Chapter 6

Economic production quantity models with rework and random preventive machine time

PP. 124-157
6.0 Introduction

In this chapter, Economic production quantity (EPQ) inventory model for trended demand has been analyzed with rework facility and stochastic preventive machine time. Due to complexity of the model search method is proposed to determine best optimal solution. Numerical examples and sensitivity analysis are carried out to validate the proposed model.

Two models are formulated under the following sections (viz.):

- **Model 6.1**  
  EPQ model for trended demand with rework and random preventive machine time

- **Model 6.2**  
  EPQ model for imperfect production processes with rework and random preventive machine time for deteriorating items and trended demand

6.1 EPQ Model for Trended Demand with Rework and Random Preventive Machine Time

6.1.1 Assumptions and Notations

The following additional assumptions and notations other than those given in A.1 and N.1 are used to formulate the proposed model:

**6.1.1.1 Assumptions**

1. Standard quality items must be greater than the demand.
2. The production and rework rates are constant.
3. Setup cost for rework process is zero or negligible.
4. Recoverable items are spawned during the production up time, and scrapped items are produced during the rework up time.
6.1.1.2 Notations

\( R(t) \) : \( a (1 + bt) \), \( a > 0 \), \( 0 \leq b < 1 \) where \( a > 0 \) is scale demand and \( 0 < b < 1 \)
denotes the rate of change of demand.

\( I_{1a} \) : serviceable inventory level in a production up time

\( I_{2a} \) : serviceable inventory level in a production down time

\( I_{3a} \) : serviceable inventory level in a rework up time

\( I_{3r} \) : serviceable inventory level from rework up time

\( I_{4r} \) : serviceable inventory level from rework process in rework down time

\( I_{r1} \) : recoverable inventory level in a production up time

\( I_{r3} \) : recoverable inventory level in a rework up time

\( TI_{1a} \) : total serviceable inventory in a production up time

\( TI_{2a} \) : total serviceable inventory in a production down time

\( TI_{3a} \) : total serviceable inventory in a rework up time

\( TI_{3r} \) : total serviceable inventory from a rework up time

\( TI_{4r} \) : total serviceable inventory from rework process in a rework down time

\( TTI_{r1} \) : total recoverable inventory level in a production up time

\( TTI_{r3} \) : total recoverable inventory level in a rework up time

\( T_{1a} \) : production up time

\( T_{2a} \) : production down time
\( T_{3r} \) : rework up time

\( T_{4r} \) : rework down time

\( T_{sb} \) : total production down time

\( T_{1aub} \) : production up time when the total production down time is equal to the upper bound of uniform distribution parameter

\( I_m \) : inventory level of serviceable items at the end of production up time

\( I_{mr} \) : maximum inventory level of recoverable items in a production up time

\( I_w \) : total recoverable inventory

\( P \) : production rate

\( P_1 \) : rework process rate

\( x \) : product defect rate

\( x_1 \) : product scrap rate

\( A \) : production setup cost (in $)

\( h \) : serviceable items holding cost (in $)

\( h_1 \) : recoverable items holding cost (in $)

\( S_C \) : scrap cost (in $)

\( S_L \) : lost sales cost (in $)

\( TC \) : total inventory cost (in $)

\( T \) : cycle time (in days or years)

\( TCT \) : total inventory cost per unit time for lost sales model (in $)
\[ TCT_{NL} \] : total inventory cost per unit time for without lost sales model (in $)

\[ TCT_U \] : total inventory cost per unit time (in $) for lost sales model with uniform distribution preventive maintenance time

\[ TCT_E \] : total inventory cost per unit time (in $) for lost sales model with exponential distribution preventive maintenance time

### 6.1.2 Mathematical Model

The status of the serviceable inventory is depicted in figure 6.1.1. Production occurs during \([0, T_{1a}]\) and \(x\) defective items per unit time are to be reworked. The rework process starts at the end of the predetermined production up time. The rework time ends at \(T_{3r}\) time period. The different production processes of the material and defective items results different product rates. During the rework, some rejected and scrapped items will occur. LIFO policy is assumed for the production system. So serviceable items during the rework up time are utilized before the fresh items from the production in up time. The new production run is started when the inventory level reaches zero at the end of \(T_{2a}\) time period. It may happen that the production may not start at \(T_{2a}\) time period because unavailability of the machine is randomly distributed with a probability density function \(f(t)\). The non-availability of machine may result in shortage during \(T_3\) time period. The production will resume after the \(T_3\) time period.
Figure 6.1.1 Inventory status of serviceable items with lost sales

From the above description, the inventory level in a production up time period is governed by the differential equation

\[
\frac{d I_{1a}(t_{1a})}{dt_{1a}} = P - R(t_{1a}) - x, \quad 0 \leq t_{1a} \leq T_{1a}. \quad (6.1.1)
\]

The inventory level in a rework up time is

\[
\frac{d I_{3r}(t_{3r})}{dt_{3r}} = R(t_{3r}) - x_1, \quad 0 \leq t_{3r} \leq T_{3r}. \quad (6.1.2)
\]
The inventory level in a production down time is

$$\frac{d I_{2a}}{dt_{2a}} = -R(t_{1a}), \quad 0 \leq t_{2a} \leq T_{2a}. \quad (6.1.3)$$

The inventory level in a rework down time is

$$\frac{d I_{4r}}{dt_{4r}} = -R(t_{4r}), \quad 0 \leq t_{4r} \leq T_{4r}. \quad (6.1.4)$$

Under assumption of LIFO production system, the inventory level of good items depletes at a constant rate during rework up time and down time. The inventory level is governed by

$$\frac{d I_{3a}}{dt_{3a}} = 0, \quad 0 \leq t_{3a} \leq T_{3r} + T_{4r}. \quad (6.1.5)$$

Using, $I_{1a}(0) = 0$, the solution of (6.1.1) is

$$I_{1a}(t_{1a}) = (P - a - x)t_{1a} - \frac{ab}{2}t_{1a}^2, \quad 0 \leq t_{1a} \leq T_{1a} \quad (6.1.6)$$

which is inventory level during $[0, T_{1a}]$.

Hence, the total inventory in a production up time is

$$TI_{1a} = \int_{0}^{T_{1a}} I_{1a}(t_{1a}) dt_{1a} = (P - a - x)\left(\frac{T_{1a}^2}{2} - \frac{ab}{6}T_{3a}^3\right). \quad (6.1.7)$$
Using, $I_{3r}(0) = 0$ and $I_{4r}(0) = 0$, the total inventory of serviceable items for the rework up time and rework down time is

$$TI_{3r} = \left( R - a - x_1 \right) \frac{T_{3r}^2}{2} - \frac{ab}{6} T_{3r}^3$$

(6.1.8)

$$TI_{4r} = a \left[ \frac{T_{4r}^2}{2} + \frac{b}{3} T_{4r}^3 \right]$$

(6.1.9)

respectively.

Using $I_{2a}(I_{2a}) = 0$ the total inventory level of a production down time is

$$TI_{2a} = a \left[ \frac{T_{2a}^2}{2} + \frac{b}{3} T_{2a}^3 \right].$$

(6.1.10)

The maximum inventory

$$I_m = I_{1a}(T_{1a}) = (P - a - x)T_{1a} - \frac{ab}{2} T_{1a}^2.$$  

(6.1.11)

and hence, the total inventory in a rework up time is

$$TI_{3a} = I_m(T_{3r} + T_{4r}).$$

(6.1.12)
Now, let us analyze the inventory level of recoverable items. (figure 6.1.2)

![Inventory Status of Recoverable Items](image)

**Figure 6.1.2 Inventory status of recoverable items**

The inventory level of recoverable items in a production up time is governed by the differential equation

$$\frac{dI_{r1}(t_{r1})}{dt_{r1}} = x, \quad 0 \leq t_{r1} \leq T_{1a}. \quad (6.1.13)$$

Since initially there is no recoverable item i.e. $I_{r1}(0) = 0$, the solution of (6.1.13) is

$$I_{r1}(t_{r1}) = x t_{r1}, \quad 0 \leq t_{r1} \leq T_{1a}. \quad (6.1.14)$$

Hence, total inventory of recoverable items in a production up time is

$$TIT_{r1} = \frac{x t_{1a}^2}{2}. \quad (6.1.15)$$
and maximum recoverable inventory

\[ I_{Mr} = I_{r1}(T_{1a}) = xT_{1a}. \]  \hspace{1cm} (6.1.16)

The inventory level of recoverable item in the rework up time is modeled as

\[ \frac{d I_{r3}(t_{r3})}{dt_{r3}} = -R_1, \quad 0 \leq t_{r3} \leq T_{3r} \]  \hspace{1cm} (6.1.17)

Using \( I_{r3}(t_{r3}) = 0 \) the inventory level of recoverable item in rework up time is

\[ I_{r3}(t_{r3}) = P_1(T_{3r} - t_{r3}), \quad 0 \leq t_{r3} \leq T_{r3}. \]  \hspace{1cm} (6.1.18)

Hence, the total inventory of recoverable item in the rework up time is

\[ TTI_{r3} = \frac{R_1T_{3r}^2}{2}. \]  \hspace{1cm} (6.1.19)

The number of recoverable inventory is

\[ I_{Mr} = I_{r3}(0) = P_1T_{3r}. \]

Hence,

\[ T_{3r} = \frac{I_{Mr}}{P_1}. \]  \hspace{1cm} (6.1.20)

Substituting \( I_{Mr} \) from equation (6.1.16), we get

\[ T_{3r} = \frac{xT_{1a}}{P_1}. \]  \hspace{1cm} (6.1.21)
Hence, total recoverable inventory is

\[ I_w = TTI_{r1} + TTI_{r3} = \frac{xT_{Ia}}{2} \left( 1 + \frac{x}{P} \right). \] (6.1.22)

The inventory level at the beginning of the production down time is equal to the inventory level at the end of the production up time; i.e. \( I_{Ia} (T_{Ia}) = I_{2a} (0) \)

Therefore,

\[ T_{2a} \approx \frac{1}{a} \left[ (P - a - x)T_{Ia} - \frac{ab}{2}T_{Ia}^2 \right] \] (6.1.23)

When \( t_{3r} = T_{3r} \) and \( t_{4r} = 0 \), the inventory level for serviceable item in rework process satisfies

\[ (R_1 - a - x_1)T_{3r} - \frac{ab}{2}T_{3r}^2 = a \left[ T_{4r} - \frac{b}{2}T_{4r}^2 \right] \]

Neglecting \( T_{4r}^2 \) (because \( 0 < T_{4r} < 1 \)), we get

\[ T_{4r} \approx \frac{1}{a} (P_1 - a - x_1) \frac{x}{P_1} T_{Ia} \] (6.1.24)

The total production inventory cost is sum of production set up cost, inventory cost of serviceable item, inventory cost of recoverable item and scrap cost.

\[ TC = A + h \left[ T_{Ia} + T_{3r} + T_{2a} + T_{4r} + T_{3r} \right] + hI_w + S_C x_1 T_{3r} \] (6.1.25)
and total cycle time is

\[ T = T_{1a} + T_{3r} + T_{2a} + T_{4r}. \]  \hspace{1cm} (6.1.26)

Hence, the total cost per unit time without lost sales is given by

\[ TCT_{NL} = \frac{TC}{T}. \]  \hspace{1cm} (6.1.27)

The optimal production up time for the EPQ system without lost sales can be obtained by setting

\[ \frac{dTCT_{NL}(T_{1a})}{dT_{1a}} = 0. \]  \hspace{1cm} (6.1.28)

When unavailability time of a machine is longer than the production down-time duration, lost sales will occur. So the total inventory cost is

\[
E(TC) = A + h \left[ TI_{1a} + TI_{3r} + TI_{2a} + TI_{4r} + TI_{3a} \right] + h_{1}I_{w} + S_{C} \cdot x_{1} T_{3r} \\
+ S_{L} \int_{t = T_{2a} + T_{4r}}^{\infty} R(t) \left( t - (T_{2a} + T_{4r}) \right) f(t) \, dt
\]  \hspace{1cm} (6.1.29)

and total cycle time for lost sales is

\[
E(T) = T_{1a} + T_{3r} + T_{2a} + T_{4r} + \int_{t = T_{2a} + T_{4r}}^{\infty} \left( t - (T_{2a} + T_{4r}) \right) f(t) \, dt.
\]  \hspace{1cm} (6.1.30)

Hence, the total cost per unit time for lost sales is

\[
E(TCT) = \frac{E(TC)}{E(T)}.
\]  \hspace{1cm} (6.1.31)
We discuss lost sales scenario for two distributions; viz uniform distribution and exponential distribution.

### 6.1.2.1 Uniform Distribution

Define the probability distribution function $f(t)$, when the preventive maintenance time $t$ is distributed uniformly as follows.

$$f(t) = \begin{cases} \frac{1}{\tau}, & 0 \leq t \leq \tau \\ 0, & \text{otherwise} \end{cases}$$

Substituting $f(t)$ in (6.1.31), gives total cost per unit time for uniform distribution as

$$TCT_U = \frac{A + h[T_{1a} + T_{3r} + T_{2a} + T_{4r} + T_{3a} + h_I I_w] + S_C x_1 T_{3r} + S_L \int_0^\tau \frac{a(1+bt)}{t} (t - (T_{2a} + T_{4r})) \, dt}{T_{1a} + T_{3r} + T_{2a} + T_{4r} + \int_0^\tau \frac{1}{t_{1a} + T_{4r}} (t - (T_{2a} + T_{4r})) \, dt}$$

Substituting all the time variables in (6.1.32) in terms of $T_{1a}$, the objective function; $TCT_u$ is a function of $T_{1a}$ only. The optimum value of $T_{1a}$ can be computed by setting

$$\frac{dTCT_U(T_{1a})}{dT_{1a}} = 0.$$
To derive the best solution from non-lost sales and lost sales scenarios, we propose following steps (Wee and Widyadana (2011)):

Step 1: Calculate (6.1.28), (6.1.23) and (6.1.24) and set
\[ T_{sb} = T_{2a} + T_{4r}. \]

Step 2: If \( T_{sb} < \tau \), then the obtained solution is not feasible, and go to step 3; otherwise the solution is obtained.

Step 3: Set \( T_{sb} = \tau \). Find \( T_{1aub} \) using (6.1.24) and (6.1.23). Calculate
\[ TCT_{NL} (T_{1aub}) \] using (6.1.27).

Step 4: Calculate (6.1.33), (6.1.23) and (6.1.24) and set \( T_{sb} = T_{2a} + T_{4r}. \)

Step 5: If \( T_{sb} \geq \tau \) then \( T^{*}_{1a} = T_{1aub} \) and the corresponding total cost is
\[ TCT_{NL} (T_{1aub}) \); otherwise, calculate \( TCT_{U} (T_{1a}) \).

Step 6: If \( TCT_{NL} (T_{1aub}) \leq TCT_{U} (T_{1a}) \), then \( T^{*}_{1a} = T_{1aub} \); otherwise \( T^{*}_{1a} = T_{1a} \).

### 6.1.2.2 Exponential Distribution

Define the probability distribution function \( f(t) \), when the preventive maintenance time \( t \) is distributed exponential with mean \( \frac{1}{\lambda} \) as
\[ f(t) = \lambda e^{-\lambda t}, \lambda > 0. \]
Here, the total cost per unit time for lost sale $S_L$ is

$$TCT_E = \frac{1}{T_{1a} + T_{3r} + T_{2a} + T_{4r} + \frac{1}{\lambda} e^{-\lambda(T_{2a}+T_{4r})}} \left[ A + h[TI_{1a} + TI_{3r} + TI_{2a} + TI_{4r} + TI_{3a}] + h_I I_w + S_C x_1 T_{3r} + S_L \int_{t=T_{2a}+T_{4r}}^{\infty} R(t)(t-(T_{2a}+T_{4r})) \lambda e^{-\lambda t} dt \right]$$

(6.1.34)

Arguing as in (6.1.2.1), we can obtain optimum total cost. The high non-linearity of the cost functions (6.1.27), (6.1.32) and (6.1.34) does not guarantee that the optimal solution is global. However, using parametric values, convexity of the objective function is established.

### 6.1.2.3 Numerical Examples and Sensitivity Analysis

Consider following parametric values to study the working of the proposed problem. Let $A = $200 per production cycle, $P = 10,000$ units per unit time, $a = 5,000$ units per unit time, $b = 10\%$, $x = 500$ units per unit time, $x_i = 400$ units per unit time, $h = $5 per unit per unit time, $h_I = $3 per unit per unit time, $S_L = $10 per unit, $S_C = $12 per unit and the preventive maintenance time is uniformly distributed over the interval $[0, 0.1]$. (Wee and Widyadana (2011)). Using the solution procedure outlined, the optimal production up time is $T_{1a} = 41.5$ days and corresponding minimum total cost per unit time is $TCT_U = $2,575. This establishes that some lost sales reduce the total cost per unit time. The convexity of $TCT_U$ is established in figure 6.1.3.
The sensitivity analysis is carried out by changing each of the parameters by 
$-40\%, -20\%, +20\%$ and $+40\%$. The variations in production up time $T_{1a}$ and 
total cost per unit time for one inventory parameter at a time are depicted in 
figures 6.1.4 to 6.1.7 for uniform and exponential distributions.
Figure 6.1.4 Sensitivity analysis of production up time for uniform distribution

Figure 6.1.5 Sensitivity analysis of total cost for uniform distribution
Figure 6.1.6 Sensitivity analysis of production up time for exponential distribution

Figure 6.1.7 Sensitivity analysis of total cost for exponential distribution
Figures 6.1.4 and 6.1.6 depict sensitivity analysis of production up time, $T_{1a}$ with respect to all the inventory parameters considered in the modeling when preventive maintenance time follows uniform distribution/ exponential distribution. It is observed that production up time is slightly sensitive to changes in $P$ and $a$, moderately sensitive to changes in $b$ and $\tau$, and little impact due to changes in the other inventory parameters. $T_{1a}$ has negative impact with increase in the production rate $P$ and positive impact when scale demand $a$; rate of demand $b$ increases.

The optimal total cost per unit time slightly sensitive to changes in $a, P, x$ and $L$, moderately sensitive to changes in $A, b, \tau, S_C, x_1$ and $S_L$. No change is observed in the optimal total cost per unit time for remaining inventory parameters. The optimal total cost per unit time is inversely related to $P$ and $R_1$, and directly related to other inventory parameters. (See figures 6.1.5 and 6.1.7)

6.2 EPQ Model for Imperfect Production Processes with Rework and Random Preventive Machine Time for Deteriorating Items and Trended Demand

6.2.1 Assumptions and Notations

The following additional assumptions and notations other than those given in A.1, N.1 and section 6.1.1 are used to formulate the proposed model:

6.2.1.1 Assumptions

1. Good quality items must be greater than the demand.
6.2.1.2 Notations

\( C_d \) : Cost of deteriorated units (in $)

6.2.2 Mathematical Model

Arguing as in section 6.1.2, the inventory level during production up time can be described by the differential equation

\[
\frac{d I_{1a}(t_{1a})}{dt_{1a}} = P - R(t_{1a}) - x - \theta I_{1a}(t_{1a}), \quad 0 \leq t_{1a} \leq T_{1a}. \tag{6.2.1}
\]

The inventory level during rework up time is governed by the differential equation

\[
\frac{d I_{3r}(t_{3r})}{dt_{3r}} = R_1 - R(t_{3r}) - x_1 - \theta I_{3r}(t_{3r}), \quad 0 \leq t_{3r} \leq T_{3r}. \tag{6.2.2}
\]

The rate of change of inventory level during production down time is

\[
\frac{d I_{2a}(t_{2a})}{dt_{2a}} = -R(t_{2a}) - \theta I_{2a}(t_{2a}), \quad 0 \leq t_{2a} \leq T_{2a} \tag{6.2.3}
\]

and during rework down time is

\[
\frac{d I_{4r}(t_{4r})}{dt_{4r}} = -R(t_{4r}) - \theta I_{4r}(t_{4r}), \quad 0 \leq t_{4r} \leq T_{4r}. \tag{6.2.4}
\]

Under the assumption of LIFO production system, the rate of change of inventory of good items during rework up time and down time is governed by

\[
\frac{d I_{3a}(t_{3a})}{dt_{3a}} = -\theta I_{3a}(t_{3a}), \quad 0 \leq t_{3a} \leq T_{3a} + T_{4r}. \tag{6.2.5}
\]
Using \( I_{1a}(0) = 0 \), the inventory level in a production up time is

\[
I_{1a}(t_{1a}) = \frac{1}{\theta}(P-a-x_1)(1-e^{-\theta t_{1a}}) - \frac{ab}{\theta^2}(e^{-\theta t_{1a}} + \theta t_{1a} - 1).
\]  \hspace{1cm} (6.2.6)

The total inventory in a production up time is

\[
TI_{1a} = \int_0^{t_{1a}} I_{1a}(t_{1a}) \, dt_{1a}.
\]  \hspace{1cm} (6.2.7)

Using \( I_{3r}(0) = 0 \), solution of (6.2.2) is

\[
I_{3r}(t_{3r}) = \frac{1}{\theta}(P_1-a-x_1)(1-e^{-\theta t_{3r}}) - \frac{ab}{\theta^2}(e^{-\theta t_{3r}} + \theta t_{3r} - 1).
\]  \hspace{1cm} (6.2.8)

and total inventory in a rework up time is

\[
TI_{3r} = \int_0^{t_{3r}} I_{3r}(t_{3r}) \, dt_{3r}.
\]  \hspace{1cm} (6.2.9)

Using \( I_{4r}(t_{4r}) = 0 \), solution of (6.2.4) is

\[
I_{4r}(t_{4r}) = \frac{a}{\theta}(e^{\theta(T_{4r}-t_{4r})} - 1) - \frac{ab}{\theta^2}(e^{\theta(T_{4r}-t_{4r})} - 1) + \frac{ab}{\theta}(T_{4r}e^{\theta(T_{4r}-t_{4r})} - 1).
\]  \hspace{1cm} (6.2.10)

and hence the total inventory of serviceable items during rework down time is

\[
TI_{4r} = \int_0^{t_{4r}} I_{4r}(t_{4r}) \, dt_{4r}.
\]  \hspace{1cm} (6.2.11)
Similarly, using $I_{2a}(T_{2a}) = 0$ the total inventory during production down time is

$$TI_{2a} = \int_0^{T_{2a}} I_{2a}(t_{2a}) \, dt_{2a}. \quad (6.2.12)$$

Now, the maximum inventory level is

$$I_m = I_{1a}(T_{1a}) = \frac{1}{\theta} (P - a - x) \left(1 - e^{-\theta T_{1a}}\right) - \frac{ab}{\theta^2} \left(e^{-\theta T_{1a}} + \theta T_{1a} - 1\right). \quad (6.2.13)$$

Hence, the total inventory in a rework up time is

$$TI_{3a} = I_m \left(T_{3r} + T_{4r} - \frac{\theta}{2} (T_{3r} + T_{4r})^2\right). \quad (6.2.14)$$

Next, we analyze the inventory level of recoverable items. (figure 6.1.2)

The rate of change of recoverable items in a production up time is

$$\frac{d I_{r1}(t_{r1})}{dt_{r1}} = x - \theta I_{r1}(t_{r1}), \ 0 \leq t_{r1} \leq T_{1a}. \quad (6.2.15)$$

Using $I_{r1}(0) = 0$ the inventory level of the recoverable items during the production up time is

$$I_{r1}(t_{r1}) = \frac{x}{\theta} \left(1 - e^{-t_{r1}}\right), \ 0 \leq t_{r1} \leq T_{1a} \quad (6.2.16)$$
and hence, total recoverable items in a production up time is

\[ TTI_{r1} = \int_{0}^{T_{1a}} I_{r1}(t_{r1}) dt_{r1}. \]  

(6.2.17)

Initially, the recoverable inventory is

\[ I_{Mr} = I_{r1}(T_{1a}) = \frac{x}{\theta} \left( 1 - e^{-\theta T_{1a}} \right) T_{1a} = \frac{x}{\theta} \left( T_{1a} - \frac{\theta T_{1a}^2}{2} \right). \]  

(6.2.18)

The rate of change of inventory level of recoverable item during the rework up time is governed by differential equation

\[ \frac{d I_{r3}(t_{r3})}{dt_{r3}} = -R - \theta I_{r3}(t_{r3}), \quad 0 \leq t_{r3} \leq T_{3r}. \]  

(6.2.19)

Using, \( I_{r3}(t_{r3}) = 0 \) the solution of equation (6.2.19) is

\[ I_{r3}(t_{r3}) = \frac{P_1}{\theta} \left( e^{\theta(T_{3r} - t_{r3})} - 1 \right). \]  

(6.2.20)

The total inventory of recoverable item during rework up time is

\[ TTI_{r3} = \int_{0}^{T_{3r}} I_{r3}(t_{r3}) dt_{r3}. \]  

(6.2.21)

The number of recoverable items is

\[ I_{Mr} = I_{r3}(0) = \frac{P_1}{\theta} \left( e^{\theta T_{3r}} - 1 \right). \]  

(6.2.22)
Since \( \theta T_{3r} \ll 1 \) and using Taylor’s series approximation, equation (6.2.22) gives

\[
T_{3r} = \frac{I_{Mr}}{P_1}.
\]  

(6.2.23)

Substituting \( I_{Mr} \) from equation (6.2.18) in equation (6.2.23), we get

\[
T_{3r} = \frac{x}{P_1} \left( T_{1a} - \frac{\theta T_{1a}^2}{2} \right).
\]  

(6.2.24)

Total recoverable items

\[
I_w = TTI_{r1} + TTI_{r3}.
\]  

(6.2.25)

Total number of units deteriorated is

\[
DU = \left( P - \int_0^{T_{1a}} R(t)dt \right) + \left( P_1 - \int_0^{T_{3r}} R(t)dt \right) - \int_0^{T_{2a} + T_{4r}} R(t)dt - x_1 T_{3r}.
\]  

(6.2.26)

Since the inventory level at the beginning of the production down time is equal to the inventory level at the end of the production up time minus the deteriorated units at \( T_{3r} + T_{4r} \), using Misra [1975], the approximation concept, we have

\[
T_{2a} \approx \frac{1}{a} \left( (P-a-x)T_{1a} - \frac{ab}{2} T_{1a}^2 \right) \left[ 1 - \theta(T_{3r} + T_{4r}) + \frac{1}{2} \theta(T_{3r} + T_{4r})^2 \right].
\]  

(6.2.27)

The inventory for serviceable item in rework process is

\[
I_{3r}(T_{3r}) = I_{4r}(0).
\]
With simple calculations

\[
T_{4r} \approx \frac{1}{a} (P_1 - a - x_1) T_{3r} \left(1 - \frac{1}{2} \theta T_{3r}\right).
\]  
(6.2.28)

Using equations (6.2.24) and (6.2.28), \( T_{2a} \) given in equation (6.2.27) is a function of \( T_{1a} \) only. The total production cost of inventory system is sum of production set up cost, holding cost of serviceable inventory, deteriorating cost of recoverable inventory cost and scrap cost.

Therefore,

\[
TC = A + h [TI_{1a} + TI_{3r} + TI_{2a} + TI_{4r} + TI_{3a}] + h_1 I_w + C_d DU + S_C x_1 T_{3r}
\]  
(6.2.29)

and total replenishment time is

\[
T = T_{1a} + T_{3r} + T_{2a} + T_{4r}.
\]  
(6.2.30)

The total cost per unit time without lost sales is given by

\[
TCT_{NL} = \frac{TC}{T}.
\]  
(6.2.31)

The optimal production up time for the EPQ model without lost sales is the solution of

\[
\frac{dTCT_{NL} (T_{1a})}{dT_{1a}} = 0.
\]  
(6.2.32)
Lost sales will occur when maintenance time of machine is greater than the production down-time period. So the total inventory cost in this case is

\[
E(TC) = TC + S_L \int_{t=T_2a+T_4r}^{\infty} R(t)(t-(T_2a + T_4r))f(t)\,dt. \tag{6.2.33}
\]

and the total cycle time for lost sales scenario is

\[
E(T) = T + \int_{t=T_2a+T_4r}^{\infty} (t-(T_2a + T_4r))f(t)\,dt. \tag{6.2.34}
\]

Using equations (6.2.33) and (6.2.34), the total cost per unit time for lost sales scenario is

\[
E(TCT) = \frac{E(TC)}{E(T)}. \tag{6.2.35}
\]

### 6.2.2.1 Uniform Distribution Case

Define the probability distribution function \(f(t)\), when the preventive maintenance time \(t\) follows uniform distribution as follows.

\[
f(t) = \begin{cases} 
\frac{1}{\tau}, & 0 \leq t \leq \tau \\
0, & \text{otherwise}
\end{cases}
\]
Substituting $f(t)$ in equation (6.2.35), gives total cost per unit time for uniform distribution as

$$TCT_U = \frac{1}{T_1 + T_3 + T_2 + T_4 + \frac{1}{\tau} \int_0^\tau (t - (T_2a + T_4r)) dt} \left[ A + h \left[ TI_1a + TI_3r + TI_2a + TI_4r + TI_3a \right] + h_1 I_w + C_d DU + S_C x_1 T_3r + \frac{S_L}{\tau} \int_0^\tau (1 + bt)(t - (T_2a + T_4r)) dt \right]$$

(6.2.36)

The optimal production up time for lost sales case is solution of

$$\frac{dTCT_U(T_{la})}{dT_{la}} = 0.$$  

(6.2.37)

To decide whether manufacturer should allow lost sales or not, we propose following steps (Wee and Widyadana (2011)):

Step 1: Calculate $T_{la}$ from equation (6.2.32). Hence calculate $T_{2a}$ from equation (6.2.27) and $T_{4r}$ from equation (6.2.28). Set $T_{sb} = T_{2a} + T_{4r}$.

Step 2: If $T_{sb} < \tau$, then non lost sales case is not feasible, and go to step 3; otherwise the optimal solution is obtained.

Step 3: Set $T_{sb} = \tau$. Find $T_{laub}$ using equations (6.2.27) and (6.2.28). Calculate $TCT_{NL}(T_{laub})$ using equation (6.2.31).
Step 4: Calculate $T_{1a}$ from equation (6.2.37), hence $T_{2a}$ from equation (6.2.27) and $T_{4r}$ from equation (6.2.28), and set $T_{sb} = T_{2a} + T_{4r}$.

Step 5: If $T_{sb} \geq \tau$ then optimal production up time $T_{1a}$ is $T_{1aub}$ and $TCT_{NL}(T_{1aub})$.

If $T_{sb} \leq \tau$, then, calculate $TCT_{U}(T_{1a})$ using equation (6.2.36).

Step 6: If $TCT_{NL}(T_{1aub}) \leq TCT_{U}(T_{1a})$, then optimal production up time $T_{1aub}$; otherwise it is $T_{1a}$

### 6.2.2.2 Exponential Distribution Case

Define the probability distribution function $f(t)$, when the preventive maintenance time $t$ follows exponential distribution with mean $\frac{1}{\lambda}$ as

$$f(t) = \lambda e^{-\lambda t}, \quad \lambda > 0.$$  

Here, the total cost per unit time for lost sales scenario is

$$TCT_E = \frac{TC + S_L \int_{t=T_{2a}+T_{4r}}^{\infty} R(t) \left( t - (T_{2a} + T_{4r}) \right) \lambda e^{-\lambda t} \, dt}{T + \frac{1}{\lambda} e^{-\lambda (T_{2a} + T_{4r})}}. \quad (6.2.38)$$

The optimal $T_{1a}$ can be obtained by setting

$$\frac{dTCT_{E}(T_{1a})}{dT_{1a}} = 0. \quad (6.2.39)$$
The convexity of $TCT_{NL}$, $TCT_U$ and/or $TCT_E$ has been established graphically with suitable values of inventory parameters.

### 6.2.2.3 Numerical Examples and Sensitivity Analysis

In this section, we validate the proposed model by numerical examples. First we consider uniform distribution case. Take $A = 200$ per production cycle, $P = 10,000$ units per unit time, $R_1 = 4,000$ units per unit time, $a = 5,000$ units per unit time, $b = 10\%$, $x = 500$ units per unit time, $x_1 = 400$ units per unit time, $h = 15$ per unit per unit time, $h_1 = 3$ per unit per unit time, $S_L = 10$ per unit, $S_C = 12$ per unit, $C_d = 0.01$ per unit, $\theta = 10\%$ and the preventive maintenance time is uniformly distributed over the interval $[0, 0.1]$. Following algorithm with Maple 14, the optimal production up time $T_{1a} = 0.109$ years and the corresponding optimal total cost per unit time is $TCT_U = 4.448$. The convexity of $TCT_U$ is exhibited in figure 6.2.1.
The sensitivity analysis is carried out by changing one parameter at a time by $-40\%, -20\%, +20\%$ and $+40\%$. The variations in optimal production up time and the total cost per unit time for different inventory parameters are shown in figures 6.2.2 to 6.2.5 for uniform and exponential distributions.

Figure 6.2.1 Convexity of total optimal cost with uniform distribution
Figure 6.2.2 Sensitivity analysis of production up time for uniform distribution

Figure 6.2.3 Sensitivity analysis of total cost for uniform distribution
Optimal Inventory Policies for Different Demand Structures

Figure 6.2.4 Sensitivity analysis of production up time for exponential distribution

Figure 6.2.5 Sensitivity analysis of total cost for exponential distribution

It is observed from figure 6.2.2 that the optimal production up time is slightly sensitive to changes in $P, \theta$ and $a$, moderately sensitive to changes in $\tau$ and $b$, 
and insensitive to changes in the other parameters. $T_{i_d}$ is negative related to $P$ and $\theta$ and positively related to $a$ and $\tau$. Figure 6.2.3 exhibits variations in the optimal total cost per unit time with uniform distribution. The optimal total cost is slightly sensitive to changes in $a, P, x, C_d$ and $h$; moderately sensitive to changes in $A, \tau, Sc, \theta, x_1$ and $S_L$ and insensitive to changes in other parameters.

Take mean of exponential distribution as 20. The optimum total cost is $5,116 when the optimal production up time is 0.14 years. The convexity of the total cost is shown in figure 6.2.6.

Figure 6.2.6 Convexity of total optimal cost with exponential distribution
From figure 6.2.4, for lost sales with exponential distribution, it is observed that the optimal production up time is slightly sensitive to changes in $P$ and $a$, moderately sensitive to changes in the parameter $\lambda$ and $\theta$ and insensitive to changes in the other parameters. The optimal production up time is negatively related to $P, b$ and $\lambda$ and positively related to the value of $a$. In figure 6.2.5, the variations in the optimal total cost are studied. Observations are similar to uniform distribution case.

**Conclusions**

In this chapter, rework of imperfect quality and random preventive maintenance time are incorporated in economic production quantity model when demand increases with time. The random preventive maintenance time is distributed uniformly and exponentially. The models are validated by the example. The sensitivity analysis suggests that the optimal total cost per unit time is sensitive to changes in the production rate, the demand rate, and the product defect rate in both the uniform and the exponential distributed preventive maintenance time. To combat increasing demand, the management should adopt latest machinery which decreases defective production rate, reducing rework and as a consequence, machine’s production up time can be utilized to its utmost. In second model, it is observed that the production up time is sensitive to demand rate and deterioration. It suggests that the manufacturer should control deterioration of units in inventory by using proper storage facilities. The optimal total cost per unit time is sensitive to changes in the holding cost, the product
defect rate and the production rate in both the distributions. This suggests that the manufacture should depute efficient technician to reduce preventive maintenance time.