have,

\[ N_i \mathcal{E}_g (\sigma_i^2|X) = \left(1 - \frac{1}{N_i}\right) \sum_j \mathcal{E}_g (Y_{ij}^2|X_{ij}) \]

\[ -\frac{1}{N_i} \sum_{j \neq j'} \mathcal{E}_g (Y_{ij} Y_{ij'}|X_{ij} X_{ij'}) \]

\[ = \left(1 - \frac{1}{N_i}\right) \sum_j \left[ \mathcal{I}_g (Y_{ij}|X_{ij}) + \{\mathcal{E}_g (Y_{ij}|X_{ij})\}^2 \right] \]

\[ -\frac{1}{N_i} \sum_{j \neq j'} \left[ \mathcal{C}_g (Y_{ij}, Y_{ij'}|X_{ij}, X_{ij'}) + \mathcal{E}_g (Y_{ij}|X_{ij}) \mathcal{E}_g (Y_{ij'}|X_{ij'}) \right] \]

\[ = \left(1 - \frac{1}{N_i}\right) \sum_j \{\sigma^2 X_{ij}^2 + (\alpha + \beta X_{ij})^2\} \]

\[ -\frac{1}{N_i} \sum_{j \neq j'} \{ (\alpha + \beta X_{ij})(\alpha + \beta X_{ij'}) \}. \]
After simplification, we get

\[ \mathcal{E}_{g^2} (\sigma_i^2|X) = \beta^2 \sigma_i^2(x) + \sigma^2 d_i(g) \]  \hspace{1cm} (4.2.1)

where

\[ d_i(g) = \frac{N_i - 1}{N_i} \frac{1}{N_i} \sum_j X_{ij}^g = \frac{N_i - 1}{N_i} \bar{X}_i^g \]  \hspace{1cm} (4.2.2)

and

\[ \bar{X}_i^g = \frac{1}{N_i} \sum_j X_{ij}^g \]  \hspace{1cm} (4.2.3)

is the mean of \( g \)th power of the auxiliary variable \( X \) in the \( i \)th stratum.

Thus \( \sigma_i^2 \) can be expected under the model \( \Delta_g \) to be proportional to \( \sigma_i^2(x) \) if it is proportional to \( d_i(g) \). If this condition is satisfied, the unknown \( \sigma_i^2 \) in the TNOA may be replaced by \( d_i(g) \). If the above condition is fulfilled the TNOA under the model \( \Delta_g \) reduces to the following two model-based allocations:

\[ n_i \propto N_i \sigma_i(x) \]  \hspace{1cm} (4.2.4)
\[ n_i(g) \propto N_i \sqrt{d_i(g)} = \sqrt{N_i(N_i - 1)} \bar{X}_i^g \]

\[ = \sqrt{(N_i - 1) \sum_j X_{ij}^g} \quad (4.2.5) \]

provided the corrected stratum coefficient of variation (CSCV)

\[ \theta_i(g) = \frac{\sigma_i(x)}{\sqrt{d_i(g)}} = \frac{\sigma_i(x)}{\sqrt{\frac{N_i - 1}{N_i} \frac{X_i^g}{\bar{X}_i^g}}} = \frac{\sigma_i(x) \sqrt{N_i - 1}}{\sqrt{X_i^g}} = \frac{S_i(x)}{\sqrt{X_i^g}} \quad (4.2.6) \]

are equal in all strata.

**Definition 4.2.1.** The allocations (4.2.4) and (4.2.5) under the condition that the corrected stratum coefficients of variation (CSCV) \( \theta_i(g) \) given by (4.2.6) are equal in all strata, are called \( \Delta_g \)-Auxiliary Variable Optimum Allocation (\( \Delta_g \)-AVOA) and \( \Delta_g \)-Model Optimum Allocation (\( \Delta_g \)-MOA) respectively for the strategy (St. SRSWR: \( \bar{Y}_{SUE} \)).

Next, \( d_i(g) \) may be rewritten as

\[ d_i(g) = \left( \frac{N_i - 1}{N_i} \right) \bar{X}_i^g = \left( \frac{N_i - 1}{N_i} \right) \frac{1}{N_i} \sum_{j=1}^{N_i} \left( X_{ij}^g / 2 \right)^2 \]

\[ = \left( \frac{N_i - 1}{N_i} \right) \left\{ \sigma_i^2 \left( x^{g/2} \right) + \left( \bar{X}_i^{g/2} \right)^2 \right\} \quad (4.2.7) \]

where \( \left( X_i^{g/2} \right) \) and \( \sigma_i(x^{g/2}) \) are the mean and standard deviation of the GAV \( x^{g/2} \) respectively in the \( i^{th} \) stratum.

Now from (4.2.1) and (4.2.7),

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\[ \mathcal{E}_{\delta} (\sigma_i^2 | X) = \beta^2 \sigma_i^2 (x) \]
\[ + \sigma^2 \left( \frac{N_i - 1}{N_i} \right) \left\{ \sigma_i^2 (x^{g/2}) + \left( X_i^{g/2} \right)^2 \right\} \]  
\[ = \beta^2 \sigma_i^2 (x) + \sigma^2 \left( \frac{N_i - 1}{N_i} \right) \left( X_i^{g/2} \right)^2 \left\{ C_i^2 (x^{g/2}) + 1 \right\} \]  
\[ = \beta^2 \sigma_i^2 (x) + \sigma^2 \left( \frac{N_i - 1}{N_i} \right) \sigma_i^2 (x^{g/2}) \left\{ 1 + \frac{1}{C_i^2 (x^{g/2})} \right\} \]  
\[ \text{(4.2.8)} \]
\[ \text{where} \]
\[ C_i (x^{g/2}) = \frac{\sigma_i (X_i^{g/2})}{\left( X_i^{g/2} \right)} \]

is the coefficient of variation (CV) of the GAV \( x^{g/2} \) in the \( i \)th stratum.

Now if \( C_i (x^{g/2}) \propto \text{constant} \), then from (4.2.9),
\[ \mathcal{E}_{\delta} (\sigma_i^2 | X) \propto \sigma_i^2 (x) \]
\[ \text{or} \quad \propto \frac{N_i - 1}{N_i} \left( X_i^{g/2} \right)^2 \]
\[ \text{if} \quad \sigma_i^2 (x) \propto \frac{N_i - 1}{N_i} \left( X_i^{g/2} \right)^2 . \]

Also from (4.2.10) under the assumption \( C_i (x^{g/2}) \propto \text{constant} \), we have
\[ \mathcal{E}_{\delta} (\sigma_i^2 | X) \propto \sigma_i^2 (x) \]
\[ \text{or} \quad \propto \sigma_i^2 (x^{g/2}) \left( \frac{N_i - 1}{N_i} \right) \]
\[ \text{if} \quad \sigma_i^2 (x) \propto \sigma_i^2 (x^{g/2}) \left( \frac{N_i - 1}{N_i} \right) . \]

Thus if \( C_i (x^{g/2}) \), the stratum coefficients of variation of \( x^{g/2} \), are equal
then the \( \Delta_g \)-MOA gives rise to following two equal stratum CVs (ESCVs)
Model Optimum Allocations (ESCVMOAs):

1. 

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\[ n_i(g) \propto \sum_{j=1}^{N_i} X_j^{\alpha/2} \sqrt{\frac{N_i-1}{N_i}} \]  \hspace{1cm} (4.2.11)

provided

\[ \eta_i(g) = \frac{\sigma_i(x)}{N_i} = \frac{\sigma_i(x) \sqrt{\frac{N_i}{N_i-1}}}{C_i(x)} \]  \hspace{1cm} (4.2.12)

are equal in all strata.

2.

\[ n_i(g) \propto N_i \sigma_i(x^{\alpha/2}) \sqrt{\frac{N_i-1}{N_i}} \]  \hspace{1cm} (4.2.13)

provided

\[ \zeta_i(g) = \frac{\sigma_i(x)}{\sigma_i(x^{\alpha/2})} = \frac{\sigma_i(x)}{C_i(x^{\alpha/2})} \propto \eta_i(g) \]  \hspace{1cm} (4.2.14)

are equal in all strata.

**Definition 4.2.2.** The allocation (4.2.11) is called \( \Delta_g \)-Mean-based ESCVMOA (\( \Delta_g \)-MESCVMOA) for the strategy \( \text{St. SRSWR: } \hat{Y}_{\text{St.UE}} \).

**Definition 4.2.3.** The allocation (4.2.13) is called \( \Delta_g \)-Standard Deviation-based ESCVMOA (\( \Delta_g \)-SDESCVMOA) for the strategy \( \text{St. SRSWR: } \hat{Y}_{\text{St.UE}} \).
4.2.2 Approximations to $\Delta_g$-MOA and $\Delta_g$-ESCVMOAs

If the stratum sizes $\{N_i\}$ are so large that $\frac{1}{N_i}$ can be neglected relative to unity i.e., $\left(\frac{N_i-1}{N_i}\right) \approx 1$ or equivalently that the term in $\frac{1}{N_i}$ in $d_i(g)$ viz., $\frac{X_i^g}{N_i}$ is negligible compared to $d_i(g)$ i.e., Relative Neglected Term (RNT) is such that

$$n_i(g) = \frac{X_i^g}{d_i(g)} \approx \frac{1}{N_i - 1} \approx 0 \ll 1,$$

then $d_i(g)$ may be approximated by

$$d_i'(g) = X_i^g.$$

and the $\Delta_g$-MOA given by equation (4.2.5) may be approximated by the following approximate $\Delta_g$-MOA given by

$$n_i(g) \propto N_i \sqrt{d_i'(g)} = N_i \sqrt{X_i^g} = \sqrt{N_i \sum_j X_{ij}^g} \quad (4.2.15)$$

provided

$$\theta_i'(g) = \frac{\sigma_i'(g)}{\sqrt{d_i'(g)}} = \frac{\sigma_i(g)}{\sqrt{X_i^g}}$$

$$= \frac{\sigma_i(g)}{d_i(g)} \sqrt{\frac{N_i - 1}{N_i}} = \theta_i(g) \sqrt{\frac{N_i - 1}{N_i}} \quad (4.2.16)$$

are equal in all strata.

Definition 4.2.4. The approximate allocation (4.2.15) under large strata
approximation is called $\Delta_g$-Approximate MOA ($\Delta_g$-AMOA) for the strategy (St.SRSWR: $\hat{Y}_{SU}$).

Further, if $C_i(x^{g/2})$ are equal then the approximations to $\Delta_g$-MESCVMOA in (4.2.11) and $\Delta_g$-SDESCVMOA in (4.2.13) are respectively given by the following approximate ESCVMOAs:

1. 

$$n_i(g) \propto \sum_j x_{ij}^{g/2} \quad (4.2.17)$$

if

$$\eta_i'(g) = \frac{\sigma_i(x)}{x_i^{g/2}} = \eta_i(g) \sqrt{\frac{N_i - 1}{N_i}} \quad (4.2.18)$$

are equal in all strata.

2. 

$$n_i(g) \propto N_i \sigma_i(x^{g/2}) \quad (4.2.19)$$

if

$$\xi_i'(g) = \frac{\sigma_i(x)}{\sigma_i(x^{g/2})} = \xi_i(g) \sqrt{\frac{N_i - 1}{N_i}} \quad (4.2.20)$$

are equal in all strata.

**Definition 4.2.5.** The allocation (4.2.17) under large strata approximation with assumption of Equal SCV($x^{g/2}$) is called $\Delta_g$-Mean-based ESCVMOA ($\Delta_g$-MESCVOAMOA) for the strategy (St.SRSWR: $\hat{Y}_{SU}$).

**Definition 4.2.6.** The allocation (4.2.19) under large strata approximation and with assumption of Equal SCV($x^{g/2}$) is called $\Delta_g$-SD-based ESCVMOA ($\Delta_g$-SDESCVMOA) for the strategy (St.SRSWR: $\hat{Y}_{SU}$).
Thus $\Delta_g$-MESCVA MOA for the strategy (St.SRSWR: $\hat{Y}_{St.UE}$) is the $\Delta_g$-GAVPA along with the condition (4.2.18). Also $\Delta_g$-SDESCVAMOA for the strategy (St.SRSWR: $\hat{Y}_{St.UE}$) is $\Delta_g$-GAVOA for the strategy (St.SRSWR: $\hat{Y}_{St.UE}$) along with the condition (4.2.20).

4.2.3 Particular Cases of the $\Delta_g$-Model-based Allocations

In this subsection, the $\Delta_g$-Model-based allocations i.e., $\Delta_g$-MOA and $\Delta_g$-ESCVMOAs and their approximations namely $\Delta_g$-AMOA and $\Delta_g$-ESCVAMOAs are deduced for certain specific values of interest of the superpopulation parameter $g$.

Case I: Model $\Delta_0$

From (4.2.2), we have

$$d_i(0) = \frac{N_i - 1}{N_i}$$

$$n_i(0) \propto N_i \sqrt{d_i(0)} = N_i \sqrt{\frac{N_i - 1}{N_i}} = \sqrt{N_i(N_i - 1)}$$

provided

$$\theta_i(0) = \frac{\sigma_i(x)}{\sqrt{d_i(0)}} = \frac{\sigma_i(x)}{\sqrt{(N_i - 1)/N_i}}$$

$$= \sqrt{\frac{N_i}{N_i - 1}} \sigma_i(x) = S_i(x)$$

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are equal.

\( \Delta_0 \)-MESCVMOA is identical with \( \Delta_0 \)-MOA but \( \Delta_0 \)-SDESCVMOA does not exist. Under large strata approximation, \( \Delta_0 \)-AMOA also reduces to BSSPA (proportional allocation) under the condition that standard deviations \( \sigma_i(x) \) are all equal. \( \Delta_0 \)-MESCVAMOA is identical with \( \Delta_0 \)-AMOA but \( \Delta_0 \)-SDESCVAMOA does not exist.

Thus \( \Delta_0 \)-MOA and \( \Delta_0 \)-MESCVMOA for the large strata are approximated respectively by \( \Delta_0 \)-AMOA and \( \Delta_0 \)-MESCVAMOA, both of which in turn reduce to BSSPA (proportional allocation) provided the standard deviations of the auxiliary variable for all strata are equal.

**Case II: Model \( \Delta_1 \)** For \( g = 1 \), using (4.2.2), we have

\[
d_i(1) = \left( \frac{N_i - 1}{N_i} \right) \bar{X}_i
\]

Hence the \( \Delta_1 \)-MOA is given by

\[
n_i(1) \propto N_i \sqrt{\frac{N_i - 1}{N_i}} \bar{X}_i = \sqrt{(N_i - 1)X_i}
\]

provided

\[
\theta_i(1) = \frac{\sigma_i(x)}{\sqrt{\frac{N_i - 1}{N_i}} \bar{X}_i} = \frac{S_i(x)}{\sqrt{\bar{X}_i}}
\]

are equal in all strata.

If the \( CV(\sqrt{x}) \), \( \{C_i(\sqrt{x})\} \) are assumed to be equal for all strata then
Δ₁-MESCVMOA is given by

\[ n_i(1) \propto \sum_j \sqrt{X_{ij}} \sqrt{\frac{N_i - 1}{N_i}} \]

provided

\[ \eta_i(1) = \frac{\sigma_i(x)}{(X_i^{1/2})} \sqrt{\frac{N_i - 1}{N_i}} = \frac{S_i(x)}{C_i(x)} \]

are equal in all strata.

Similarly under the assumption of equal \( C_i(\sqrt{x}) \), \( \Delta_1 \)-SDESCVMOA is given below:

\[ n_i(1) \propto N_i \sigma_i(\sqrt{x}) \sqrt{\frac{N_i - 1}{N_i}} \]

provided

\[ \zeta_i(1) = \frac{\sigma_i(x)}{\sigma_i(\sqrt{x})} \sqrt{\frac{N_i - 1}{N_i}} = \frac{S_i(x)}{\sigma_i(\sqrt{x})} = \frac{\eta_i(1)}{C_i(\sqrt{x})} \propto \eta_i(1) \]

are equal in all strata.

Now the approximation of large stratum sizes i.e.,

\[ \frac{N_i - 1}{N_i} \approx 1 \]

and the RNT is such that

\[ \tau_i(1) = \frac{1}{N_i - 1} \approx 0 \ll 1, \]

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then the $\Delta_1$-MOA may be approximated by $\Delta_1$-AMOA given by

$$n_i(1) \propto N_i \sqrt{X_i} = \sqrt{N_i X_i}$$

provided

$$\theta'_i(1) = \frac{\sigma_i(x)}{\sqrt{X_i}}$$

are equal in all strata.

Similarly under the above approximation of large stratum sizes and under the assumption of equal SCV of $\sqrt{x}$ viz., $C_i(\sqrt{x})$, the above $\Delta_1$-MESCVMOA and $\Delta_1$-SDESCVMOA may be approximated by their approximate allocations $\Delta_1$-MESCVMOA and $\Delta_1$-SDESCVMOA given below:

**$\Delta_1$-MESCVMOA:**

$$n_i(1) \propto \sum_j \sqrt{X_{ij}}$$

provided

$$\eta'_i(1) = \frac{\sigma_i(x)}{\left(X_i^{1/2}\right)}$$

are equal in all strata.

**$\Delta_1$-SDESCVMOA:**

$$n_i(1) \propto N_i \sigma_i(\sqrt{x})$$

if
are equal in all strata.

**Case III: Model $\Delta_2$** Substituting $g = 2$, in (4.2.2), we have

\[
d_i(2) = \left( \frac{N_i - 1}{N_i} \right) \bar{X}_i^2 = \left( \frac{N_i - 1}{N_i} \right) \frac{1}{N_i} \sum_j X_{ij}^2 = \left( \frac{N_i - 1}{N_i} \right) \left\{ \sigma_i^2(x) + \bar{X}_i^2 \right\}
\]

\[
= \left\{ \sigma_i^2(x) + \bar{X}_i^2 \right\} - \frac{1}{N_i} \left\{ \sigma_i^2(x) + \bar{X}_i^2 \right\}
\]

Now using (4.2.1), we get

\[
E_{\delta_2}(\sigma_i^2|X) = (\beta^2 + \sigma^2)\sigma_i^2(x) + \sigma^2 \left[ \bar{X}_i^2 - \frac{1}{N_i} \left\{ \sigma_i^2(x) + \bar{X}_i^2 \right\} \right]
\]

\[
= (\beta^2 + \sigma^2)\sigma_i^2(x) + \sigma^2 \bar{X}_i^2 \left[ 1 - \frac{1 + C_i^2(x)}{N_i} \right]
\]

\[
= (\beta^2 + \sigma^2)\sigma_i^2(x) + \sigma^2 \bar{X}_i^2 \{ 1 - \kappa_i(x) \}
\]

where

\[
\kappa_i(x) = \frac{1 + C_i^2(x)}{N_i} = \frac{\sum_j X_{ij}^2}{X_i^2}.
\]

Thus

\[
E_{\delta_2}(\sigma_i^2|X) \propto \sigma_i^2(x)
\]

or

\[
\propto \bar{X}_i^2 \{ 1 - \kappa_i(x) \}
\]

if

\[
\frac{\sigma_i^2(x)}{\bar{X}_i^2 \{ 1 - \kappa_i(x) \}} \propto \text{constant}
\]
or if

\[ \frac{C_i(x)}{\{1 - \kappa_i(x)\}} \propto \text{constant.} \]

Thus \( \Delta_2\)-MOA is given by

\[ n_i(2) \propto X_i \sqrt{1 - \kappa_i(x)} \]

provided

\[ \theta_i(2) = \frac{C_i(x)}{\sqrt{1 - \kappa_i(x)}} \]

are equal in all strata.

The two \( \Delta_2\)-MOAs under the assumption of equal \( SCV(x^{\theta/2}) \), viz., \( \Delta_2\)-MESCVMOA and \( \Delta_2\)-SDESCVMOA are given below:

1. \( \Delta_2\)-MESCVMOA:

\[ n_i(2) \propto X_i \sqrt{\frac{N_i - 1}{N_i}} \]

provided

\[ \eta_i(2) = \frac{\sigma_i(x)}{\bar{X}_i \sqrt{\frac{N_i - 1}{N_i}}} = C_i(x) \sqrt{\frac{N_i}{N_i - 1}} = C_i'(x) \]

are equal in all strata.

2. \( \Delta_2\)-SDESCVMOA:
\[ n_i(2) \propto N_i \sigma_i(x) \sqrt{\frac{N_i - 1}{N_i}} \]

provided

\[ \zeta_i(2) = \frac{\sigma_i(x)}{\sigma_i(x) \sqrt{\frac{N_i - 1}{N_i}}} = \sqrt{\frac{N_i}{N_i - 1}} \]

are equal in all strata.

Therefore, if the stratum sizes are so large that

(i) \( \frac{N_i - 1}{N_i} \approx 1 \), or

(ii) the relative neglected term (RNT) is such that

\[ \tau_i(2) = \frac{1}{N_i - 1} \approx (0) \ll 1 \], or

(iii) \( \kappa_i(2) \approx 0 \ll 1 \), or

(iv) \( \kappa_i(x) \propto \text{constant } \forall i \),

i.e., even if \( \{\kappa_i(x)\} \) are not negligible, it is sufficient that they are equal,

then the \( \Delta_2\)-MOA may be approximated by \( \Delta_2\)-AMOA given below:

\[ n_i(2) \propto X_i \]

provided

\[ \theta'_i(2) = C_i(x) \]
are equal in all strata.

Similarly $\Delta_2$-MESCVMOA and $\Delta_2$-DESCVMOA, under the large sized strata may be approximated as follows:

1. $\Delta_2$-MESCVMOA:

$$n_i(2) \propto X_i$$

provided

$$\eta_i(2) = \frac{\sigma_i(x)}{X_i} = C_i(x)$$

are equal in all strata.

Thus $\Delta_2$-MESCVMOA is Rao's Auxiliary Variable Proportional Allocation (RAVPA). Therefore RAVPA follow as a particular case of $\Delta_2$- MESCVMOA.

2. $\Delta_2$-DESCVMOA:

$$n_i(2) \propto N_1 \sigma_i(x)$$

provided

$$\zeta_i(2) = \frac{\sigma_i(x)}{\sigma_i(x)} = 1$$

are equal i.e., which is automatically satisfied.

The assumption of Equal SCV($x$) is the only condition for this allocation. $\Delta_2$-
SDESCVAMOA is nothing but Hanurav’s Auxiliary Variable Optimum Allocation (HAVOA). Therefore HAVOA is a particular case of $\Delta_g$-SDESCVAMOA.

**Theorem 4.2.7.** The Tschuprow-Neyman optimum allocation under the superpopulation model $\Delta_g$ for stratified sampling strategy (St. SRSWR: $\tilde{Y}_{SRAE}$) reduces either to Auxiliary Variable Optimum Allocation (AVOA) i.e.,

$$n_i \propto N_i \sigma_i(x)$$

or equivalently to $\Delta_g$-Model Optimum Allocation

$$n_i \propto N_i \sqrt{d_i(g)} = N_i \sqrt{\left(\frac{N_i - 1}{N_i}\right) \bar{X}_i^g} = \sqrt{(N_i - 1) \sum_j X_{ij}^g}$$

provided the corrected coefficient of variation (CCV)

$$\theta_i(g) = \frac{\sigma_i(x)}{\sqrt{\frac{N_i - 1}{N_i} X_i^g}} = \frac{S_i(x)}{\sqrt{X_i^g}}, \quad i = 1, 2, \ldots, k$$

where

$$\bar{X}_i^g = \frac{1}{N_i} \sum_j X_{ij}^g$$

are equal in all strata.
Corollary 4.2.8. If the stratum sizes \( \{N_i\} \) are so large that the terms in \( \frac{1}{N_i} \) can be neglected relative to unity i.e., \( \frac{1}{N_i} \approx 0 \) or equivalently the relative neglected term (RNT)

\[
\tau_i(g) = \frac{\bar{x}_{i}^{g}/N_i}{d_i(g)} = \frac{1}{N_i - 1} \approx 0 \ll 1
\]

then the \( \Delta_q \)-MOA in the theorem 4.2.7 may be approximated by the \( \Delta_q \)-Approximate Model Optimum Allocation (\( \Delta_q \)-AMOA) given by

\[
n_i(g) \propto N_i \sqrt{\bar{x}_{i}^{g}} = \sqrt{\sum_i \bar{x}_{ij}^{g}}
\]

provided

\[
\theta_i'(g) = \sigma_i(x) \sqrt{\bar{x}_{i}^{g}} = \theta_i(g) \sqrt{\frac{N_i - 1}{N_i}}, \quad i = 1, 2, \ldots, k
\]

are equal in all strata.

Theorem 4.2.9. If \( C_i(x^{g/2}) \), the coefficient of variation of the Generalized Auxiliary Variable \( x^{g/2} \) in the \( i^{th} \) stratum, \( i = 1, 2, \ldots, k \) are equal in all strata then the two \( \Delta_q \)-Equal Coefficient of Variation (\( x^{g/2} \))- Model Optimum Allocations (\( \Delta_q \)-ESCVMOAs) based on TNOA for stratified strategy (St.SRSWR: \( \bar{Y}_{St,UB} \)) are given as follows:

1. \( \Delta_q \)-Mean-based ESCVMOA(\( \Delta_q \)-MESCVMOA)
\[ n_i(g) \propto \sum_j X_{ij}^{g/2} \sqrt{\frac{N_i - 1}{N_i}} \]

provided

\[ \eta_i(g) \frac{\sigma_i(x)}{\sqrt{\frac{N_i - 1}{N_i} (X_i^{g/2})}} = \frac{S_i(x)}{\left( X_i^{g/2} \right)} \]

\[ = C_i(x) \frac{X_i}{\left( X_i^{g/2} \right)} \]

are equal in all strata,

where

\[ \overline{X_i^{g/2}} = \frac{1}{N_i} \sum_j X_{ij}^{g/2}. \]

2. \( \Delta_g \)-SD-based ESCVMOA (\( \Delta_g \)-SDESCVMOA)

\[ n_i(g) \propto N_i \sigma_i(x^{g/2}) \sqrt{\frac{N_i - 1}{N_i}} \]

provided

\[ \zeta_i(g) = \frac{\sigma_i(x)}{\sigma_i(x^{g/2}) \sqrt{\frac{N_i - 1}{N_i}}} = \frac{S_i(x)}{\sigma_i(x^{g/2})} = \frac{\eta_i(g)}{C_i(x^{g/2})} \propto \eta_i(g) \]

are equal in all strata.
Corollary 4.2.10. If the stratum sizes \( \{N_i\} \) are so large that the terms in \( \frac{1}{N_i} \) can be neglected relative to unity i.e., \( \frac{1}{N_i} \approx 0 \) or equivalently the relative neglected term (RNT)

\[
\tau_i(g) = \frac{X_i^g/N_i}{d_i(g)} = \frac{1}{N_i - 1} \approx 0 \ll 1
\]

and if \( C_i(x^{g/2}) \), \( i = 1, 2, \ldots, k \), are all equal then \( \Delta_g\text{-MESCVMOA} \) and \( \Delta_g\text{-SDESCVMOA} \) in the above theorem 4.2.9 may be approximated by the \( \Delta_g\text{-MESCVAMOA} \) and \( \Delta_g\text{-SDESCVAMOA} \) respectively which are given below:

1. \( \Delta_g\text{-MESCVAMOA} \)

\[
\eta_i = \sum_j X_{ij}^{g/2}
\]

provided

\[
\eta_i(g) = \frac{\sigma_i(x)}{(X_i^{g/2})}, \quad i = 1, 2, \ldots, k
\]

where

\[
X_i^{g/2} = \frac{1}{N_i} \sum_j X_{ij}^{g/2}
\]

are equal in all strata.
2. $\Delta_g$-SDESCVAMOA:

$$n_i(g) \propto N_i \sigma_i(x^{g/2})$$

provided

$$\zeta_i(g) = \frac{\sigma_i(x)}{\sigma_i(x^{g/2})}, \quad i = 1, 2, \ldots, k$$

are equal in all strata.

**Theorem 4.2.11.** If any one of the following conditions are satisfied

(i) the stratum sizes are so large that $\frac{N_i - 1}{N_i} \approx 1$

(ii) the Relative Neglected Term (RNT)

$$\tau_i(g) = \frac{\frac{1}{N_i^2} \sum_j X_{ij}^2}{\frac{N_i - 1}{N_i^2} \sum_j X_{ij}^2}$$

$$= \frac{1}{N_i - 1} \approx 0 \ll 1$$

(iii) The $C_i(x)$, coefficient of variation of $x$, is such that

$$\kappa_i(x) = \frac{1 + C_i^2(x)}{N_i} \approx 0$$

(iv) $\kappa_i(x) \propto \text{constant} \forall i$

i.e., even if $\{\kappa_i(x)\}$ are not negligible, it is sufficient that they are equal and if $\{C_i(x)\}$ are equal in all strata then the TNOA for stratified strategy (St. SRSWR: $\bar{Y}_{SLE}$) under the model $\Delta_2$ reduces to
1. the Rao's Auxiliary Variable Proportional Allocation (RAVPA) i.e,

\[ n_i(2) \propto X_i. \]

2. the Hanurav's Auxiliary Variable Optimum Allocation (HAVOA) i.e.,

\[ n_i(2) \propto N_i \sigma_i(x). \]

### 4.3 Stratified SRSWOR Design

#### 4.3.1 TNOA under Superpopulation Model

When the simple random sampling without replacement (SRSWOR) is adopted within each stratum, the TNOA is given by

\[ n_i = n \frac{N_i S_i}{\sum_{i} N_i S_i} \quad (4.3.1) \]

where

\[ S_i^2 = \frac{N_i}{N_i - 1} \sigma_i^2 = \frac{1}{N_i - 1} \sum_j (Y_{ij} - \bar{Y}_i)^2 \]

is the within stratum variance of the study variable in the \( i^{th} \) stratum.

Now under the model \( \Delta_a \), using (4.2.1), we have
\[ \mathcal{E}_{\delta}(S_i^2 | X) = \left( \frac{N_i}{N_i - 1} \right) \mathcal{E}_{\delta_i}(\sigma^2_i | X) \]

\[ = \left( \frac{N_i}{N_i - 1} \right) \left[ \beta^2 \sigma^2_i(x) + \sigma^2 d_i(g) \right] \]

\[ = \beta^2 \sigma^2_i(x) + \sigma^2 D_i(g) \]  \hspace{1cm} (4.3.2)

where

\[ D_i(g) = \frac{N_i}{N_i - 1} d_i(g) = \frac{X_i^g}{N_i}. \]  \hspace{1cm} (4.3.3)

Now we observe from the equation (4.3.2) that

\[ \mathcal{E}_{\delta}(S_i^2 | X) \propto S_i^2(x) \]

or equivalently,

\[ \propto D_i(g) \]

if \( S_i^2(x) \propto D_i(g) \).

Thus \( \{S_i^2\} \) is expected, under the model \( \Delta_g \), to be proportional to \( \{S_i^2(x)\} \) if \( \{S_i^2(x)\} \) is, in turn proportional to \( \{D_i(g)\} \) in each stratum. If it is so, the known \( \{S_i^2(x)\} \) may be substituted for the unknown \( \{S_i^2\} \) in the TNOA which gives rise to the following two allocations based on the Model \( \Delta_g \):

1. \[ n_i \propto N_i S_i(x) \]  \hspace{1cm} (4.3.4)

2. \[ n_i(g) \propto N_i \sqrt{D_i(g)} = N_i \sqrt{X_i^g} \]  \hspace{1cm} (4.3.5)
provided the corrected coefficient of variation (CCV)

\[ \theta_i(g) = \frac{S_i(x)}{\sqrt{D_i(g)}}, \quad i = 1, 2, \ldots, k, \]  

(4.3.6)

are equal in all strata.

Definition 4.3.1. The allocations (4.3.4) and (4.3.5) under the condition that the corrected coefficient of variation (CCV) given by (4.3.6) are equal in all strata, are called \( \Delta_g \)-Auxiliary Variable Optimum Allocation (\( \Delta_g \)-AVOA) and \( \Delta_g \)-Model Optimum Allocation (\( \Delta_g \)-MOA) respectively for the strategy (St. SRSWOR: \( \hat{Y}_{S_{lUE}} \)).

Thus the substitution of \( \{S_i(x)\} \) for the unknown \( \{S_i\} \) in the TNOA is justified if the CCV \( \{\theta_i(g)\} \) are equal in all strata.

\( \Delta_g \)-MOAs under the ESCV Assumption

Next, \( D_i(g) \) may be rewritten as

\[ D_i(g) = \frac{X_i^g}{N_i} = \frac{1}{N_i} \sum_{j=1}^{N_i} \left( \frac{X_i^{g/2}}{x_i^{g/2}} \right)^2 = \sigma_i^2 \left( \frac{x_i^{g/2}}{x_i^{g/2}} \right) + \left( \frac{X_i^{g/2}}{x_i^{g/2}} \right)^2 \]  

(4.3.7)

where \( \left( \frac{X_i^{g/2}}{x_i^{g/2}} \right) \) and \( \sigma_i(x_i^{g/2}) \) are the mean and standard deviation of the GAV \( x_i^{g/2} \) respectively in the \( i^{th} \) stratum.

Now, using equation (4.3.7), \( \varepsilon_{\delta_g}(S_i^2|x) \) in equation (4.3.2) may be rewritten as

\[ \varepsilon_{\delta_g}(S_i^2|x) = \beta^2 S_i^2(x) + \sigma^2 \left\{ \sigma_i^2 \left( \frac{x_i^{g/2}}{x_i^{g/2}} \right) + \left( \frac{X_i^{g/2}}{x_i^{g/2}} \right)^2 \right\} \]  

(4.3.8)
\[
\beta^2 S_i^2(x) + \sigma^2 \left( \frac{x_i^{g/2}}{X_i^{g/2}} \right)^2 \left\{ C_i^2(x_i^{g/2}) + 1 \right\} = \beta^2 S_i^2(x) + \sigma^2 \sigma_i^2(x_i^{g/2}) \left\{ 1 + \frac{1}{C_i^2(x_i^{g/2})} \right\} \tag{4.3.9}
\]

where

\[
C_i(x_i^{g/2}) = \frac{\sigma_i(x_i^{g/2})}{\left( \frac{x_i^{g/2}}{X_i^{g/2}} \right)}
\]

is the coefficient of variation of \(x_i^{g/2}\) in the \(i^{th}\) stratum.

Now if \(C_i(x_i^{g/2}) \propto \text{constant}\), then from equation (4.3.9),

\[
\mathcal{E}_{\delta_i} (S_i^2 | X) \propto S_i^2(x)
\]

or \(\propto \left( \frac{x_i^{g/2}}{X_i^{g/2}} \right)^2\)

if \(S_i^2(x) \propto \left( \frac{x_i^{g/2}}{X_i^{g/2}} \right)^2\)

and also from equation (4.3.10),

\[
\mathcal{E}_{\delta_i} (S_i^2 | X) \propto S_i^2(x) \propto \sigma_i^2(x_i^{g/2})
\]

if \(S_i^2(x) \propto \sigma_i^2(x_i^{g/2})\).

If the coefficients of variation of \(x_i^{g/2}\) viz., \(\{C_i(x_i^{g/2})\}\) are equal in all the strata, then under this assumption, we obtain the following two allocations from \(\Delta_g\)-MOA:

1.

\[
n_i(g) \propto N_i \left( \frac{x_i^{g/2}}{X_i^{g/2}} \right) = \sum_{j=1}^{N_i} X_i^{g/2}
\]  

(4.3.11)

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provided

\[ \eta_i(g) = \frac{S_i(x)}{X_i^{g/2}} = C'_i(x) \frac{\bar{X}_i}{X_i^{g/2}} \quad (4.3.12) \]

are equal in all strata.

2.

\[ \eta_i(g) \propto N_i \sigma_i(x^{g/2}) \quad (4.3.13) \]

provided

\[ \zeta_i(g) = \frac{S_i(x)}{\sigma_i(x^{g/2})} = \frac{\eta_i(g)}{C_i(x^{g/2})} \propto \eta_i(g) \quad (4.3.14) \]

[since the ass: \( C_i(x^{g/2}) \propto \text{constant} \)]

are equal in all strata.

**Definition 4.3.2.** The allocation (4.3.11) proportional to \( \sum_j X_{ij}^{g/2} \) is called \( \Delta_g \)-Mean-based Equal Coefficient of Variation Model Optimum Allocation (\( \Delta_g \)-MESCVMOA) for the strategy (St.SRSWOR: \( \tilde{Y}_{St.UE} \)).

**Definition 4.3.3.** The allocation (4.3.13) proportional to \( N_i S_i(x^{g/2}) \) is called \( \Delta_g \)-Standard Deviation-based Equal Coefficient of Variation Model Optimum Allocation (\( \Delta_g \)-SDESCVMOA) for the strategy (St.SRSWOR: \( \tilde{Y}_{St.UE} \)).
We note that $\Delta_g$-MESCVMOA for the strategy (St.SRSWOR: $\hat{Y}_{st,UE}$) is $\Delta_g$-GAVPA along with the condition (4.3.12). Similarly $\Delta_g$-SDESCVMOA for the strategy (St.SRSWOR: $\hat{Y}_{st,UE}$) is $\Delta_g$-GAVOA for the strategy (St.SRSWR: $\hat{Y}_{st,UE}$) along with the condition (4.3.14). Thus the substitution of $\{S_i(x)\}$ in place of $\{S_i\}$ in the TNOA under the assumption of equality of $C_i(x^{g/2})$ is justified if either $\{\eta_i(g)\}$ or equivalently $\{\zeta_i(g)\}$ are equal in all strata. The two allocations $\Delta_g$-MESCVMOA and $\Delta_g$-SDESCVMOA are identical and their conditions are equivalent under the assumption that the coefficient of variation of $x^{g/2}$ are equal in all strata.

When $g$ is known, there are two options. First the MOA proportional to $N_i \sqrt{X_i^g}$ may be computed instead of the unknown TNOA. Alternatively, if the CV($x^{g/2}$) is almost same in all the strata then either of the two ESCV-MOAs may be used. But when $g$ is unknown the only alternative left is to use $\Delta_g$-AVOA proportional to $N_i \, S_i(x)$. These allocations $\Delta_g$-MOAs and $\Delta_g$-ESCVMOAs are empirically compared with live data in section 4.5.

4.3.2 Approximations to $\Delta_g$-MOAs and $\Delta_g$-ESCVMOAs

The expression $D_i(g)$ in equation (4.3.7), may be written as

$$D_i(g) = S_i^2(x^{g/2}) + \left( \frac{X_i^{g/2}}{N_i} \right)^2 - \frac{S_i^2(x^{g/2})}{N_i}.$$ 

If the strata are so large that the term in $1/N_i$ viz., $\frac{S_i^2(x^{g/2})}{N_i}$ can be neglected.
as compared to \( D_i(g) \), the Relative Neglected Term (RNT) is given by

\[
t_i(g) = \frac{S_i^2(x^{g/2})}{N_i D_i(g)}.
\]

Now if the RNT is such that

\[
t_i(g) \approx 0 \ll 1,
\]

then \( D_i(g) \) may be approximated by

\[
D_i'(g) = S_i^2(x^{g/2}) + \left(\frac{X_i^{g/2}}{x^{g/2}}\right)^2.
\]

Consequently an approximation to \( \Delta_g \)-MOA is obtained by substituting \( D_i'(g) \) for \( D_i(g) \) in \( \Delta_g \)-MOA as given below:

\[
n_i(g) \propto N_i \sqrt{D_i'(g)} = N_i \sqrt{S_i^2(x^{g/2}) + \left(\frac{X_i^{g/2}}{x^{g/2}}\right)^2}
\]

provided

\[
\theta_i'(g) = \frac{S_i(x)}{\sqrt{D_i'(g)}} = \frac{S_i(x)}{\sqrt{S_i^2(x^{g/2}) + \left(\frac{X_i^{g/2}}{x^{g/2}}\right)^2}}, \quad i = 1, 2, \ldots, k
\]

are all equal.

**Definition 4.3.4.** The allocation (4.3.16) is called \( \Delta_g \)-Approximate MOA (\( \Delta_g \)-AMOA) for the strategy (St.SRSWOR: \( \hat{Y}_{\text{St.UE}} \)).
Thus if the stratum sizes are so large that the RNT $t_i(g)$ is negligible compared to unity then the $\Delta_g$-MOA may be approximated by the $\Delta_g$-AMOA.

$\Delta_g$-AMOAs under the ESCV Assumption

Now

$$\mathcal{E}(S_i^2|X) \approx \beta^2 S_i^2(x) + \sigma^2 \left\{ S_i^2(x^{g/2}) + \left(\frac{X_{i}^{g/2}}{s_i}\right)^2 \right\}$$

(4.3.18)

$$= \beta^2 S_i^2(x) + \sigma^2 \left\{ S_i^2(x^{g/2}) + 1 \right\}$$

(4.3.19)

$$= \beta^2 S_i^2(x) + \sigma^2 S_i^2(x^{g/2}) \left\{ 1 + \frac{1}{C_i(x^{g/2})} \right\}$$

(4.3.20)

where

$$C_i(x^{g/2}) = \frac{S_i(x^{g/2})}{(x_{i}^{g/2})} = C_i(x^{g/2}) \sqrt{\frac{N_i}{N_i - 1}}$$

(4.3.21)

may also be considered as the coefficient of variation of $x^{g/2}$. Cochran(1977) [8] used $C_i(x)$ as the CV of $x$ instead of $C_i(x)$.

Further if $\{C_i(x^{g/2})\}$, coefficient of variation of $x^{g/2}$, are equal in all strata, then following two allocations under the assumption of equal SCV ($x^{g/2}$) can be obtained from $\Delta_g$-AMOA as follows:

1. $\Delta_g$-MESCVAOMOA:

$$n_i(g) \propto \sum_j X_{ij}^{g/2}$$

(4.3.22)
provided

\[ \eta_i(g) = \frac{S_i(x)}{X_i^{5/2}} \quad (4.3.23) \]

are equal in all strata.

2. \( \Delta_g \)-SDESCVAMOA:

\[ n_i(g) \propto N_i S_i(x^{5/2}) \quad (4.3.24) \]

provided

\[ \zeta'_i(g) = \frac{S_i(x)}{S_i(x^{5/2})} = \frac{\eta_i(g)}{\zeta(x^{5/2})} \propto \eta_i(g) \]

\[ = \sqrt{\frac{N_i-1}{N_i}} \zeta_i(g) \quad (4.3.25) \]

are equal in all strata.

**Definition 4.3.5.** The allocation (4.3.22) is called \( \Delta_g \)-Mean-based ESCVAMOA (\( \Delta_g \)-MESCVAMOA) for the strategy (St.SRSWOR: \( \hat{Y}_{St.UE} \)).

**Definition 4.3.6.** The allocation (4.3.24) is called \( \Delta_g \)-Standard Deviation-based ESCVAMOA (\( \Delta_g \)-SDESCVAMOA) for the strategy (St.SRSWOR: \( \hat{Y}_{St.UE} \)).

Thus \( \Delta_g \)-MESCVAMOA for the strategy (St.SRSWOR: \( \hat{Y}_{St.UE} \)) is the \( \Delta_g \)-GAVPA alongwith the condition (4.3.23). Also
\( \Delta_g \text{-SDESCVAMOA} \) for the strategy (St.SRSWOR: \( \hat{\Phi}_{St,U_E} \)) is \( \Delta_g \text{-GAVOA} \) for the strategy (St.SRSWOR: \( \hat{\Phi}_{St,L_E} \)) alongwith the condition (4.3.25).

Further we note that \( \Delta_g \text{-MESCVMOA} \) is identical with \( \Delta_g \text{-MESCVMOA} \) except the assumption. The approximation \( 1/N_i \approx 0 \) affects only the assumption of equal \( C_i(x^{g/2}) \) for \( \Delta_g \text{-MESCVMOA} \) to make it \( C'_i(x^{g/2}) \) for \( \Delta_g \text{-MESCVMOA} \). The BSSPA and RAVPA follow as particular cases of GAVPA for \( g = 0 \) and 2 respectively. Thus GAVPA is the general allocation for populations with any \( g \).

The HAVOA \( \propto N_i S_i(x) \) follow as a particular case of GAVOA for \( g = 2 \) and thus is the general allocation like GAVPA for population with any \( g \).

Thus we have the following theorems:

**Theorem 4.3.7.** *Tscheplow-Neyman’s optimum allocation under the super-population model \( \Delta_g \) for stratified sampling strategy (St.SRSWOR: \( \hat{\Phi}_{St,U_E} \)) reduces either to \( \Delta_g \text{-Auxiliary Variable Optimum Allocation (\( \Delta_g \text{-AVOA} \))}

\[
n_i \propto N_i S_i(x)
\]

or equivalently to \( \Delta_g \text{-Model Optimum Allocation (\( \Delta_g \text{-MOA} \))}

\[
n_i(g) \propto N_i \sqrt{\frac{1}{N_i} \sum_j X_{ij}^g} = \sqrt{N_i \sum_j X_{ij}^g}
\]

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under the condition that the corrected coefficient of variation (CCV)

\[ \theta_i(g) = \frac{S_i(x)}{\sqrt{\frac{1}{N_i} \sum_j X_{ij}^2}}, \quad i = 1, 2, \ldots, k \]

are equal in all strata.

Corollary 4.3.8. If the term in \( \frac{1}{N_i} \) viz., \( \frac{S_i^2(x^{i/2})}{N_i} \) is negligible compared to \( D_i(g) \) or in other words, the RNT is close to zero in comparison to unity i.e.,

\[ \frac{S_i^2(x^{i/2})}{D_i(g)} \approx 0 \ll 1, \]

where

\[ D_i(g) = \frac{1}{N_i} \sum_j X_{ij}^2 \]

then the \( \Delta_g\)-MOA may be approximated by \( \Delta_g\)-Approximate MOA (\( \Delta_g\)-AMOA) given by

\[ n_i(g) \propto N_i \sqrt{D_i'(g)} \]

under the condition that

\[ \theta_i'(g) = \frac{S_i(x)}{\sqrt{D_i'(g)}} \]

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are equal in all strata, where

\[
D_i(g) = S_i^2(x_i^{q/2}) + \left(\frac{X_i^{q/2}}{x_i^{q/2}}\right)^2.
\]

**Theorem 4.3.9.** If the coefficient of variation of \( x_i^{q/2} \), \( C_i(x_i^{q/2}) \), \( i = 1,2,\ldots,k \), are equal then the \( \Delta g \)-MOA for the strategy (St.SRSWOR: \( \bar{y}_{St.UE} \)) gives rise to the following two \( \Delta g \)-Equal Coefficient of Variation Model Optimum Allocations (\( \Delta g \)-ESCVAMOAs):

1. **\( \Delta g \)-MESCVAMOA:**

\[
\eta_i(g) \propto \sum_j X_{ij}^{q/2}
\]

provided

\[
\eta_i(g) = C_i(x) \frac{\bar{X}_i}{(X_i^{q/2})}
\]

are equal in all strata.

2. **\( \Delta g \)-SDESCVAMOA:**
\[ n_i(g) \propto N_i S_i(x^{g/2}) \]

provided

\[ c_i'(g) = \frac{S_i(x)}{S_i(x^{g/2})} \propto n_i(g) \]

are equal in all strata.

**Corollary 4.3.10.** Under the large strata approximation i.e., assuming

\[
\frac{1}{N_i} \approx 0 \text{ or } \frac{N_i - 1}{N_i} \approx 1
\]

if

\[ c_i'(x^{g/2}) = \frac{S_i(x^{g/2})}{(X_i^{g/2})}, \quad i = 1, 2, \ldots, k \]

are all equal in all strata,

then two ESCV\((x^{g/2})\) MOAs (ESCVMOAs) \(\Delta_g\)-MESCVMOA and \(\Delta_g\)-SDESCVMOA may be approximated by \(\Delta_g\)-MESCVMOA and \(\Delta_g\)-SDESCVAMOA respectively.

1. \(\Delta_g\)-MESCVMOA:
\[ n_i(g) \propto \sum_j X_{ij}^{g/2} \]

provided \( \eta_i(g) \) are equal in all strata.

2. \( \Delta_g \)-SDESCVAMOA:

\[ n_i(g) \propto N_i S_i(x^{g/2}) \]

provided

\[ \zeta'_i(g) = \zeta_i(g) \sqrt{\frac{N_i - 1}{N_i}} \]

are equal in all strata.

4.3.3 Particular Cases of the \( \Delta_g \)-Model-based Allocations

In this subsection, we deduce below the \( \Delta_g \)-Model-based allocations for some particular values of the superpopulation parameter \( g \).

Case I: Model \( \Delta_0 \)

Now by substituting \( g = 0 \) in equation (4.3.3), we have

\[ D_i(0) = 1 \]

Hence \( \Delta_0 \)-MOA is given by
\[ n_i(0) \propto N_i \]

provided

\[ \theta_i(0) = S_i(x) \]

are equal in all strata.

Thus \( \Delta_0\)-MOA deduces to Bowley's Stratum Size Proportional Allocation (BSSPA) if the standard deviation of the auxiliary variable \( x \) are equal in all strata.

Further, we have, for \( g = 0 \)

\[ C_i(x^{g/2}) = \frac{\sigma_i(x^{g/2})}{\left( \overline{x_i^{g/2}} \right)} = 0 \]

and \( \Delta_g\)-MESCVMOA under \( \Delta_0 \) deduces to

\[ n_i(0) \propto \sum_j x_{ij}^{g/2} = N_i \]

provided

\[ \eta_i(0) = \frac{S_i(x)}{\left( \overline{x_i^{g/2}} \right)} = S_i(x) \propto \text{constant.} \]

Thus \( \Delta_0\)-MESCVMOA also deduces to BSSPA and is same as \( \Delta_0\)-MOA.

It is obvious because the condition of Equal SCV(\( x^{g/2} \)) is redundant. \( \Delta_0\)-MESCVMAMOA is the same as \( \Delta_0\)-MESCVMOA which reduces to the \( \Delta_0\)-MOA.
MOA. $\Delta_0$-SDESCVAMOA does not exist just as $\Delta_0$-SDESCVMOA does. Thus the only model based allocation under the model $\Delta_0$ is $\Delta_0$-MOA which reduces to BSSPA under the condition of constant variance of the auxiliary variable. Under the model $\Delta_0$, $S_i^2$ is expected to be proportional to $S_i^2(x)$ if $S_i^2(x)$ in turn is proportional to constant i.e., if they are equal. Since $\{S_i^2\}$ are expected to be proportional to $S_i^2(x)$, therefore $\{S_i^2\}$ are expected to be equal in all strata. Therefore $\Delta_y$-MOA is obtained by replacing $\{S_i\}$ in TNOA by $\{S_i(x)\}$ i.e.,

$$n_i(0) \propto N_i S_i(x)$$

or

$$n_i(0) \propto N_i$$ if $S_i(x)$ are equal.

$$n_i(0) \propto N_i S_i(x)$$ if $S_i(x)$ are equal

$$\Rightarrow n_i(0) \propto N_i$$

Thus if the strata are constructed such that the within stratum variability of the auxiliary variable is similar in all the strata i.e., stratum variances are equal then the MOA based on TNOA reduces to BSSPA. Under the model $\Delta_0$, $\Delta_0$-SDESCVMOA and $\Delta_0$-SDESCVAMOA don’t exist.

Similar to the case of SRSWR, the allocation $\Delta_0$-SDESCVMOA, and its approximation under large size strata $\Delta_0$-SDESCVAMOA, do not exist in case of SRSWOR also.

**Case II: Model $\Delta_1$**

$\Delta_1$-MOA
For \( g = 1 \), \( D_i(1) = \bar{X}_i \) and therefore \( \Delta_1\text{-MOA} \) is given by

\[
n_i(1) \propto \frac{n_i \sqrt{\bar{X}_i}}{\sqrt{N_i \bar{X}_i}}
\]

provided

\[
\theta_i(1) = \frac{S_i(x)}{\sqrt{\bar{X}_i}} = \frac{S_i(x)}{\bar{X}_i} \frac{\bar{X}_i}{\sqrt{\bar{X}_i}} = C_i(x) \sqrt{\bar{X}_i}
\]

are equal in all strata.

If we assume that \( C_i(\sqrt{x}) \) are equal then \( \Delta_1\text{-MSCVMOA} \) is given by

\[
n_i(1) \propto \sum_j X_{ij}^{1/2}
\]

provided

\[
\eta_i(1) = \frac{S_i(x)}{(X_i^{1/2})}
\]

are equal in all strata.

Similarly under the assumption of equal \( SCV(\sqrt{x}) \), \( \Delta_1\text{-SDESCVMOA} \)
is deduced as follows:

$$n_i(1) \propto N_i \sigma_i(\sqrt{x})$$

provided

$$\zeta_i(1) = \frac{S_i(x)}{\sigma_i(\sqrt{x})}$$

are equal in all strata.

Now if the approximation that the term in $1/N_i$ viz., $S_i^2(\sqrt{x})/N_i$ is negligible compared to stratum mean $\bar{X}_i$ i.e., the RNT

$$t_i(1) = \frac{S_i(\sqrt{x})}{N_i \bar{X}_i} \approx 0 \ll 1$$

then $\Delta_1$-MOA may be approximated by the following $\Delta_1$-AMOA:

$$n_i(1) \propto N_i \sqrt{D_i'(1)} = N_i \sqrt{S_i^2(\sqrt{x}) + \left(X_i^{1/2}\right)^2}$$

under the condition that
\[ \theta_i(1) = \frac{S_i(x)}{\sqrt{D_i(1)}} \]

are equal in all strata.

Similarly under the above approximation of large stratum sizes and under the assumption of equal SCV(\(\sqrt{x}\)) the above \(\Delta_1\)-MESCVMOA and \(\Delta_1\)-SDESCVMOA may be approximated by the following allocations:

\(\Delta_1\)-MESCVMOA

\[ n_i(1) \propto \sum_j X_{ij}^{1/2} \]

provided

\[ \eta_i(1) = \frac{S_i(x)}{\left(X_i^{1/2}\right)} \]

are equal.

\(\Delta_1\)-SDESCVMOA

\[ n_i(1) \propto N_i S_i(\sqrt{x}) \]

provided

\[ \zeta_i(1) = \frac{S_i(x)}{S_i \sqrt{x}} \]
are equal.

We note that the allocations in $\Delta_1$-MESCVAMOA is same as that in $\Delta_1$-MESCVMOA except that the assumption and condition are slightly different.

Case III: Model \( \Delta_2 \)

\( \Delta_2 \)-MOA

Substituting \( g = 2 \) in equation (4.3.7), we have

\[
D_i(2) = \frac{1}{N_i} \sum_j X_{ij}^2 = \sigma_i^2(x) + \bar{X}_i^2
\]

\[= (1 - \frac{1}{N_i})S_i^2(x) + \bar{X}_i^2 = S_i^2(x) + (\bar{X}_i^2 - \frac{S_i^2(x)}{N_i}).\]

Now substituting \( D_i(2) \) in equation (4.3.8), we have

\[
E_\delta(S_i^2|X) = (\beta^2 + \sigma^2)S_i^2(x) + \sigma^2 \left( \bar{X}_i^2 \sim \frac{S_i^2(x)}{N_i} \right)
\]

\[= (\beta^2 + \sigma^2)S_i^2(x) + \sigma^2 \bar{X}_i^2 \left( 1 - \frac{C_i^2(x)}{N_i} \right).\]

Thus

\[
E_\delta(S_i^2|X) \propto S_i^2(x)
\]

or

\[
\propto \bar{X}_i^2 \left( 1 - \frac{C_i^2(x)}{N_i} \right)
\]

if

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Hence the $\Delta_2$-MOA deduces to

$$n_i(2) \propto X_i \sqrt{1 - \frac{C_i^2(x)}{N_i}}$$  \hspace{1cm} (4.3.26)$$

provided

$$\eta_i(2) = \frac{C_i'(x)}{\sqrt{1 - \frac{C^2_i(x)}{N_i}}}$$  \hspace{1cm} (4.3.27)$$

are equal in all strata.

**Definition 4.3.11.** The allocation (4.3.26) is proportional to corrected stratum totals (CSTPA) of the auxiliary variable and is termed as $\Delta_g$-Corrected Auxiliary Variable Proportional Allocation ($\Delta_g$-CAVPA) for the strategy (St.SRSWOR: $\hat{Y}_{St.UE}$).

Next, for deducing the $\Delta_g$-ESCVMOAs, the assumption of equal SCV of the generalized auxiliary variable reduces to that of the auxiliary variable. Under this assumption, the two allocations are as follows:

**$\Delta_g$-MESCVMOA.**

If $C_i(x)$, the coefficient of variation of the auxiliary variables are equal, then

$$n_i(2) \propto X_i$$
provided

\[ \eta_h(2) = \frac{S_i(x)}{X_i} = C_i'(x) \]

are equal in all strata.

Thus if the CV(x) is equal for all strata, \( \Delta_2\)-MOA reduces to Rao's auxiliary variable proportional allocation (RAVPA).

\( \Delta_\gamma \)-SDESCVMOA

\[ n_i(2) \propto N_i \sigma_i(x) \]

provided

\[ \zeta_i(2) = \frac{S_i(x)}{\sigma_i(x)} = \sqrt{\frac{N_i}{N_i - 1}} \]

are equal in all strata.

The condition for this allocation is independent of \( x \) and will be automatically satisfied for equal sized strata.

Now \( C_i'^2(x) \) is expected to be negligible compared to the stratum sizes \( N_i \) as the square of the CV(x) would be relatively very small compared to relatively large stratum sizes. Therefore if \( C_i'^2(x)/N_i \) can be ignored, then
the $\Delta_2$-MOA may be approximated by $\Delta_2$-AMOA:

$$n_i(2) \propto X_i$$

provided

$$\eta_i(2) \approx C'_i(x)$$

are equal in all strata.

We note that even if $\{C''_i(x)/N_i\}$ are not negligible, it is sufficient that

$$C''_i(x)/N_i \propto \text{constant}$$

or

$$C'_i(x) \propto \sqrt{N_i}$$

for CAVPA to reduce to RAVPA.

Thus if any one of the following conditions is satisfied then the $\Delta_2$-MOA may be approximated by $\Delta_2$-AMOA:
Further, besides the above conditions, if the coefficient of variation \( C'_i(x) = \frac{S_i(x)}{X_i} \) are equal in all strata, then the allocations viz., \( \Delta_g \)-MESCV.MOA and \( \Delta_g \)-SDESCV.MOA may be approximated by the following two approximate allocations \( \Delta_g \)-MESCVAMOA and \( \Delta_g \)-SDESCVAMOA respectively:

1. \( \Delta_g \)-MESCVAMOA:

\[
n_i(2) \propto X_i
\]

provided

\[
\eta_i(2) = \frac{S_i(x)}{X_i} = C'_i(x)
\]

are all equal in all strata.
Thus for this allocation, assumption and condition are identical. This allocation reduces to RAVPA.

2. $\Delta_g$-SDESCVAMOA:

$$n_i(2) \propto N_i S_i(x)$$

provided

$$\zeta_i(2) = \frac{S_i(x)}{S_i(x)} = 1$$

are equal, which is satisfied. This allocation becomes HAVOA.

The result in the special Case-III of the Model $\Delta_2$ for $g = 2$ is so often the case met with in practice for survey populations that we present in a separate theorem.

**Theorem 4.3.12.** If any one of the following conditions is satisfied:

(i) the stratum sizes are so large that $\frac{N_i - 1}{N_i} \approx 1$

(ii) the Relative Neglected Term (RNT)

$$t_i(g) = \frac{S_i^2(x)/N_i}{\frac{1}{N_i} \sum_j X_{ij}^2} \approx 0 \ll 1$$

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(iii) \( C'_i(x) \) the coefficient of variation of \( x \), is such that

\[
\frac{C'^2_i(x)}{N_i} \approx 0 \ll 1
\]

(iv) \( \frac{C'_i(x)}{N_i} \propto \text{constant i.e., are equal in all strata.} \)

and, if \( C'_i(x) = \frac{S_i(x)}{X_i} \) are equal in all strata then the TNOA for SRSWOR under the model \( \Delta_2 \) reduces to the following allocations:

1. the allocation proportional to stratum total of the auxiliary variable i.e., Rao's Auxiliary Variable Proportional Allocation (RAVPA).

2. the allocation obtained by replacing \( S_i \) by \( S_i(x) \) in TNOA i.e., Hanurav's Auxiliary Variable Optimum Allocation (HAVOA) without any other condition.

Many survey populations follow the model \( \Delta_2 \) with \( g = 2 \) i.e., the model variance proportional to \( x^2 \). For such populations, the Tschuprow-Neyman optimum allocation reduces to RAVPA proportional to stratum total of the auxiliary variable or the HAVOA proportional to \( N_i \) \( S_i(x) \) provided the coefficient of variation of \( X \) are equal in all strata (Rao, 1968) [49]. Raj and Chandhok (1998) [43] compared RAVPA with BSSPA and found that the former is more useful than the latter specially for skewed populations. Similarly HAVOA is also expected to perform better than BSSPA for skewed populations. But between RAVPA and HAVOA, which one will be better than the other and for what kind of populations, is not known.
4.4 Applications and Implications of the Conditions and Assumption

Several questions may be raised now:
1. Whether is the condition /assumption satisfied and to what extent is it satisfied?
2. What is the effect of the extent, to which it is satisfied, on the efficiency of the allocations?
3. By examining the condition/assumption, can we decide about the allocation at the planning stage of the survey?

We take up these questions i.e., one after another, first for the condition in the next section followed by those for assumptions.

4.4.1 Effect of Conditions on Efficiency

Among the conditions \( \theta_i, \eta_i \) and \( \zeta_i \), we explore the solutions of the above questions for \( \theta_i(g) \) only. The same treatment will apply to the other conditions. To answer the first question, we need a measure for departure from equality of \( \{\theta_i(g)\} \). Consider the standard deviation of the \( \theta_i(g) \)-values

\[
\sigma_{\theta(g)} = \sqrt{\frac{1}{k} \sum_{i=1}^{k} \{\theta_i(g) - \bar{\theta}(g)\}^2}
\]

where

\[
\bar{\theta}(g) = \frac{1}{k} \sum_{i=1}^{k} \{\theta_i(g)\}
\]
is the mean of $\theta(g)$ - values, measures the root mean squared deviation from their arithmetic mean. However, compared to the absolute measure, the relative measure, the coefficient of variation of $\theta(g)$-values, $C_\theta(g) = \frac{\sigma_\theta(g)}{\bar{\theta}(g)}$ would be a better measure to assess the level of satisfaction of the condition. If all the $\{\theta_i(g)\}$ are equal, an ideal situation or level of satisfaction of condition, then $C_\theta(g)$ will be zero. Thus the lower value of $C_\theta(g)$ indicate the better level of satisfaction whereas a larger value will reflect the poor level of satisfaction of the condition. Thus the coefficient of variation of $\theta(g)$-values will be a suitable measure to assess the level of satisfaction of the condition.

Next the answer to the second question may be obtained by calculating the coefficient of correlation between $C_\theta(g)$-values and the relative efficiency of an allocation w.r.t. TNOA. The high negative correlation will reflect the better the condition is satisfied the higher will be efficiency. On the otherhand, high positive correlation will point to the lower efficiency for lower $C_\theta(g)$. Poor correlation will show that the allocation is robust w.r.t. the condition.

The level of satisfaction of the conditions $\eta_i(g)$ may be examined exactly in the same way as the condition $\theta_i(g)$.

Finally let us come to the third question. If $g$ is unknown, then MOA may be calculated for different values of $g$. The value of $g$ which yields the lowest $C_\theta(g)$, the efficiency of the allocation is expected to be the highest. Thus a suitable value of $g$ may be chosen by comparing $C_\theta(g)$. If $g$ is known, coefficient of variation of the condition quantities $\theta(g)$, $\eta(g)$ and $\zeta(g)$ may be compared for the corresponding allocations. The allocation with the lowest condition CV is expected to be the most efficient.
If $g$ is not known which is quite often the case in practice, the condition CVs may be calculated for different values of $g$ for each one of the allocations. First the value of $g$ may be identified for each allocation for which the condition is the minimum. Then the minimum CVs for each allocation may be compared to choose the allocation with the lowest condition CV. The allocation with a $g$-value for which condition CV is minimum, is expected to be the most efficient allocation. Thus the condition may be useful for choosing the best allocation.

### 4.4.2 Effect of ESCV-Assumption on the Efficiency

The coefficient of variation of a variable is quite a stable quantity from stratum to stratum. The coefficient of variation would be much more stable for different powers of the variable i.e.,

$$C_1(x) \approx C_1(x^{g/2}) \text{ for } 0 < g \leq 2.$$  

The checking of the assumption of equal SCV ($x^{g/2}$) may be done exactly in the same way as it is done for the condition. The coefficient of variation of assumption \{C($x^{g/2}$)\} values denoted by CV[C($x^{g/2}$)] may be calculated. The lower the CV[C($x^{g/2}$)] the better is the assumption satisfied.

The two allocations derived under the same assumption of equal $C(x^{g/2})$ viz., $\Delta_g$-MESCVMOA and $\Delta_g$-SDESCVMOA are therefore expected to be robust w.r.t the assumption. However still, if we want to study the influence of the assumption on efficiency, the coefficients of correlation may be cal-
culated between the assumption $CV[C(x^{g/2})]$ and the efficiency for the two ESCV allocations. The interpretation of these correlation coefficients would be similar to those for conditions e.g., high negative correlation will indicate that the better the assumption is satisfied the higher will be the efficiency.

4.5 Empirical Illustration

The results derived in section 4.3 on various model-based allocations obtained from TNOA under the superpopulation model for stratified strategy (St.SRSWOR: $\hat{Y}_{St.U,E}$) are illustrated by two populations.

The Population-I (Pop.-I) is given in Table-A1 of Appendix A. It is taken from Murthy (1967) [39] consists of data on number of workers (x) and output (y) for 80 factories in a region. The Population-II (Pop.-II) is given in Table-A2 of Appendix-A which is taken from Cochran (1977) [8]. It consists of 1930 census population y and 1920 census population (x) of 64 large cities. Both the populations are divided into 2 strata by equalizing $\sum_j X_{ij}^g$ for different values of $g$: Stratification type (ST)-A equalizing $N_i (g = 0)$, ST-B equalizing $\sum_j \sqrt{X_{ij}} (g = 0.5)$, ST-C equalizing $X_i (g = 1)$, ST-D equalizing $\sum_j X_{ij}^{1.5} (g = 1.5)$ and ST-E equalizing $\sum_j X_{ij}^2 (g = 2)$. The sample size for Pop.-I is 5 and that for Pop.-II is 4 and thus sampling fraction has been kept at $(1/16)^{th}$ in both the populations so as to avoid any differences in efficiencies on account of different sampling fractions.
4.5.1 \(g\)-Independent Allocations

The variances of the estimates of population total for TNOA and Relative Efficiencies (REs) of the three \(g\)-independent allocations viz., AVOA, AVPA and BSSPA for the five types of STs for the two populations viz., Pop-I and II are presented in tables 4.5.1 (I) and 4.5.1 (II) respectively given below:

Table - 4.5.1(I)

REs of STs w.r.t. ST-A and
REs of Allocations w.r.t. TNOA for Pop.-I

<table>
<thead>
<tr>
<th>ST No.</th>
<th>Size</th>
<th>(n_i)</th>
<th>(/n)</th>
<th>(\mu_{/}^{\text{V}(Y)})</th>
<th>TNOA</th>
<th>AVOA</th>
<th>AVPA</th>
<th>BSSPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>40</td>
<td>2.45</td>
<td>1073</td>
<td>0.69</td>
<td>0.4750</td>
<td>1.08</td>
<td>0.6820</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>40</td>
<td>2.55</td>
<td>1</td>
<td>4.31</td>
<td>0.9995</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>56</td>
<td>3.96</td>
<td>1321</td>
<td>2.31</td>
<td>0.6834</td>
<td>1.60</td>
<td>0.4771</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>24</td>
<td>1.04</td>
<td>0.81</td>
<td>2.68</td>
<td>0.9587</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>65</td>
<td>4.52</td>
<td>1976</td>
<td>3.86</td>
<td>0.9054</td>
<td>2.49</td>
<td>0.5861</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>15</td>
<td>0.48</td>
<td>0.54</td>
<td>1.13</td>
<td>0.9450</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>70</td>
<td>4.92</td>
<td>2222</td>
<td>4.77</td>
<td>0.9802</td>
<td>3.86</td>
<td>0.7864</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>10</td>
<td>0.08</td>
<td>0.48</td>
<td>0.23</td>
<td>0.8970</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>73</td>
<td>4.87</td>
<td>2693</td>
<td>4.75</td>
<td>0.9876</td>
<td>3.60</td>
<td>0.7449</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>7</td>
<td>0.13</td>
<td>0.39</td>
<td>0.25</td>
<td>0.9527</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Variance in units of \(10^6\).
**Table - 4.5.1(II)**

REs of STs w.r.t. ST-A and

REs of Allocations w.r.t. TNOA for Pop.-II

<table>
<thead>
<tr>
<th>St. No.</th>
<th>St. Size</th>
<th>TNOA $V(\hat{y})^*$</th>
<th>AVOA</th>
<th>AVPA</th>
<th>BSSPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>ST 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>32</td>
<td>0.96</td>
<td>1950</td>
<td>0.41</td>
<td>.9555</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
<td>3.33</td>
<td>1</td>
<td>3.59</td>
<td>1.02</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>42</td>
<td>1.33</td>
<td>1457</td>
<td>1.23</td>
<td>.9972</td>
</tr>
<tr>
<td>2</td>
<td>22</td>
<td>2.67</td>
<td>1.33</td>
<td>2.76</td>
<td>1.54</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>49</td>
<td>2.10</td>
<td>1203</td>
<td>2.24</td>
<td>.9952</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>1.89</td>
<td>1.62</td>
<td>1.76</td>
<td>1.97</td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>54</td>
<td>2.88</td>
<td>1379</td>
<td>3.08</td>
<td>.9840</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>1.12</td>
<td>1.41</td>
<td>0.91</td>
<td>1.50</td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>58</td>
<td>2.88</td>
<td>1852</td>
<td>3.67</td>
<td>.9653</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>1.12</td>
<td>1.05</td>
<td>0.33</td>
<td>1.04</td>
</tr>
</tbody>
</table>

* Variance in units of $10^4$.

All the allocations derived in subsections 4.3.1 to 4.3.3 for SRSWOR case are calculated for five values of $g$ viz., $g=0, 0.5, 1, 1.5$ and 2. For the 5 STs along with their condition CV, assumption CV and approximations i.e., RNT, $t_i(g)$ and are given in tables 4.5.2(I-A-E) and 4.5.2(IIA-2E) of Appendix-B for the population I and II respectively. Table 4.5.1(I) shows that the lowest ST-A equalizing stratum sizes $N_i$ is the best with least variance under optimum allocation for the Pop.-I. The STs equalizing higher powers are becoming less and less efficient and ST-E is the least efficient. For Pop.-II from table 4.5.1(II), we see that the intermediate ST-C equalizing stratum totals of the
auxiliary variable is the best and the efficiency decreases on both sides i.e., low as well as high stratifications become lesser efficient, making ST-A and E the least efficient ones for this population.

4.5.2 $g$-Dependent Allocations

The REs of all the allocations derived in subsections 4.3.1 to 4.3.3 for SRSWOR case are calculated for five values of $g$ viz., $g = 0, 0.5, 1, 1.5$ and $2$ for the five STs alongwith their conditions CV, assumption CV and approximations {i.e., RNT, $t_4(g)$} and are given in tables 4.5.2(IA-IE) and 4.5.2(IIA-IIIE) for the population I and II respectively. The following interpretations are given for illustration of the results based on the tables 4.5.2(IA-IE) and 4.5.2(IIA-IIIE).

(a) Behaviour of Efficiencies of Allocations

The REs of most of the allocations in Pop.-I are decreasing with increasing $g$ except those based on $S_i(x^{g/2})$ viz., $\Delta_g$-SDESCVMOA or SDESCVAMOA in ST-E only for which the efficiency is increasing with increasing $g$. In case of Pop.-II, the REs of the allocations are increasing with increasing $g$ except for some allocations for higher values of $g$ in STs A, D and E. The exceptions are the allocations based on $S_i(x^{g/2})$ viz., $\Delta_g$-SDESCVMOA or $\Delta_g$-SDESCVAMOA for $g = 0.5$ and $1$ in ST-A., $\Delta_g$-MOA and $\Delta_g$-MESCVMOA based on $\sum_j X_{ij}^{g/2}$ and their approximations for the ST-D and E only. The allocations for this population are attaining their highest RE for $g = 2$ whereas
for Pop.-I the allocations are the most efficient for the lowest i.e., $g = 0$. It appears that Pop.-I has value of $g$ very close to zero where as for the Pop.-II, the most suitable value of the model parameter $g$ is 2.

(b) Effect of Approximation

The relative neglected terms (RNTs) $\{t_i(g)\}$ are negligible relative to unity as the highest values for any stratum and for any ST are 0.0059 and 0.0061 for the two populations. Since the RNT is negligible for these small populations than it can safely be considered negligible in case of actual survey populations in which stratum sizes would be large.

In case of Pop.-I, approximate allocations are more efficient then those without approximation for all STs and all $g$ barring a few exceptions. For instance MOA is superior to AMOA only for higher $g$ in ST-A and SDESCVAMOA is more efficient than SDESCVAMOA in all STs except ST-A. for Pop.-II approximate allocations are better than or same as the corresponding without approximate ones for all $g$'s in all STs with only exception for MOA with $g = 2$ in ST-D for which MOA is marginally superior to AMOA.

We have seen that the RNT is expected to be negligible and its effect on the efficiency would also be negligible. Therefore approximate allocations may be used without much loss of efficiency in practice.
(c) **Effect of Condition on RE of Allocations**

In Pop.-I, the CVs of the condition measure $\theta_i(g)$ are decreasing and the RE of the allocations are increasing for all STs for all allocations in ST-A and B except for allocations $\Delta_g$-SDESCVMOA and $\Delta_g$-SDESCVAMOA in all the STs. This indicates that the better the condition is satisfied the higher is the efficiency of the allocations. Similarly for Pop.-II, the RE of an allocation depends on the level of satisfaction of the condition for all STs and all allocations except MOA, $\Delta_g$-MESCVMOA and their approximates for higher ST-D and $\Delta_g$-SDESCVMOA and $\Delta_g$-SDESCVAMOA in ST-A only. This fact is immediately clear from the tables 4.5.3(I and II) giving correlation coefficient between the CV($\theta_i$) and the RE of the allocations, which are very high negative values for both the populations except lower STs A and B in case of Pop.-I with a low $g$ value.

![Table - 4.5.3(I)](image)

**Table - 4.5.3(I)**

*Correlation Coefficient between Condition CV and Allocation RE for Pop.-I*

<table>
<thead>
<tr>
<th>Allocation</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOA</td>
<td>0.9897</td>
<td>0.6096</td>
<td>-0.9531</td>
<td>-0.9965</td>
<td>-0.9786</td>
</tr>
<tr>
<td>MESCVMOA</td>
<td>0.9881</td>
<td>0.4704</td>
<td>-0.9691</td>
<td>-0.9911</td>
<td>-0.9684</td>
</tr>
<tr>
<td>SDESCVMOA</td>
<td>0.9994</td>
<td>0.9964</td>
<td>0.9664</td>
<td>0.9961</td>
<td>0.6028</td>
</tr>
<tr>
<td>AMOA</td>
<td>0.9898</td>
<td>0.6141</td>
<td>-0.9530</td>
<td>-0.9966</td>
<td>-0.9787</td>
</tr>
<tr>
<td>MESCVAAMOA</td>
<td>0.9881</td>
<td>0.4704</td>
<td>-0.9691</td>
<td>-0.9911</td>
<td>-0.9684</td>
</tr>
<tr>
<td>SDESCVAAMOA</td>
<td>0.9994</td>
<td>0.9966</td>
<td>0.9697</td>
<td>0.9964</td>
<td>0.7837</td>
</tr>
</tbody>
</table>
Table - 4.5.3(II)

Correlation Coefficient between Condition CV and Allocation RE for Pop.-II

<table>
<thead>
<tr>
<th>Allocation</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOA</td>
<td>-0.9750</td>
<td>-0.9772</td>
<td>-0.9834</td>
<td>-0.3170</td>
<td>-0.7758</td>
</tr>
<tr>
<td>MESCVMOA</td>
<td>-0.9801</td>
<td>-0.9794</td>
<td>-0.9790</td>
<td>-0.3359</td>
<td>-0.7398</td>
</tr>
<tr>
<td>SDESCVMOA</td>
<td>0.7051</td>
<td>-0.9282</td>
<td>-0.9812</td>
<td>-0.9890</td>
<td>-0.9930</td>
</tr>
<tr>
<td>AMOA</td>
<td>-0.9749</td>
<td>-0.9770</td>
<td>-0.9831</td>
<td>-0.3170</td>
<td>-0.6755</td>
</tr>
<tr>
<td>MESCVAMOA</td>
<td>-0.9801</td>
<td>-0.9794</td>
<td>-0.9790</td>
<td>-0.3359</td>
<td>-0.7398</td>
</tr>
<tr>
<td>SDESCVAMOA</td>
<td>0.7051</td>
<td>-0.9233</td>
<td>-0.9792</td>
<td>-0.9873</td>
<td>-0.9920</td>
</tr>
</tbody>
</table>

The better the condition is satisfied i.e., the lower the condition CV for any allocation and for any $g$ compared to other allocations or other values of $g$, the higher is the efficiency of the allocation. This is true for all values of $g$ and all allocations in all the STs. Thus we compare the condition CV for different values of $g$ and for different allocations and select a particular allocation with suitable $g$ for which the condition CV is the lowest so as to expect the highest efficiency.

Therefore it appears that except for lower STs like A and B for populations with a very low values of $g$ like Pop.-I, the condition CV could serve as a good indicator to choose the suitable allocation with right value of $g$ at the planning stage of the survey in practice specially when $g$ is not known, which is usually the case met with in practice.
(d) Role of ESCV-Assumption

We observe that either of the two ESCV-allocations deduced from MOA under the assumption of equal SCV($x^{q/2}$), viz.,

$$\Delta_g - \text{MESCVMOA} \propto \sum_j X_{ij}^{q/2}$$

and

$$\Delta_g - \text{SDESCVMOA} \propto N_i S_i(x^{q/2}),$$

is better than the parent allocation (MOA) itself for all STs in both the populations. For the Pop.-I, the CV of the assumption CV($x^{q/2}$) is increasing for all STs except A and for $g \geq 1.5$ for $\Delta_g$-MESCVMOA and for $g \leq 0.5$ for $\Delta_g$-SDESCVMOA, i.e., the assumption is deteriorating for higher values of $g$, and the efficiency is also declining. Similar is the case with the approximate allocations. In case of Pop.-II also, except for $\Delta_g$-MESCVMOA for ST-A and for ST-D for $g \leq 1$ and E for $g \leq 0.5$, if the assumption CV is improving, the efficiency of the allocation is also increasing. However, the allocation $\Delta_g$-SDESCVMOA is behaving differently except for ST-B in this population. Approximate allocations exactly behave in similar manner.
Table - 4.5.4(I)

Correlation Coefficient between Assumption CV and Allocation RE for Pop.-I

<table>
<thead>
<tr>
<th>Allocation</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
</tr>
</thead>
<tbody>
<tr>
<td>MESCVMOA</td>
<td>0.9904</td>
<td>-0.9999</td>
<td>-0.9802</td>
<td>-0.9750</td>
<td>-0.6293</td>
</tr>
<tr>
<td>SDESCVMOA</td>
<td>0.9631</td>
<td>-0.9928</td>
<td>-0.8982</td>
<td>-0.9555</td>
<td>0.2066</td>
</tr>
<tr>
<td>MESCVAMOA</td>
<td>0.9904</td>
<td>-0.9999</td>
<td>-0.9802</td>
<td>-0.9750</td>
<td>-0.6293</td>
</tr>
<tr>
<td>SDESCVAMOA</td>
<td>0.9631</td>
<td>-0.9932</td>
<td>-0.9049</td>
<td>-0.9567</td>
<td>-0.0499</td>
</tr>
</tbody>
</table>

Table - 4.5.4(II)

Correlation Coefficient between Assumption CV and Allocation RE for Pop.-II

<table>
<thead>
<tr>
<th>Allocation</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
</tr>
</thead>
<tbody>
<tr>
<td>MESCVMOA</td>
<td>0.9991</td>
<td>-0.9793</td>
<td>0.9496</td>
<td>0.0205</td>
<td>-0.9389</td>
</tr>
<tr>
<td>SDESCVMOA</td>
<td>-0.5851</td>
<td>-0.9142</td>
<td>0.9869</td>
<td>0.9939</td>
<td>0.9980</td>
</tr>
<tr>
<td>MESCVAMOA</td>
<td>0.9991</td>
<td>-0.9793</td>
<td>0.9495</td>
<td>0.0203</td>
<td>-0.9391</td>
</tr>
<tr>
<td>SDESCVAMOA</td>
<td>-0.5851</td>
<td>-0.9086</td>
<td>0.9850</td>
<td>0.9923</td>
<td>0.9971</td>
</tr>
</tbody>
</table>

The interdependence of the assumption and the efficiency of allocations under this ESCV(\(x^2/\alpha\)) assumption can also be examined by the correlation coefficient between them from tables 4.5.4(I and II). The high negative correlation coefficient between them for the Pop.-I for all STs except high positive correlation coefficient for ST-A and low correlation coefficient for \(\Delta_g\)-SDESCVMOA for ST-E confirm the relation between them. For Pop.-II however, \(\Delta_g\)-SDESCVMOA for ST-A, \(\Delta_g\)-MESCVMOA for ST-E and B have high negative correlation. By and large, the level of satisfaction of the assumption is a good indicator of the efficiency. Thus we can compare the
assumption for different $g$ for the two allocations, for whichever value of $g$ the assumption is better satisfied, will have higher efficiency. Thus it will help in choosing the optimum value of $g$ to give the highest efficiency for these two allocations.

(e) Overall Comparison of Allocations

All the model-based allocations have been arranged in decending order of efficiency for maximum range of $g$ values for each of the STs for both the populations. Overall ranking of the allocations has also been done according to the broad groups of STs. The ranking of allocations are given in the following tables 4.5.5(I) and (II).

Table - 4.5.5(I)

ST-wise Ranking of Allocations
w.r.t. RE for Pop.- (I)

<table>
<thead>
<tr>
<th>Rank</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>ME/MEA</td>
<td>SDE</td>
<td>SDE</td>
<td>SDE</td>
<td>SDE(1-2)</td>
<td>SDE</td>
</tr>
<tr>
<td>2</td>
<td>ME/MEA</td>
<td>SDEA</td>
<td>SDEA</td>
<td>SDEA</td>
<td>SDEA</td>
<td>SDEA</td>
</tr>
<tr>
<td>3</td>
<td>M</td>
<td>AM</td>
<td>AV</td>
<td>AV</td>
<td>AV</td>
<td>AV</td>
</tr>
<tr>
<td>4</td>
<td>AM M</td>
<td>AM</td>
<td>AM</td>
<td>AM</td>
<td>AM</td>
<td>AM</td>
</tr>
<tr>
<td>5</td>
<td>SDE/SDEA</td>
<td>ME/MEA</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>M</td>
</tr>
<tr>
<td>6</td>
<td>SDE/SDEA</td>
<td>ME/MEA</td>
<td>ME/MEA</td>
<td>ME/MEA</td>
<td>ME/MEA</td>
<td>ME/MEA</td>
</tr>
<tr>
<td>7</td>
<td>AV</td>
<td>AV</td>
<td>ME/MEA</td>
<td>ME/MEA</td>
<td>ME/MEA</td>
<td>ME/MEA</td>
</tr>
</tbody>
</table>
Table - 4.5.5(II)

ST-wise Ranking of Allocations w.r.t. RE for Pop.-(II)

<table>
<thead>
<tr>
<th>Rank</th>
<th>ST-A</th>
<th>ST-B</th>
<th>ST-C</th>
<th>ST-D</th>
<th>ST-E</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>SDE/ SDEA</td>
<td>AV</td>
<td>AV</td>
<td>AV</td>
<td>M/AM</td>
<td>AV (1-1.5)</td>
</tr>
<tr>
<td>2</td>
<td>SDE/ SDEA</td>
<td>SDEA</td>
<td>ME/MEA</td>
<td>ME/MEA</td>
<td>M/AM</td>
<td>SDEA (A,B)</td>
</tr>
<tr>
<td>3</td>
<td>AV</td>
<td>SDE</td>
<td>ME/MEA</td>
<td>ME/MEA</td>
<td>ME/MEA</td>
<td>ME/MEA (C,D)</td>
</tr>
<tr>
<td>4</td>
<td>AM</td>
<td>AM</td>
<td>AM</td>
<td>AM</td>
<td>ME/MEA</td>
<td>AM (1-1.5)</td>
</tr>
<tr>
<td>5</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>AV</td>
<td>M</td>
</tr>
<tr>
<td>6</td>
<td>ME/ MEA</td>
<td>ME/MEA</td>
<td>SDE/SDEA</td>
<td>SDEA</td>
<td>SDEA</td>
<td>ME/MEA (A,B)</td>
</tr>
<tr>
<td>7</td>
<td>ME/ MEA</td>
<td>ME/MEA</td>
<td>SDE/SDEA</td>
<td>SDEA</td>
<td>SDE</td>
<td>ME/MEA (A,B)</td>
</tr>
</tbody>
</table>

In the Pop.-I, the allocations based on $S_i(x_{\theta/2}^2)$ viz., $\Delta_g$-SDESCVMOA and $\Delta_g$-SDESCVAMOA are the best followed by MOA and AMOA in the allocations based on $\sum_{j} X_{ij}^{g/2}$ viz., $\Delta_g$-MESCVMOA and $\Delta_g$-MESCVAMOA are the worst for all the STs except ST-A for which the order of allocation in respect of efficiency is reversed i.e., $\Delta_g$-MESCVMOA and $\Delta_g$-MESCVAMOA become the best. The MOA and its approximate AMOA appear in the mid-
dle of all the allocations in respect of efficiency i.e., they are better than
the worst between mean - based (Δg-MESCVMOA and Δg-MESCVAMOA)
and standard deviation based (Δg-SDESCVMOA and Δg-SDESCVAMOA)
allocations. Thus for populations with low g like the Pop.-I, mean-based-
allocation may be used for the lowest ST-A and SD-based allocation should
be preferable for all other types of STs. For the Pop.-II with very high value
of g close to 2, the SD-based allocations viz., (Δg-SDESCVMOA and Δg-
SDESCVAMOA) are the most efficient followed by MOA and AMOA and
the Mean-based viz., (Δg-MESCVMOA) and (Δg-MESCVAMOA) are the
worst for the lower STs A and B. But for the middle and higher STs C, D and
also the low value of g for ST-E, as in case of Pop.-I the order of allocation is
reversed, the mean-based allocations are the most efficient followed by MOA
and AMOA and (Δg-SDESCVMOA and Δg-SDESCVAMOA) respectively.
In case of ST-E, the allocations are very sensitive for different values of g,
the ranking in terms of efficiency for g = 0.5 is the same as that for higher
STs C and D, whereas for g = 2, it is similar to that for low STs A and B
and that for middle value of g viz., 1 ≤ g ≤1.5, the behaviour of allocation
is altogether different.

The nature of the two populations is different and hence the behaviour
of allocations are opposite in nature, the allocations which are the best for
higher STs, for Pop.-I, are the best for the lower STs of Pop.-II.
4.5.3 Conclusion

We have seen that the approximation of neglecting the term in \( \frac{1}{N_i} \), as the stratum sizes are usually large in survey population in practice, does not have any substantial effect on the efficiency of an allocation. Thus we discard the allocations \( \Delta_g \)-Model Optimum Allocation (\( \Delta_g \)-MOA), \( \Delta_g \)-Mean-based Equal Stratum Coefficients of Variation Model Optimum Allocation (\( \Delta_g \)MESCVMOA) and \( \Delta_g \)-Standard Deviation-based Equal Stratum Coefficients of Variation Model Optimum Allocation (\( \Delta_g \)-SDESCVMOA) and we need consider only the approximate allocations for final analysis. We have also seen that the allocations under the ESCV-assumption viz., either \( \Delta_g \)-MESCVMOA proportional to \( \sum_j x_{ij}^{g/2} \) or \( \Delta_g \)-SDESCVMOA proportional to \( N_i S_i(x^{g/2}) \) is more efficient than the \( \Delta_g \)-Approximate MOA (\( \Delta_g \)-AMOA) from which they were derived for all the STs for both the populations considered for illustration.

The allocation \( \Delta_g \)-MESCVMOA or Generalized Auxiliary Variable Proportional Allocation (GAVPA) is better than the other viz., Generalized Auxiliary Variable Optimum Allocation (GAVOA) for lowest ST-A for the population with low \( g \) and for all high STs C, D and the highest ST-E with low \( g = 0.5 \) for populations with \( g \) close to 2. On the other hand, GAVOA would be more suitable for all higher STs for populations with a very low \( g \) and for low STs A, B and also highest ST-E with highest \( g = 2 \). Since the survey populations are large, the approximation of large stratum sizes may be used. Even if strata are not large but the number of strata is more due to heterogeneity of the population, the effect of approximation may not be
substantial. Therefore approximate allocation may be used in practice. Also assumption based allocations are better than MOA, so only

\[ \Delta_g\text{-MESCVMOA} = \sum_{ij} \frac{X_{ij}^g}{2} \]

and \( \Delta_g\text{-SDESCVAMOA} \propto N_i \sigma_i(x^{g/2}) \) are left for making choice. The choice may be made by comparing the condition CV of the two, the lower the CV, the higher efficiency may be expected.