Chapter 3

Inventory models for trended demand under two-level trade credits

PP. 68-84
3.0 Introduction

In chapter 2, we studied that, to attract the buyers, the vendor uses promotional tool viz. permissible delay period when the buyer’s order quantity is more than pre-specified quantity. In this chapter, we have studied two inventory systems for trended demand. The demand is assumed to be price-sensitive and time dependent. Here, the supplier offers a fixed credit period to the retailer and the retailer also offers credit period to the customers which is more practical scenario in the market. Our goal is to maximize the profit of the retailer.

Two models are formulated under the following sections (viz.):

Model 3.1 Optimal pricing and ordering policies for inventory system with two-level trade credits under price-sensitive trended demand

Model 3.2 Optimal pricing and ordering policies for deteriorating items with two-level trade credits under price-sensitive trended demand

3.1 Optimal Pricing and Ordering Policies for Inventory System with Two-Level Trade Credits Under Price-Sensitive Trended Demand

In the traditional model, it is assumed that the retailer should pay for items as soon as the items are received. To offer credit period is considered to be an effective promotional tool for increasing profit for both the players. In the age of competition, by offering credit period without interest, sales are promoted. In this section, we studied an inventory system with trended demand under two-level trade credit financing. Supplier offers credit period $M$ to retailer and retailer gives credit period $N$ to his customers. The goal is to maximize the total profit per unit time of the retailer with respect to pricing and ordering.
3.1.1 Assumptions and Notations

The proposed problem is formulated using following assumptions and notations.

3.1.1.1 Assumptions

The following assumptions are used in this chapter:

1. The demand rate \( R(s, t) \) is function of time and the retail price of the retailer, (say) \( R(s, t) = a(1 + bt)s^{-\eta} \) (3.1.1)

   where \( a > 0, 0 < b < 1 \) and \( \eta > 1 \).

2. The offer of credit period \( N \) to the customers by the retailer results inflow of revenue in \([N, T+N]\) for the retailer. This assumption is also taken by Teng and Goyal (2007).

3. When \( M \leq T+N \), the account is to be settled at \( M \) and for the unsold items, the retailer would pay interest during \([M, T+N]\) at the rate \( I_e \).

   When \( M > T+N \), the retailer will settle the account at \( M \) and does not pay any interest charges during the cycle.

4. The retailer generates revenue by selling items and earns interest during \([N, M]\) with rate \( I_e \).

3.1.1.2 Notations

\( R(s, t) \) : Price-sensitive linear demand rate. The functional form is \( R(s,t) = a (1+bt) s^{-\eta} \) where \( a > 0 \) is known and constant scale demand; \( 0 < b < 1 \) is the rate at which demand is increasing linearly with time; \( s \) is retail price and \( \eta > 1 \) is price mark-up
\[ M \]: Credit period offered by the supplier to the retailer (in years)

\[ N \]: Credit period offered by the retailer to the customer (in years)

\[ I_e \]: Interest earned / $ per year

\[ I_c \]: Interest charged / $ for unsold items per year by the supplier;

Note: \( I_c > I_e \)

\[ \pi(s, T) \]: Retailer’s total profit per unit time (in $)

### 3.1.2 Mathematical Model

The inventory level at any instant of time \( t \) is governed by the differential equation

\[
\frac{dI(t)}{dt} = -R(s, t); \quad 0 \leq t \leq T
\]

(3.1.2)

with the boundary condition \( I(T) = 0 \). The solution of equation (3.1.2) is

\[
I(t) = as^{-\eta} \left[ T - t + \frac{b}{2} (T^2 - t^2) \right]
\]

(3.1.3)

Using \( I(0) = Q \), the retailer’s purchase quantity per cycle is

\[
Q = I(0) = as^{-\eta}T \left[ 1 + \frac{bT}{2} \right]
\]

(3.1.4)

The sales revenue is

\[
SR = \int_{0}^{T} R(s,t) \, dt = s^{-\eta+1}aT \left[ 1 + \frac{bT}{2} \right]
\]

The ordering cost is

\[ OC = A \]
The purchase cost of $Q$-units is

$$PC = CQ = C a s^{-\eta} T \left[ 1 + \frac{bT}{2} \right]$$

The holding cost is

$$HC = h a s^{-\eta} T^2 \left[ \frac{1}{2} + \frac{bT}{3} \right]$$

For interest charges and earned, the following two cases are discussed depending on the duration of $M$ and $N$.

(A) Suppose $M \geq N$

Case 1 $M \geq T + N$

Since $M \geq T + N$, the retailer has no unsold items in the inventory system, so the interest charges in the cycle is zero i.e. $IC_i = 0$. The retailer generates revenue from the beginning of the cycle and settles the account at time $N$. So the retailer can earn interest at the rate $I_e$ per dollar per year starting from $N$ through $M$.

Therefore, the interest earned per cycle is

$$IE_1 = s I_e \left[ \int_0^T \int_0^t R(s,u) du dt + \int_0^T R(s,t) dt \right]$$

$$= a s^{-\eta+1} I_e T \left[ (M - N) \left( 1 + \frac{bT}{2} \right) - \frac{T}{2} - \frac{bT^2}{3} \right]$$

Case 2 $M \leq T + N$

In this case, the retailer does not have sufficient fund to settle the account at $M$, because customer will be paying at time $T + N$. So the retailer will have to pay interest charges during $[M, T + N]$ for unsold items at an interest rate $I_c$ per dollar per year. Hence, the interest charges to be paid in each cycle is
Here, the retailer earns interest on the generated revenue during \([N, M]\), which is given by

\[
IE_2 = s I_e \int \int_N^M R(s, u - N)du \, dt = s I_e \int_0^{M-N} R(s, u)du \\
= s^{-\eta+1} a I_e \left[ T^2 \left( \frac{bT}{6} + \frac{1}{2} \right) + \frac{b}{2} \left( M^2 - N^2 \right) + M - N - bMN \right]
\]

Hence, the average profit per unit time for retailer is

\[
\pi(s, T) = \begin{cases} 
\pi_1(s, T), & 0 \leq T \leq M - N \\
\pi_2(s, T), & T \geq M - N 
\end{cases} 
\quad \text{(3.1.5)}
\]

where \(\pi_1(s, T) = \frac{1}{T} \{ SR - PC - OC - HC - IC_1 + IE_1 \} \) \quad \text{(3.1.6)}

and \(\pi_2(s, T) = \frac{1}{T} \{ SR - PC - OC - HC - IC_2 + IE_2 \} \) \quad \text{(3.1.7)}

**B) Suppose** \(M \leq N\)

Here, the retailer does not earn any interest \(i.e. \ IE_3 = 0 \).

The interest is to be paid for all the items. Hence, the interest charged per cycle is

\[
IC_3 = C I_c \left( N - M \right) Q + \int_N^{T+N} I(t - N)dt
\]

The average profit per unit time is

\[
\pi_3(s, T) = \frac{1}{T} \{ SR - PC - OC - HC - IC_3 + IE_3 \} 
\quad \text{(3.1.8)}
\]
The objective is to maximize the average profit per unit time \( \pi_i(s, T) \), \( i = 1, 2, 3 \) with respect to retail price and cycle time. The highly non-linearity of the objective functions in equations (3.1.6-3.1.8) renders to obtain the closed form solution. We analyze the model with parametric data in the next section.

### 3.1.2.1 Computational Algorithm

Differentiate \( \pi_i(s, T) \), \( i = 1, 2, 3 \) with respect to \( s \) and \( T \) and equate with zero.

Now follow the steps given below.

Step 1: Assign values to all inventory parameters.

Step 2: For \( M \geq N \), solve simultaneously \( \frac{\partial \pi_1}{\partial s} = 0 \) and \( \frac{\partial \pi_1}{\partial T} = 0 \) and also

\[
\frac{\partial \pi_2}{\partial s} = 0 \quad \text{and} \quad \frac{\partial \pi_2}{\partial T} = 0
\]

If \( M \geq T + N \) then compute \( \pi_1 \) from equation (3.1.6) else \( \pi_2 \) from equation (3.1.7). After having \((s, T)\), the retailers purchase quantity can be obtained using equation (3.1.4).

Step 3: For \( M < N \), solve \( \frac{\partial \pi_3}{\partial s} = 0 \) and \( \frac{\partial \pi_3}{\partial T} = 0 \) simultaneously and find the average profit per unit time \( \pi_3 \) from equation (3.1.8) and purchase quantity using equation (3.1.4).

### 3.1.2.2 Numerical Examples

We study two numerical examples to demonstrate the models for \( M \geq N \) and \( M < N \) respectively.
Example 3.1.1: Consider, $A = 50$ per order, $C = 6$ per unit, $a = 10^4$ units, $h = 2.5$ per unit/year, $b = 10\%$, $\eta = 1.2$, $I_e = 10\% / $/year, $I_c = 15\% / $/year, $M = 0.8$ year and $N = 0.6$ year.

In this case, $\pi_1$ is $3,963.43 at (T = T_1 = 0.492, s = s_1 = 39.91)$

while $\pi_2$ is $4,294.15 at (T = T_2 = 0.455, s = s_2 = 37.36)$.

Therefore, the maximum profit for the retailer is for purchasing units $Q^* = 60.43$ units during cycle is $\pi^* ( = \pi_2^*) = 4,294.15$. The concavity of the average total profit per unit time is shown in figure 3.1.1 w. r. t. retail price $s$ for obtained $T = 0.455$ years. 3-D plot of the average total profit per unit time shows concavity w. r. t. selling price and replenishment time in figure 3.1.2.

![Figure 3.1.1 Concavity of total profit w. r. t. s for $M < T + N$ for obtained $T$](image1)

![Figure 3.1.2 Concavity of total profit for cycle time $T$ and retail price $s$](image2)
Example 3.1.2: Taking, \( A = 40 \) per order, \( C = 3 \) per unit, \( h = 1 \) unit/year, \( a = 10^4 \) units, \( b = 1\% \), \( \eta = 2.2 \), \( I_e = 15\% / \$ / \text{year} \), \( I_c = 10\% / \$ / \text{year} \), \( M = 0.6 \text{ year} \) and \( N = 0.8 \text{ year} \).

We have \( M < N \). Then from step 3 of the procedure above, we get \( \pi_3 \) is \$418.72 at \( (T = T_3 = 0.565, s = s_3 = 6.28) \).

Now we are interested to study how variations in inventory parameters affect decision variables and profit function. For that we perform sensitivity analysis. We vary the parameters given in example 3.1.1 by -40%, -20%, 20% and 40%. The effects of variations in retail price, cycle time, order quantity and profit are shown in figures 3.1.3 - 3.1.6. The following observations are derived for the decision maker.

Managerial Decisions

![Sensitivity analysis for retail price](image)

Figure 3.1.3 Sensitivity analysis for retail price (\( s \))

It is observed from figure 3.1.3 that selling price decreases drastically w. r. t. price mark up \( \eta \) and increases significantly w. r. t purchase cost \( C \), scale
demand $a$, the credit period $M$ offered by the supplier to the retailer and interest earned by the retailer on the generated revenue. The remaining inventory parameters have negligible impact on retail price. This suggests that by offering credit period retailer can increase demand.

![Figure 3.1.4 Sensitivity analysis for cycle time ($T$)](image)

It is observed from figure 3.1.4 that cycle time is decreasing significantly for ordering cost, purchase cost, delay period $N$ offered by the retailer to the customer. It increases sharply for scale demand $a$, price mark-up, the delay period $M$ offered by the supplier to the retailer, interest earned. This suggests that retailer should put order frequently when the delay period $M$ is offered by the supplier to the retailer.
Optimal Inventory Policies for Different Demand Structures

Figure 3.1.5 Sensitivity analysis for purchase quantity \((Q)\)

It is observed from figure 3.1.5 that the purchase quantity increases when delay period, scale demand \(a\), price mark-up \(\eta\) and interest earned \(I_e\) increases while decreases when purchase cost \(C\), ordering cost \(A\) and \(N\) decreases. This suggests that by purchasing more quantity the retailer get more interest and delay period to settle the account.

Figure 3.1.6 Sensitivity analysis for total average profit per unit time \((\pi)\)
It is observed from figure 3.1.6 that the average total profit per unit time increases sharply with increase in $a$, and increases moderately with increase in $b$, $I_e$ and credit period offered by the supplier to the retailer $M$ and decreases significantly w. r. t. increase in $A, C, N, \eta$. This suggests that the retailer can save in ordering cost by ordering more items and giving smaller delay period to the customers.

**3.2 Optimal Pricing and Ordering Policies for Deteriorating Items with Two-Level Trade Credits Under Price-Sensitive Trended Demand**

In this section, we develop inventory system for price-sensitive time dependent demand with two-level trade credits for deteriorating items. The units in inventory system are subject to deterioration at a constant rate. Retailer gets credit period $M$ from supplier and passes credit period $N$ to his customers. The aim is to make the total profit maximum of the retailer with respect to pricing and ordering per unit time.

**3.2.1 Assumptions and Notations**

The proposed model is developed using assumptions and notations defined in section 3.1.1 with one additional constraint.

**3.2.1.1 Assumptions**

The following assumptions are used in this chapter:

1. The units in inventory are deteriorating at a constant rate `$\theta$`, $0 < \theta < 1$. 
3.2.1.2 Notations

$\theta$ : Rate of deterioration of units in inventory model; $0 < \theta < 1$

3.2.2 Mathematical Model

The inventory level at any instant of time $t$ is governed by the differential equation

$$\frac{dI(t)}{dt} = -R(s,t) - \theta I(t), \ 0 \leq t \leq T$$  \hspace{1cm} (3.2.1)

with boundary condition $I(T) = 0$. The solution of differential equation (3.2.1) is

$$I(t) = a s^{-\eta} \left[ \frac{(1+bT)e^{\theta(T-t)}}{\theta} - 1 - bt + b \left( 1 - e^{\theta(T-t)} \right) \frac{1}{\theta^2} \right]$$  \hspace{1cm} (3.2.2)

Using $I(0) = Q$, the retailer’s purchase quantity per cycle is

$$Q = I(0) = a s^{-\eta} \left[ \frac{(1+bT)e^{\theta T}}{\theta} - 1 + b \left( 1 - e^{\theta T} \right) \frac{1}{\theta^2} \right]$$  \hspace{1cm} (3.2.3)

The sales revenue is

$$SR = s \int_{0}^{T} R(s,t) dt = s^{-\eta+1} aT \left[ 1 + \frac{bT}{2} \right]$$

The ordering cost is $OC = A$

The purchase cost of $Q$-units is $PC = CQ$

The holding cost is $HC = h \int_{0}^{T} I(t) dt$

For calculating interest charges and earned, we have the following two cases depending on the duration of $M$ and $N$. 
(A) Suppose $M \geq N$

**Case 1 $M \geq T + N$**

Since $M \geq T + N$, the retailer has no item left to sell in the inventory system, so the interest charges in the cycle is zero i.e. $IC_1 = 0$. The retailer generates income from the commencement of the cycle and settles the account at time $N$. So the retailer gets interest at the rate $I_c$ per dollar per year starting from $N$ through $M$.

Therefore, the interest earned per cycle is

$$IE_1 = s I_c \left[ \int_0^T R(u) du \right] \left[ (M - T - N) \int_0^T R(t) dt \right] = \frac{a s^{-\eta+1}}{6} I_c T \left[ 3(bT + 2)(M - N) \right] (3T + 2bT)$$

**Case 2 $M \leq T + N$**

Here, the retailer does not have adequate fund to settle the account at $M$ as customer will be paying at time $T + N$. So the retailer will have to pay interest charges during $[M,T + N]$ for unsold items at an interest rate $I_c$ per dollar per year. Hence, the interest payable in each cycle is

$$IC_2 = CI_c \int_M^{T+N} I(t-N) dt = CI_c \int_M^{T} I(t) dt$$

Here, the retailer earns interest on the collected revenue during $[N,M]$, which is given by

$$IE_2 = sI_c \int_N^{M+N} R(u-N) du \right] \int_N^N R(t) dt = \frac{s^{-\eta+1}}{6} a I_c \left[ T^2 (3 + bT) + 3b \left( M^2 - N^2 \right) + 6(M - N) - 6bMN \right]$$
Hence, the average profit per unit time for retailer is

\[
\pi(s,T) = \begin{cases} 
\pi_1(s,T), & 0 \leq T \leq M - N \\
\pi_2(s,T), & T \geq M - N 
\end{cases}
\]

where

\[
\pi_1(s,T) = \frac{1}{T} \{SR - PC - OC - HC - IC_1 + IE_1\}
\]

and

\[
\pi_2(s,T) = \frac{1}{T} \{SR - PC - OC - HC - IC_2 + IE_2\}
\]

(B) Suppose \( M \leq N \)

Here, the retailer does not earn any interest i.e. \( IE_3 = 0 \).

Retailer has to pay interest for all the items. Therefore, the interest charged per cycle is

\[
IC_3 = C \int \left( (N - M)Q + \int_{N}^{T+N} I(t - N) dt \right)
\]

The average profit per unit time is

\[
\pi_3(s,T) = \frac{1}{T} \{SR - PC - OC - HC - IC_3 + IE_3\}
\]

The goal is to maximize the average profit per unit time \( \pi_i(s,T), i = 1, 2, 3 \) with respect to retail price and cycle time. The extremely non-linearity of the objective functions in equations (3.2.4-3.2.6) renders to obtain the closed form solution.

3.2.2.1 Computational Algorithm

Differentiate \( \pi_i(s,T), i = 1, 2, 3 \) with respect to \( s \) and \( T \), compare with zero and follow the steps given below.

Step 1: Allocate values to all inventory parameters.
Step 2: For $M \geq N$, solve simultaneously $\frac{\partial \pi_1}{\partial s} = 0$ and $\frac{\partial \pi_1}{\partial T} = 0$ and also

$$\frac{\partial \pi_2}{\partial s} = 0 \text{ and } \frac{\partial \pi_2}{\partial T} = 0.$$ 

If $M \geq T + N$ then compute $\pi_1$ from equation (3.2.4) else $\pi_2$ from equation (3.2.5). After having $(s, T)$, the retailers purchase quantity can be obtained using equation (3.2.3).

Step 3: For $M < N$, solve $\frac{\partial \pi_3}{\partial s} = 0$ and $\frac{\partial \pi_3}{\partial T} = 0$ simultaneously and get the average profit per unit time $\pi_3$ from equation (3.2.6) and purchase quantity using equation (3.2.3).

### 3.2.2.2 Numerical Examples

We study two numerical examples to demonstrate the models for $M \geq N$ and $M < N$ respectively.

**Example 3.2.1:** Consider, $A = $50 per order, $C = $6 per unit, $a = 10^4$ units, $h = $2.5/unit/year, $b = 10\%$, $\eta = 1.2$, $I_e = 10\% / $/year, $I_c = 15\% / $/year, $\theta = 0.2$, $M = 0.8$ year and $N = 0.6$ year.

In this case, $\pi_1$ is $4,073.49$ at $(T = T_1 = 0.41, s = s_1 = $39.65)$

while $\pi_2$ is $4,251.27$ at $(T = T_2 = 0.559, s = s_2 = $40.6$)$.

So, the retailer's optimum profit is $\pi^* (= \pi_2^*) = $4,251.27 for purchasing $Q^* = 71.46$ units. In figure 3.2.1, 3-D plot of the average total profit per unit time demonstrates concavity with respect to retail price and cycle time.
**Example 3.2.2:** Taking, $A = $ 50 per order, $C = $3 per unit, $h = $2.5/unit/year, 
$a = 10^4$ units, $b = 10\%$, $\eta = 2.2$, $I_e = 15\% / $ / year, $I_c = 10\% / $ / year, 
$\theta = 0.2$, $M = 0.7$ year and $N = 0.8$ year. We have $M < N$. Then from step 3 of 
the above algorithm, we get $\pi_3$ is $337.79$ at $(T = T_3 = 0.46, s = s_3 = 7.06)$. 

Now we want to study how variations in inventory parameters influence profit 
function. For that we carry out sensitivity analysis. We vary the parameters 
given in example 3.2.1 by -40%, -20%, 20% and 40%. The outcome of variations 
in profit is demonstrated in figure 3.2.2.
From figure 3.2.2, it is observed that the average total profit per unit time increases drastically with respect to increase in $a$, and increases slowly with increase in $b$, $I_e$ and credit period to the retailer given by the supplier $M$ and decreases highly with increase in $A$, $C$, $N$, $\eta$. This advises that by ordering more and taking advantage of smaller delay period frequently, the retailer can reduce ordering cost.

**Conclusions**

In this chapter, inventory models are derived to decide the optimal pricing and ordering policies which maximizes total profit per unit time of the retailer. The demand is price-sensitive time dependent increasing. The decision policies and managerial comments are displayed from numerical examples. It is recommended that the retailer should take advantage of delay period by putting smaller orders frequently and some credit should be passed on to his customers. This benefits retailer by getting account settled earlier and reducing risk of default from customer end.