Chapter 7

Stability of relativistic electromagnetic flat-top solutions

In Chapter 6, detailed characterization of the 1-D laser pulse solitons along with the eigen spectrum of their formation in the parameter space in the absence and presence of ion dynamics has been presented. The inclusion of ion response in the study of relativistically intense electromagnetic laser pulse propagation in plasma yields certain new solitonic structures. A flat-top slow moving structure (for which the various fields have flat profile over a wide spatial range) is one such solution. In the present Chapter, the evolution of this particular flat-top soliton solution is studied in detail with the help of coupled fluid Maxwell set of equations. The study shows that the flat-top solution is unstable. The instability is characterized as the backward Brillouin instability for which the electron quiver velocity plays the role of the effective temperature.
7.1 Introduction

The interaction mechanism of the intense electromagnetic pulses with plasma is rich in a variety of nonlinear physics phenomena. Some of which has been explored [3, 7–14, 58] and a lot still needs to be examined. Several authors have sought exact nonlinear solutions in the form of propagating envelope solitary pulses for the coupled system of light field and the electron fluid. Some of these nonlinear solutions move very slowly and/or are even stationary. For these slowly moving solutions (and also in the eventuality of a major breakthrough leading to a next generation of high power lasers), the heavier ion species would also respond. Work along this direction has been initiated in some recent studies [42–55, 59].

It has been shown that the inclusion of ion response rules out the existence of static single peak solutions that have been obtained when electron species alone is considered [43]. The single peak solutions, for a continuum band in the parameter space of $\lambda$ (associated with laser frequency defined in earlier studies [11]) vs group propagation speed $\beta$, do not touch the $\beta = 0$ axis, when ion response is incorporated. The single peak solutions now start from a small finite value of the curve defined by a critical value of $\beta = \beta_c$ (which is dependent on electron to ion mass ratio as well as the parameter $\lambda$). It has been shown in some papers [46, 48] that the gap from $\beta = 0$ to $\beta_c$ supports dark solitonic structures. This transition from the dark to the bright soliton, however, does not occur drastically. Within a single peak and the dark soliton solutions, there exist new variety of spatially extended solutions having flat spatial profile at the centre. These solutions appear to be a deformation of the single peak solutions, as though their maxima have been spatially extended. The ion and electron density profiles are also observed.
to be identical for these solution, and hence the entire structure is essentially quasineutral.

Figure 7.1: The plot of critical group velocity $\beta = \beta_c$ above which the single peak solutions are permissible for various values of $\alpha$ parameter.

We present in this chapter the numerical fluid simulation studies for these flat-top solutions. It is shown that they survive for a long time but later develop an instability. The instability eventually leads to the complete disintegration of the structure. The instability is shown to be linked to a Brillouin scattering process. In a cold plasma, there are no modes associated with ions. In this case, however, the scattering generates a quasi-ion mode where the role of temperature is provided by the quiver velocity of electrons in the electromagnetic field.
7.2 The flat-top nonlinear solution

The governing equations for an interacting laser plasma system discussed in Chapter 6 clearly represents a rich nonlinear set of equation. The numerical solutions of the coupled set of Eqs. (6.7) and (6.8) using the expression for \( u_e \) and \( u_i \) from Eqs. (6.9) and (6.10) leads to several varieties of exact one dimensional localized solutions. For a fixed value of the group speed \( \beta \), only certain specific values of \( \lambda \) are permitted for obtaining solutions. These solutions have been obtained earlier by several authors [10–13, 42–45, 48].

In the presence of ion response, no stationary solution can be found. In fact, the continuum band (in \( \lambda \) vs \( \beta \) space) of single peak solutions [43] now has a forbidden gap from \( \beta = 0 \) to a critical value \( \beta_c \). The curve \( \beta = \beta_c \) as a function of \( \lambda \) for three different values of \( \alpha \) is shown in Fig. 7.1 (in the \( \lambda \) vs \( \beta \) space) where \( \alpha \) is the mass ratio, \( m_e/m_i \).

It can be observed from the figure (Fig. 7.1) that for any given value of \( \lambda \), forbidden gap shrinks as we decrease \( \alpha \). The plot for three different values of \( \alpha \), viz., 0.1, 0.01 and 0.0005 (the realistic value for electron-proton plasma) denoted by black line with triangles, red line with filled circles and solid blue line respectively have been shown in the figure.

For a fixed \( \alpha \), the single peak solutions exist only for \( \beta > \beta_c \) for any given \( \lambda \). At \( \beta = \beta_c \), just as the single peak solution ceases to exist, a new variety of solutions emerge. These solutions are termed as the flat-top solutions [57] as they have a central spatial region of flat spatial profile for all fields as shown in Fig. 7.2 by the solid lines in various subplots corresponding to the profile of the vector potential \( R \), scalar potential \( \varphi \), the electron and ion densities \( n_e, n_i \) and
their velocities $u_e, u_i$. It can be seen that the structure is quasineutral as $n_e$ and $n_i$ are same. The amplitude of $R$ is considerably much higher than that of $\varphi$. The flat-top structure in this plot corresponds to $\beta_e = 0.11$, $\lambda = 0.99680694$ and $\alpha = 0.01$. The width and the amplitude of these flat-top solitons get decided by the values of the spectral parameters $\beta$ and $\lambda$. A flat-top solution with a lower

Figure 7.2: The profile of flat-top solution in various field are shown by solid lines in different subplot of the figure. The value of $\beta = 0.11$ and $\lambda = 0.99680694$ for this solution. For the same mass ratio $\alpha = 0.01$ and group velocity $\beta = 0.11$, as the value of $\lambda$ is increased to $0.99682$ and $0.997$ the single peak solutions shown by dashed and dashed dot lines are obtained.
value of $\beta$ (higher $\lambda$) has smaller amplitude and larger width than a solution with higher value of $\beta$ (smaller $\lambda$). We compare two such solutions in Fig. 7.3.

The left column of subplots corresponds to $\beta = 0.11$ and $\lambda = 0.99680694$ whereas the parameter values for the right column of subplots are $\beta = 0.14$ and $\lambda = 0.9660155$. These solutions correspond to $\alpha = 0.01$ which is unrealistically high. In Fig. 7.4, we show how the flat-top solutions get modified as $\alpha$ is reduced to its realistic value.

The plot corresponds to a fixed value of $\lambda = 0.9167206$, the group velocity $\beta$ is of course different when $\alpha$ is changed.
Figure 7.4: A comparison of flat-top solutions for different values of $\alpha$ parameter at fixed group velocity $\beta = 0.17$. Subplots (a) and (b) show the structure of scalar and vector potential respectively.
These flat-top solutions, in eigen space, provide a transition boundary between the localized single peak solutions and indefinitely extended wave-front solutions (shocks or dark solitons) [48, 60]. In fact, it is found that there occurs a smooth transition from a single peak solution with given group velocity to a flat-top solution with the same group velocity as the value of $\lambda$ is decreased. This is clearly evident from the other plots shown in Fig. 7.2. The profiles shown by dashed and the dash-dot lines correspond to those solutions for which $\beta = 0.11$ (the same value as the flat-top structure), however, the value of $\lambda$ is 0.99682 and 0.997 respectively.

The formation of the flat-top solutions at the transition boundary between single peak bright solitons and the dark structures at $\beta = \beta_c$ can be illustrated by a small mathematical analysis. We also provide an explanation for a particular relationship that $\lambda$ and $\beta$ have to satisfy for the formation of such flat-top structures. Such an analytical study is, however, carried out in the small amplitude limit. The observations show that the small amplitude solutions (both flat-top and single peak solitons) are essentially quasineutral, i.e. $n_e \approx n_i = n$. This also implies $u_e \approx u_i = u$ from the continuity equation of the two species.

Using the quasi-neutrality condition and eliminating $\varphi$ from $\gamma_e(1 - \beta u_e) - \varphi = 1$ and $\gamma_i(1 - \beta u_i) + \varphi\alpha = 1$ we get

$$\gamma_i + \alpha \gamma_e(1 - \frac{\beta^2(n - 1)}{n}) - (1 + \alpha) = 0$$

At low amplitude, the density response being weak one chooses, $n = 1 + \epsilon$, where $\epsilon$ is a small parameter. We then obtain an expression for $\epsilon$ from Eq. (7.1) as

$$\epsilon = [(1 + \alpha) - (\gamma_i + \alpha \gamma_e)]/[(1 - \beta^2)(\gamma_i + \alpha \gamma_e) - (1 + \alpha)]$$
In the weakly relativistic limit, we then expand \( \gamma_e \approx 1 + R^2/2 + ... \) and \( \gamma_i \approx 1 \) for the small amplitude flat-top soliton solutions. This yields

\[
\epsilon = \frac{\alpha R^2}{2\beta^2(1 + \alpha)} + \frac{\alpha^2 R^4(1 - \beta^2)}{4\beta^4(1 + \alpha)^2} + .... \tag{7.2}
\]

\[
\frac{n}{\gamma_e} = 1 - \frac{R^2}{2} + \frac{\alpha R^2}{2\beta^2(1 + \alpha)} + \frac{\alpha^2 R^4}{4\beta^4(1 + \alpha)^2}(1 - \beta^2) - \frac{\alpha R^4}{4\beta^2(1 + \alpha)} + .... \tag{7.3}
\]

\[
\frac{n}{\gamma_i} = 1 + \frac{\alpha R^2}{2\beta^2(1 + \alpha)} + \frac{\alpha^2 R^4}{4\beta^4(1 + \alpha)^2}(1 - \beta^2) + .... \tag{7.4}
\]

Figure 7.5: A comparison of \( \lambda \) vs \( \beta \) curve for the analytical and numerical values at \( \alpha = 0.01 \).

We take the conventional case of \( M = 0 \), and use the above expansion for \( n/\gamma_e \) and \( n/\gamma_i \) in Eq. (6.2). In an earlier work by Poornakala et al. [46] the low
amplitude dark and bright solitonic structures were obtained by seeking such an expansion for a finite temperature plasma. In that work, only terms upto order $R^3$ were retained to get an expression of the form $R'' + AR + BR^3 = 0$. Considering the limit of a cold plasma, their study shows that the transition from dark to bright form of the solution occurred when $\beta = \beta_c = \sqrt{\alpha}$. At $\beta = \beta_c$, it can be shown that the coefficient $B$ goes to zero. Thus, as $\beta^2 - \alpha$ changes sign, the sign of coefficient $B$ changed giving rise to bright and dark solitonic structures. However, we show here that if we retain higher order terms in the expansion, flat-top solutions form at this boundary. Thus, retaining the next higher term in the expansion we obtain the following equation

$$R'' + \frac{R}{1 - \beta^2} \left[ \frac{\lambda^2}{1 - \beta^2} - (1 + \alpha) \right] + \frac{R^3}{2(1 - \beta^2)} \left[ \frac{\beta^2 - \alpha}{\beta^2} \right] + \frac{R^5}{4(1 - \beta^2)} \left[ \frac{\beta^2 \alpha (1 + \alpha) - \alpha^2}{\beta^4 (1 + \alpha)} \right] = 0$$

which has the form of $R'' + AR + BR^3 + CR^5 = 0$, where the coefficients are

$$A = \frac{\lambda^2 - (1 + \alpha)(1 - \beta^2)}{(1 - \beta^2)^2}$$

$$B = \frac{\beta^2 - \alpha}{2 \beta^2 (1 - \beta^2)}$$

and

$$C = \frac{\beta^2 \alpha (1 + \alpha) - \alpha^2}{4 \beta^4 (1 - \beta^2) (1 + \alpha)}$$

It should be noted that by ignoring terms of higher power in $\alpha$, we regain the expression obtained by Poornakala et al. [46]. We integrate once to achieve

$$\frac{R'^2}{2} + A \frac{R^2}{2} + B \frac{R^4}{4} + C \frac{R^6}{6} = K \quad (7.5)$$
The constant $K$ can be chosen to be zero for localized bright solutions including the flat-top structures which vanish at $\pm \infty$. Hence, using $K = 0$ and making the substitution $f = R^2$ we get

$$\frac{f'^2}{f^2} + \frac{8C}{6} f^2 + 2Bf + 4A = 0 \quad (7.6)$$

This is an elliptic differential equation. The positive solution of this equation is

$$f(\xi) = \frac{2f_1f_2}{(f_1 + f_2) - (f_1 - f_2) \cosh \left(2\kappa \sqrt{\frac{f_1 f_2}{\xi}}\right)} \quad (7.7)$$

where $\kappa = \sqrt{C/3}$ and $f_1, f_2$ are the solutions of the quadratic equation:

$$\frac{8C}{6} f^2 + 2Bf + 4A = 0 \quad (7.8)$$

When $f_1 \to f_2$, we get flat-top solutions as has been shown by Akhmediev et al. [61]. For any other arbitrary value of $f_1$ and $f_2$ one obtains single-peak soliton solutions. When $f_1 \to f_2$, we have $B^2 = 16AC/3$ which in turn gives us:

$$\lambda^2 = (1 + \alpha)(1 - \beta^2) \left[1 - \frac{3}{16\alpha \beta^2 (1 + \alpha)} \left(\beta^2 - \alpha\right)^2\right] \quad (7.9)$$

This is the eigen value condition for small amplitude flat-top structures. This eigen value condition ($\lambda$ vs $\beta$ curve) in comparison with the numerical values for the flat-top structures has been shown in Fig. 7.5. It can be seen from the plot that the eigen values obtained analytically are consistent with the numerical values in the small $\beta$ limit. This is in accordance with the approximations of weakly relativistic solitons. Deviation at large $\beta$ values is due to the approximations made in deriving the analytical values. The flat-top solutions form in the neighbourhood of the
Figure 7.6: A plot of $\beta^2$ vs $\alpha$ for the flat-top solutions showing a linear relation between $\beta^2$ and $\alpha$. 
condition $\beta^2 \to \alpha$. This has been shown in Fig. 7.6, where the plot of $\beta^2$ vs $\alpha$ for various flat-top solutions have been shown as hollow circles.

The points fall on a straight line as expected. If we look at the coefficients $A$, $B$ and $C$ carefully, we find that these coefficients have the same order for any value of $\alpha$ for $\beta^2 \to \alpha$, where the flat-top structures form. Thus, at $\beta \approx \sqrt{\alpha}$, the boundary which separates dark and bright solitons in reference [46] there exists an infinitesimal domain of $\beta_2 = \alpha + \delta$ where the flat-top solitons form.

We next address the question of the stability of these flat-top solutions. Analytically this can be addressed by using the Vakhitov-Kolokolov criteria [62, 63]. In this case, the soliton solution will be unstable if

$$\frac{dP_0}{dA} > 0$$  \hspace{1cm} (7.10)

where $P_0 = \int_{-\infty}^{\infty} R^2 d\xi$. The expression for $P_0(\lambda)$ for the analytical flat-top soliton solution from Eq. (7.7) can be evaluated and is given by

$$P_0(\lambda) = \frac{2k_1}{\kappa}$$  \hspace{1cm} (7.11)

where $k_1$ is a positive constant. Here, we have taken positive values of $f_1$ and $f_2$ and $f_1 < f_2$. Thus

$$\frac{dP_0}{dA} = \frac{2k_1}{\kappa^2} \left( \frac{1}{2\sqrt{3C}} \right) \frac{dC}{dA} > 0$$  \hspace{1cm} (7.12)

Hence, according to the condition (7.10), the flat-top soliton solutions turn out to be unstable. We have also evaluated $P_0$ for the exact numerical soliton solutions and checked its variation with respect to the parameter $A$. This has been shown in Fig. 7.7. The curve of $P_0$ vs $A$ clearly demonstrates that $dP_0/dA > 0$, showing that the solutions can be unstable.
Figure 7.7: A plot of $P_0$ vs $A$ for the flat-top solutions showing a positive slope.

In the next section, we describe the dynamical trait of this particular variety of solution in detail. Our numerical simulation studies demonstrate that the flat-top solutions persist for several plasma periods. However, later they exhibit a development of an instability as a result of back scattering process.

7.3 Dynamical evolution of the flat-top solutions

For the numerical simulation studies, electron and ion continuity and parallel momentum equations have been solved using the flux corrected scheme of Boris et al. [64]. The second order time differentiation for the vector potential has been tackled by separating it into two first order equations. We choose the field profile of the flat-top solution as our initial condition for investigation.

The various stages of the evolution of the fields $R$ and $\varphi$ for a particular flat-top
Figure 7.8: The evolution of the fields $R$ and $\varphi$ for a flat-top solution with $\beta = 0.11, \lambda = 0.996807$ has been shown at various times. The appearance of the unstable perturbations at the front edge of the structure and then propagating backwards can be observed. The entire structure disintegrates subsequently as a result of this instability.

solution with $\beta = 0.11, \lambda = 0.996807$ has been shown in the subplots of Fig. 7.8. One observes that the solution propagates without perceptible distortion for $> 100$ plasma periods. At a later time it can be observed that the front end of the solution gets distorted. The disturbance seems to travel backwards, grows and engulfs the entire solution at later time. In Fig. 7.9, the amplitude of the perturbed fields $R$ and $\varphi$ at $t = 100$ electron plasma periods has been shown.

The thin dashed dotted line shows the electron density profile of the original
Figure 7.9: A comparison of the perturbed scalar $\varphi$ and vector $R$ potential in space which shows that their length scales are typically identical. The thin dashed dot line shows the original flat-top structure. It can be seen that the perturbations typically maximize at the front edge of the flat-top structure.
flat-top solutions. This has been shown to place the location of the perturbation with respect to the original structure at this time. It is interesting to note that the perturbed scalar $\varphi$ and vector $R$ potentials have typically identical scales.

Figure 7.10: Schematic showing the mechanism of forward and backward scattering in 1-D

Let us now comment on the possible instability mechanism which is responsible for the break up of the flat-top solutions. If we treat the light wave associated with the structure as the pump wave, then under the constraint of 1-D dynamics it can either suffer a forward and/or backward scattering [see Fig. 7.10]. Such a scattering process can generate either a plasma wave and/or a wave associated
with ion dynamics. In the case of forward scattering (the scattered radiation being of almost similar frequency), the scalar potential reflecting the scattered plasma and/or ion wave will have a wave length which would be much longer than the wavelength of the scattered radiation field. In our case, we see from Fig. 7.9 that this is not the case. The two scales are almost identical. This suggests that it is a backward scattering process. We now address the question whether the instability scatters a plasma wave and/or a wave associated with ions. The plot in Fig. 7.11 shows that the ion and electron perturbed densities are in phase.

This suggests that the scattering is from a slow wave associated with ion response. There is, however, no conventional ion wave that can be supported in a cold plasma medium. We feel that the quiver velocity of the electrons play the role of effective temperature for the ion wave produced in the medium which scatters the pump radiation. This suggests that the instability associated with the flat-top solitons is essentially a Brillouin backscattering process.

To put this assertion on a firmer footing, we evaluate the numerical growth rate for various flat-top solutions identified by various distinct values of the group velocity. This has been shown in Fig. 7.12 by triangular data points. We have also
Figure 7.11: The perturbed ion and electron densities during the linear phase of the instability has been shown. The figure clearly shows that the density perturbations are in phase.
Figure 7.12: A comparison of the numerically obtained growth rate with the analytical expressions [15–17].
alongside shown the analytical growth rate of the Brillouin backscattering process obtained from the expression of Liu et al. [15]. In the strong field limit, the growth rate for quasi-mode of Brillouin scattering is given by:

\[
\Gamma_{Krue} = \left(\frac{\sqrt{3}}{2}\right)\left(k_0^2 v_{os}\omega_{pl} \omega_0^2 / 2\omega_0\right)^{1/3}
\]

\[
\Gamma_{Kaw} = 2^{1/3}\left(v_{os}^2 \omega_{pl} \omega_0^2 / \omega_0\right)^{1/3}
\]

\[
\Gamma_{Liu} = \left(\sqrt{3}/2\right)^{1/3}\left(k_0 v_{os}\omega_{pl}\omega_0^2 \right)^{1/3}
\]

The analytical and numerical growth rates show a decent match. We notice that the approximate analytical growth rate expression obtained by different authors [15–17] for this instability, differ from each other typically by similar order, due to the nature of the approximation. In the light of which the agreement between numerical and analytical estimates are fairly reasonable.

### 7.4 Summary

The nonlinear exact solutions of the coupled laser plasma system obtained by ignoring ion response predict the existence of stationary as well as slowly moving structures. For these structures, it would be incorrect to a-priori neglect the ion dynamical response. The incorporation of ion response rules out the existence of static solutions. Solutions are permissible only beyond a certain critical value of group speed. For group speeds below this value there are no permissible bright solitons. At the critical group velocity, a new kind of structures are permitted by the equations. These structures have a flat and broad spatial profile for all concerning fields. A detailed characterization of these flat-top solutions has been provided in this chapter.
We have investigated in this chapter the dynamical evolution of these flat-top solutions. It is observed that these solutions survive for several plasma periods but ultimately develop an instability which breaks the structure. This particular destabilization process has been identified as the backward Brillouin scattering process.

It should be noted that the coupled laser plasma system permits a wide variety of solutions. These structures can have practical relevance provided questions related to their accessibility, stability and the time scale of growth for unstable case are understood and explored thoroughly. Our dynamical evolution study has been motivated towards addressing these issues. In an earlier work [56], it has been shown that the high amplitude multiple peak solutions are unstable to forward Raman scattering process. In the present work, we have shown that the flat-top variety of solutions observed with ion dynamical response develop a backward Brillouin scattering instability.