1. \([a, b]\) is non-empty iff \(a \leq b\)

Proof: Suppose \([a, b]\) is non-empty and \(c \in [a, b]\).

Then

\[a \leq c \leq b\]

and hence,

\[a \leq b\] \hspace{1cm} (3.1)

Now conversely, if \(a \leq b\)
then

\[a \leq a \leq b\]

Thus

\[a \in [a, b]\] \hspace{1cm} (3.2)

and so \([a, b]\) is non-empty. (3.1) and (3.2) give the required result.

2. Suppose \([a, b]\) is non-empty, then

(a) \([a, b] \subseteq [c, d]\) iff \(c \leq a \leq b \leq d\)

Proof: Suppose

\([a, b] \subseteq [c, d]\)

Since \(a \in [a, b]\), clearly

\[a \in [c, d]\]

Therefore,
Similarly,
\[ c \leq b \leq d \quad (3.4) \]

The hypothesis \([a, b]\) is non-empty implies that \(a \leq b\) (from result 1) and so by (3.3) and (3.4),
\[ c \leq a \leq b \leq d \quad (3.5) \]

Conversely, if \(c \leq a \leq b \leq d\) and if \(x \in [a, b]\) then
\[ a \leq x \leq b \]

and
\[ c \leq x \leq d \]

Therefore
\[ x \in [c, d] \]

and hence
\[ [a, b] \subseteq [c, d] \quad (3.6) \]

(3.5) and (3.6) give the required result.

(b) \([a, b] \subseteq [c, d]\) iff either \(c < a\) and \(b \leq d\) or \(c \leq a\) and \(b < d\)

*Proof*: Suppose
\[ [a, b] \subseteq [c, d] \]

By result 2(a),
\[ c \leq a \leq b \leq d \]

Unless \(c < a\) or \(b < d\),
\[ c = a \text{ and } b = d \]

which is a contradiction.

174
Hence either
\[ c < a \text{ and } b \leq d \text{ or } c \leq a \text{ and } b < d \]  \hspace{1cm} (3.7)

Conversely, if \( c < a \leq b \leq d \), then by result 2(a),
\[ [a, b] \subseteq [c, d] \]

Now, \( c \in [c, d] \) but \( c \notin [a, b] \). So, \([a, b] \subset [c, d]\)

Again, if \( c \leq a \leq b < d \), then by result 2(a),
\[ [a, b] \subseteq [c, d] \]

Now, \( d \in [c, d] \) but \( d \notin [a, b] \). So, \([a, b] \subset [c, d]\).

Therefore, if either \( c < a \leq b \leq d \) or \( c \leq a \leq b < d \),
\[ [a, b] \subseteq [c, d] \] \hspace{1cm} (3.8)

(3.7) and (3.8) give the required result.

3. If the intersection of k intervals is non-empty, then it is an interval with endpoints from the k intervals.

Proof. Let the k intervals be \([a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]\).

Let \( c = \max \{a_i | 1 \leq i \leq k\} \) and \( d = \min \{b_i | 1 \leq i \leq k\} \).

Since \( \cap_{i=1,k} [a_i, b_i] \) is non-empty, let
\[ x \in \cap_{i=1,k} [a_i, b_i] \]
\[ \Rightarrow x \in [a_i, b_i] \ \forall \ i = 1, 2, \ldots, k \] \hspace{1cm} (3.9)
\[ \Rightarrow x \geq a_i \ \forall \ i = 1, 2, \ldots, k \]
\[ \Rightarrow x \geq \max \{a_i | 1 \leq i \leq k\} \]
\[ \Rightarrow x \geq c \] \hspace{1cm} (3.10)
Also, from (3.9),

\[ x \leq b_i \quad \forall \ i = 1, 2, \ldots, k \]

\[ \Rightarrow x \leq \min \{b_i \mid 1 \leq i \leq k\} \]

\[ \Rightarrow x \leq d \quad (3.11) \]

From (3.10) and (3.11), \( c \leq x \leq d \) and hence \( x \in [c, d] \).

Therefore,

\[ \cap_{i=1}^{k} [a_i, b_i] \subseteq [c, d] \quad (3.12) \]

The hypothesis \( \cap_{i=1}^{k} [a_i, b_i] \) is non-empty implies that \([c, d]\) is non-empty.

Now, let

\[ y \in [c, d] \quad (3.13) \]

\[ \Rightarrow y \geq c = \max \{a_i \mid 1 \leq i \leq k\} \]

\[ \Rightarrow y \geq a_i \quad \forall \ i = 1, 2, \ldots, k \quad (3.14) \]

Also, from (3.13),

\[ y \leq d \]

\[ \Rightarrow y \leq \min \{b_i \mid 1 \leq i \leq k\} \]

\[ \Rightarrow y \leq b_i \quad \forall \ i = 1, 2, \ldots, k \quad (3.15) \]

From (3.14) and (3.15), it follows that

\[ a_i \leq y \leq b_i \quad \forall \ i = 1, 2, \ldots, k \]

\[ \Rightarrow y \in [a_i, b_i] \quad \forall \ i = 1, 2, \ldots, k \]

\[ \Rightarrow y \in \cap_{i=1}^{k} [a_i, b_i] \]

Hence,

\[ [c, d] \subseteq \cap_{i=1}^{k} [a_i, b_i] \quad (3.16) \]

From (3.12) and (3.16),
\[ [c, d] = \bigcap_{m=1}^{k} [a_i, b_i] \]

Now, since the endpoint \((c, [') is one of the endpoints \((a_1, [')\ldots(a_k, [') and the endpoint \((d, [') is one of the endpoints \((b_1, [')\ldots(b_k, [')\), therefore, the result holds.