Appendix I

1. $[a, b]$ is non-empty iff $a \leq b$

Proof: Suppose $[a, b]$ is non-empty and $c \in [a, b]$.

Then

$$a \leq c \leq b$$

and hence,

$$a \leq b \tag{3.1}$$

Now conversely, if $a \leq b$

then

$$a \leq a \leq b$$

Thus

$$a \in [a, b] \tag{3.2}$$

and so $[a, b]$ is non-empty. (3.1) and (3.2) give the required result.

2. Suppose $[a, b]$ is non-empty, then

(a) $[a, b] \subseteq [c, d]$ iff $c \leq a \leq b \leq d$

Proof: Suppose

$$[a, b] \subseteq [c, d]$$

Since $a \in [a, b]$, clearly

$$a \in [c, d]$$

Therefore,

$$c \leq a \leq d \tag{3.3}$$

Similarly,

$$c \leq b \leq d \tag{3.4}$$

The hypothesis [a, b] is non-empty implies that $a \leq b$ (from result 1) and so by (3.3) and (3.4),

$$c \leq a \leq b \leq d \tag{3.5}$$

Conversely, if $c \leq a \leq b \leq d$ and if $x \in [a, b]$ then

$$a \leq x \leq b$$

and

$$c \leq x \leq d$$

Therefore

$$x \in [c, d]$$

and hence

$$[a, b] \subseteq [c, d] \tag{3.6}$$

(3.5) and (3.6) give the required result.

(b) $[a, b] \subset [c, d]$ iff either $c < a$ and $b \leq d$ or $c \leq a$ and $b < d$

Proof: Suppose

$$[a, b] \subset [c, d]$$

By result 2(a),

$$c \leq a \leq b \leq d$$

Unless $c < a$ or $b < d$,

$$c = a \text{ and } b = d$$

which is a contradiction.

Hence either

$$c < a \text{ and } b \le d \text{ or } c \le a \text{ and } b < d \qquad (3.7)$$

Conversely, if $c < a \le b \le d$, then by result 2(a),

$$[a, b] \subseteq [c, d]$$

Now, $c \in [c, d]$ but $c \notin [a, b]$. So, $[a, b] \subset [c, d]$

Again, if $c \le a \le b < d$, then by result 2(a),

$$[a, b] \subseteq [c, d]$$

Now, $d \in [c, d]$ but $d \notin [a, b]$. So, $[a, b] \subset [c, d]$.

Therefore, if either $c < a \le b \le d$ or $c \le a \le b < d$,

$$[a, b] \subset [c, d] \qquad (3.8)$$

(3.7) and (3.8) give the required result.

3. If the intersection of k intervals is non-empty, then it is an interval with endpoints from the k intervals.

Proof: Let the k intervals be $[a_1, b_1], [a_2, b_2], \ldots [a_k, b_k]$.

Let $c = \max \{a_i \mid 1 \le i \le k\}$ and $d = \min \{b_i \mid 1 \le i \le k\}$.

Since $\cap_{i=1,k} [a_i, b_i]$ is non-empty, let

$$x \in \cap_{i=1,k} [a_i, b_i]$$

$$\Rightarrow x \in [a_i, b_i] \ \forall i = 1, 2, .. k \qquad (3.9)$$

$$\Rightarrow x \ge a_i \ \forall i = 1, 2, .. k$$

$$\Rightarrow x \ge \max\{a_i \mid 1 \le i \le k\}$$

$$\Rightarrow x \ge c \qquad (3.10)$$

Also, from (3.9),

$$x \leq b_i \ \forall \ i = 1, 2,..k$$

$$\Rightarrow \ x \leq \min\{b_i \mid 1 \leq i \leq k\}$$

$$\Rightarrow \ x \leq d \qquad (3.11)$$

From (3.10) and (3.11), $c \leq x \leq d$ and hence $x \in [c, d]$.

Therefore,

$$\cap_{i=1,k} [a_i, b_i] \subseteq [c, d] \qquad (3.12)$$

The hypothesis $\cap_{i=1,k} [a_i, b_i]$ is non-empty implies that [c, d] is non-empty.

Now, let

$$y \in [c, d] \qquad (3.13)$$

$$\Rightarrow \ y \geq c = \max\{a_i \mid 1 \leq i \leq k\}$$

$$\Rightarrow \ y \geq a_i \ \forall \ i = 1, 2,..k \qquad (3.14)$$

Also, from (3.13),

$$y \leq d$$

$$\Rightarrow \ y \leq \min\{b_i \mid 1 \leq i \leq k\}$$

$$\Rightarrow y \leq b_i \ \forall \ i = 1, 2,..k \qquad (3.15)$$

From (3.14) and (3.15), it follows that

$$a_i \leq y \leq b_i \ \forall \ i = 1, 2,..k$$

$$\Rightarrow \ y \in [a_i, b_i] \ \forall \ i = 1, 2,..k$$

$$\Rightarrow \ y \in \cap_{i=1,k} [a_i, b_i]$$

Hence,

$$[c, d] \subseteq \cap_{i=1,k} [a_i, b_i] \qquad (3.16)$$

From (3.12) and (3.16),

$$[c, d] = \cap_{i=1,k} [a_i, b_i]$$

Now, since the endpoint (c, '[') is one of the endpoints (a_1, '[')....(a_k, '[') and the endpoint (d, ']') is one of the endpoints (b_1, ']').....(b_k, ']'), therefore, the result holds.