CHAPTER IV

INTERMITTENCY AND
MULTIFRACTALITY

IN THE FRAGMENTATION OF Mg
PROJECTILE NUCLEI
4.1 Introduction

Nucleus-nucleus collisions at intermediate energy offer various possibilities to produce hot nuclei which often undergo break-up into smaller pieces having charges $Z_{pp} \geq 2$, resulting nuclear multifragmentation, where the excitation energy is near the binding energy. Within this region lies the possibility of seeing a phase transition and critical phenomena in the nuclear matter limit. One of the most important challenges of heavy ion physics is the identification and characterization of nuclear liquid-gas phase transition believed to be the underlying mechanism of nuclear multifragmentation process [1-2]. The striking characteristics of the systems undergoing continuous phase transition that might have taken place in the final stage of fragmentation of heavy ion collisions are believed to be the occurrences of fluctuations of the fragments charge (mass) distribution, that exist on all length scales in a small range of the control parameter. Such fluctuation may diverse or even tend to vanish near some critical value of the control parameter. A number of works on multifragmenting system at low energy $\sim 1$ AGeV [3-7] over all impact parameters exhibited large scale fluctuations. In the study of heavy ion collision, numerous techniques have been developed to analyze the fluctuations and the correlations for various physical quantities. In particular, one of the most powerful and promising possibilities seems to be the analysis of event-by-event data in terms of intermittency which is a statistical concept initially developed to study turbulent flows [8-9]. Intermittency in physical systems is studied by examining the scaling properties of the moments of the distributions of relevant variables over a range of scales. Białas and Peschanski first introduced the concept of intermittency to the study of dynamical fluctuations in the density distributions of particles produced in high energy collisions. To examine the intermittent pattern of fluctuations Białas and Peschanski proposed the method of estimation of scaled factorial moments which has the advantage of quantifying dynamical fluctuations without the spurious influence of statistical fluctuations [8,10-12].

To understand the underlying physics of hadronization in QGP-hadron type phase transition, several works have been subsequently carried out to study the dynamical fluctuation of the produced particles using SFM technique. There exists
an abundant evidence of power law behaviour of produced particles in experimental data of $e^+e^-$ annihilation [13-15], muon-hadron [16-17], hadron-nucleus [18-19], and nucleus-nucleus collision [18-22]. Later the technique of scaled factorial moment was also applied to study non-statistical fluctuations in the emission process of slow and fast target fragments. The investigations have been carried out with $^{16}$O-AgBr (2.1 AGeV), $^{16}$O-AgBr (60 AGeV), $^{32}$S-AgBr (200 AGeV), $^{28}$Si-AgBr (14.5 AGeV), $^{24}$Mg-AgBr (4.5 AGeV) [23-27]. All these analyses show the presence of intermittent type of fluctuations in the emission of slow target fragments.

However, no such studies have been reported so far regarding the spatial distribution of projectile fragments. Projectile fragments have the momentum per nucleon almost equal to that of the parent nucleus and hence they are essentially emitted inside a narrow forward angle around the direction of the incident beam and remain relativistic. Hence unlike the target fragments, the heavy fragments of the projectile nucleus are very closely spaced having a very small angular separation. It has therefore always been a challenging task to study any such physical quantity that is related to the spatial distribution of the PFs. To understand thoroughly the complicated mechanism of heavy ion collision one has to take into account how the produced particles as well as the fragments of both TFs and PFs coming out of an interaction are distributed in phase space.

In general, the constituents of the spectator part of the projectile as well as the spectator part of the target can be well separated at energies like this work. But the experimental work on high energy A-A collisions carried out with electronic detectors to study PFs have limited coverage in the pseudo rapidity range. Nuclear emulsion on the other hand is a global $4\pi$ detector and has the best spatial resolution (0.1 mrad) among all the detectors currently in use in experimental high energy physics [28-29]. Even the largest collaboration [30] that has been formed to study different characteristics of projectile fragments with active detectors has reported their limitation over the angular acceptance. Because of this advantage nuclear emulsion has been found to be a useful tool particularly to study those properties which are related to the spatial distribution of emitted particle. Intermittency in the high energy A-A collisions is one such property. In this work an attempt has been
made to study the emission spectra of projectile fragments in the light of intermittency (self similarity).

The concept of intermittency is in turn intimately connected to the fractal geometry of the object under investigation and hence the dynamics of the underlying physical process [31]. Mandelbrot [32], the pioneer, showed a new way of looking into the world of apparent irregularities or fractals. Fractal geometry allows us to mathematically describe systems that are intrinsically irregular at all scales. A fractal structure has the property that, if one magnifies a small portion of it, it still shows the same complexity as the entire system. Such behaviour of fractals is called 'scale symmetry'. Usually the term fractal is used to characterize systems with properties of self-similarity in general. If these properties can be described by a single exponent, one has a simple or homogeneous fractal, a monofractal. In a more complex case, the term multi-fractality is used when discussing generalized scaling. The most notable property of fractals is their dimensions. A formalism for treating fractal dimension and its generalization had already been developed and has been applied effectively to the study of intermittent behavior in turbulent fluid. The intermittency exponent $\phi_q$ characterizes the fractal structure of the distribution via the anomalous fractal dimensions $d_q$ as $d_q = \phi_q / (q - 1)$, where $q$ is the order of the moment. The anomalous fractal dimensions describe how the distribution changes with increasing resolution and reflect the fractal and multifractal structure of particle emission.

As mentioned earlier, there are phenomenological hints for intermittent behavior in the emission pattern of charged secondaries emitted from high energy nuclear collisions. It was Carruthers and Ming [33] who, possibly for the first time, investigated the fractal dimension in hadronic multiparticle production. Later Dremin [34] suggested the study of correlation dimension; Lipa and Buschbeck [35] considered other generalized dimensions. Hwa [36-37] then pointed out that in none of the above mentioned investigations a formalism could be developed for a systematic study of the fractal properties that can provide an effective means of describing a highly non-uniform rapidity distribution of produced particles. He then identified a new set of moment, called generalized moment, $G_q$ that can be
determined from particle multiplicities in narrow rapidity windows and drew some important inference on the nature of the dimensions $D_q$ that are generalizations of the fractal dimensions for multifractal sets. Hwa also discussed the general properties of the spectrum of scaling indices and indicated how it can provide an effective means of describing a highly non-uniform rapidity distribution.

Here in this chapter an attempt has been made to interpret the observed power law dependence of normalized factorial moments on the phase space bin size, which is a signature of the self similarity in fluctuation pattern of particle multiplicity, in terms of fractal geometry.

Meanwhile, Satz and coworkers [38-39] studied the moments of the distribution of the size of spin clusters in the two-dimensional Ising model and found that intermittency occurs around and can be associated with the critical point of that system. In particular, they point to the usefulness of the (factorial) moments as a method for studying fragmentation or decay mechanisms [40].

Later Płoszajczak and Tucholski first introduced the SFM analysis for the study of dynamical fluctuations in fragment size distributions [3, 41-43] in the break-up of high energy nuclei in the nuclear emulsion. They studied the break-up of $^{197}$Au nuclei at around 1 GeV/nucleon, and showed that the factorial moments of the charge distribution of the fragments increased like a power law with the increasing charge resolution, thus exhibiting the property of self similarity or otherwise the intermittency and concluded that the study of intermittency in nuclear fragmentation is relevant in the search for critical phenomena. Thus, it has been believed that cluster size distributions are intermittent at the critical point. A similar analysis, confirming the existence of intermittency in nuclear fragmentation, was later applied to the break-up of $^{238}$U and $^{131}$Xe nuclei with energies a few GeV/nucleon [4-9]. In this work an attempt has also therefore been made to study the possible signature of non thermal phase transition in the light of intermittency and self similarity in spatial as well as in the fragments charge distribution for 4.5 AGeV minimum biased $^{24}$Mg-Em interactions.
4.2 Mathematical formalism for intermittency analysis

Before presenting the details of the method of analysis related to the estimation of SFM one important technical difficulty in studying this parameter should be addressed.

In the study of the fluctuation in phase space variable, scaled factorial moments are estimated following two techniques - one, called horizontal averaging, takes into account the non-statistical part of fluctuation in spatial distribution of particles in an event while the other, called vertical averaging, characterizes dynamical fluctuation in event space. The technique of vertical averaging has the limitation of losing information about the fluctuation in spatial distribution in an event. The method of horizontal averaging though takes into account fluctuations in density distribution in phase space in an event, has the limitation of its dependence on the shape of the single particle density distribution spectrum and therefore needs to be corrected by a factor called Fialkowski factor [44] to make it shape independent [44-45]. However, the shape dependence of the horizontally averaged scaled factorial moments can also be eliminated by converting single particle density distribution spectrum from $\eta (\cos \theta)$ space to a distribution in $\chi(\eta)$, where $\chi(\eta)$ is a new cumulative variable [35,45-47] defined as:

$$\chi (\eta ) = \frac{\int_{\eta_{\text{min}}}^{\eta} \rho (\eta ) d\eta}{\int_{\eta_{\text{max}}}^{\eta_{\text{max}}} \rho (\eta ) d\eta}$$  (4.1)

In the above equation, the numerator corresponds to the total number of PFs which have $\eta$ values less than or equal to a particular value of $\eta$ and the denominator corresponds to the total number of PFs in the entire sample of data. Obviously, the new variable $\chi(\eta)$ should vary from 0 to +1.
4.2.1 Horizontally scaled factorial moment

Following the technique of Bialas and Peschanski [8,10] in the study of intermittency, the pseudorapidity interval $\Delta \eta$ is divided into $M$ bins of equal size, $\delta \eta = \Delta \eta / M$.

Let $n_m$ be the number of particles in the $m^{th}$ bin, where $m$ can take the values from 1 to $M$.

For a single event, the $q^{th}$ order scaled factorial moment is defined as [8,10]:

$$
F_q = M^{q-1} \sum_{m=1}^{M} \frac{n_m (n_m - 1) \ldots (n_m - q + 1)}{n(n-1) \ldots (n-q+1)}
$$

(4.2)

where $n$ is the total number of particles in the event in the pseudorapidity interval, and $M$ is the number of bins in which $\chi(\eta)$ space is divided into bins of equal size:

$$
d\chi = \frac{\chi_{\text{max}} - \chi_{\text{min}}}{M} = \frac{1}{M}
$$

(4.3)

Here, $n = \sum_{m=1}^{M} n_m$.

For an ensemble of events having varying multiplicities, the expression for scaled factorial moment is modified as [48]:

$$
F_q = M^{q-1} \frac{1}{<n>^q} \sum_{m=1}^{M} \frac{n_m (n_m - 1) \ldots (n_m - q + 1)}{<n>}
$$

(4.4)

Here, $<n>$ represents the mean multiplicity of projectile fragments of the population in the full $\chi(\eta)$ space and is defined as:

$$
<n> = \frac{1}{N_{\epsilon \nu}} \sum_{i=1}^{N_{\epsilon \nu}} n_i
$$

(4.5)
On averaging over the number of events in the data sample, the horizontally averaged normalized or scaled factorial moments is expressed by the following relation:

\[
<F>_q = \frac{1}{<n>_q^s} \left< \frac{1}{M} \sum_{m=1}^{M} n_m (n_m - 1) \cdots (n_m - q - 1) \right>
\]  \hspace{1cm} (4.6)

where

\[
n = \frac{1}{M} \sum_{m=1}^{M} n_m
\]

Now dividing \( \chi(\eta) \) space into \( M \) bins one can find out the values of \(<F>_q \) and plot them for different values of \( M \).

4.2.2 Vertical scaled factorial moments

The vertical scaled factorial moments are defined by the relation:

\[
F_q^v = \frac{1}{<n>_q^s} \sum_{m=1}^{M} \frac{n_m (n_m - 1) \cdots (n_m - q + 1)}{<n_m>^q}
\]  \hspace{1cm} (4.7)

On averaging over all the events in the data sample we get the vertically averaged normalized or scaled factorial moment of \( q^{th} \) order which is given by:

\[
<F>_q = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{N_{ev}} \sum_{i=1}^{N_{ev}} n_m(n_m - 1) \cdots (n_m - q + 1) <n_m>^q
\]  \hspace{1cm} (4.8)

where, \( <n_m> = \frac{1}{N_{ev}} \sum_{i=1}^{N_{ev}} n_m \)

is the average multiplicity of the \( m^{th} \) bin over the whole data sample comprising the number of events, \( N_{ev} \).

It has been shown by Bialas and Peshanski that for purely statistical fluctuations \(<F>_q \) is essentially independent of the bin size or the number of bins.
M. However for the presence of any dynamical contribution to the fluctuation, the scaled factorial moment should follow a power law of the form \( F_q \propto M^{\phi_q} \).

Thus a linear plot of \( \ln < F_q > \) against \( \ln M \) with positive slope confirms self similar pattern in the emission spectra of the projectile fragments. The positive exponents, \( \phi_q \), referred to as intermittency indices, characterize the strength of the intermittency signal.

The intermittency indices can be obtained from the asymptotic behaviour represented by:

\[
\phi_q = - \frac{\Delta \ln < F_q >}{\Delta \ln M}
\] (4.9)

The power law behavior of the scaled factorial moments is envisaged [48-49] to be due to the fractal nature of the multi-particle spectra. Lipa and Buschbeck [35] have correlated the scaling behaviour of the factorial moments to the physics of fractal and multifractal objects through the relation:

\[
d_q = \frac{\phi_q}{q-1}
\] (4.10)

where \( d_q \) is called the anomalous dimension. It is used for the description of the fractal objects. Thus, using the above relation, the anomalous dimension \( d_q \), can be calculated directly from the intermittency index \( \phi_q \). The order independence of \( d_q \) indicates monofractal behaviour of multiparticle spectra, whereas an increase of \( d_q \) with \( q \) indicates the presence of multifractality in the emission spectrum.

4.3 Mathematical formalism for fractal and multifractal analysis

The self similarity observed in the power law dependence of scaled factorial moments reveals a connection between intermittency and fractality. The particle number density in each phase space bin depends on whether the resolution of the binning is larger or smaller than the angular separation between neighboring particles [50]. It has been found that if the resolution is of the order of the average...
separation of two neighboring particles in the phase space, then the binning of the phase space with that resolution may result in some empty bins. Considering the empty bins in the distribution are analogous to the holes then the set of non-empty bins would form a fractal set [50].

In fractal approach, it has been suggested that the nuclear interactions can be treated as geometrical objects with non-integer dimensions. Out of various methods that have been proposed to investigate the fractal structure, Hwa [36] was the first to provide the idea of the generalized multifractal moments $G_q$ to study the multifractality and self-similarity in multiparticle production. If the particle production process exhibits self-similar behavior, then a modified form of $G_q$ moment in terms of step function [37] shows the remarkable power law dependence on phase space bin size.

With this concept of fractality, the fractal moments $G_q$ have been defined to evaluate parameters which characterize the fractal properties.

In this approach, as mentioned earlier, a given pseudorapidity interval in an event, $\Delta \chi = \chi_{\text{max}} - \chi_{\text{min}}$ is divided into ‘M’ bins of equal size:

$$d\chi = \frac{\Delta \chi}{M}$$ (4.11)

If $n_m$ denotes the number of particles in the $m^{\text{th}}$ bin, and if non empty bins are also included, then ‘m’ would run from 1 to M. The total number of particles in an event is calculated using the following relation:

$$n = \sum_{m=1}^{M} n_m$$ (4.12)

The fraction of particles in the $m^{\text{th}}$ bin is given by, $p_m = n_m / n$. The quantity $p_m$ is a small number and fluctuates distinctly from bin to bin. The multifractal moment $G_q$ introduced by Hwa is now defined [36, 50] as:
where the summation is carried out over all the non empty bins only which constitute a fractal set and \( q \) is the order of the moment. Because of the very nature of the formulation of \( G_q \), i.e., summation over non empty bins only, the fractal moments can be calculated for any positive or negative integral or non integral order (value of \( q \)) and thus may take a dominant role over other multiplicity moments in revealing the dynamics of multiparticle production through the study of fluctuations in the density of produced particles. When averaged over all the events in a data sample in which the total number of events is \( N_{eV} \), \( <G_q> \) is expressed as:

\[
< G_q > = \frac{1}{N_{eV}} \sum_{i=1}^{N_{eV}} G_q
\]  

(4.14)

A given rapidity distribution is said to exhibit self-similar behaviour and hence of fractal nature if \( <G_q> \) exhibits a power law behaviour [50-51] over a range of small \( d\chi \) in the following manner:

\[
< G_q > \propto (d\chi)^{\tau_q}
\]  

(4.15)

The exponent \( \tau_q \) may be determined from the observed linear dependence of \( \ln <G_q> \) on \( \ln \chi \) using the relation:

\[
\tau_q = \lim_{dx \to 0} \frac{\Delta \ln < G_q >}{\Delta \ln d\chi}
\]  

(4.16)

One of the most basic properties of the fractals which describe the scaling behaviour is the generalized dimension \( D_q \), introduced by Hentschel and Procaccia [52]. \( \tau_q \) is related to \( D_q \) for all values of \( q \), by the following relation:

\[
D_q = \frac{\tau_q}{q-1}
\]  

(4.17)
Here, $D_0$ is the fractal dimension, $D_1$ is the information dimension and $D_2$ is the correlation dimension [36, 53-54]. If $D_q$ decreases with the increase of $q$, the emission pattern is said to be multifractal. On the other hand, if $D_q$ remains constant, then the emission pattern is referred to as monofractal [52,55].

Later by introducing a step function to suppress the low multiplicity events for which the statistical fluctuation is large, Hwa and Pan [37] proposed a modified $G_q$ moment to investigate the fractal properties of the emission spectra of different charged secondaries. In this investigation, the generalized moments $G_q$ is first estimated in $\chi (\cos \theta)$ space using [24, 56] the relation:

$$G_q = \sum_{m=1}^{M} \left( \frac{n_m}{N_{q'}} \right)^q \theta (n_m - q)$$

(4.18)

Here $\theta(n_m - q)$ is a step function such that $\theta(n_m - q) = 1$ for $n_m \gg q$, and $\theta(n_m - q) = 0$ for $n_m < q$.

The vertically averaged horizontal moment, $<G_q>$, is then calculated as:

$$<G_q> = \frac{1}{N_{q'}} \sum_{i=1}^{N_{q'}} G_q$$

(4.19)

A power law dependence of $<G_q>$ on the phase space bin size, or on the number of phase space bins $M$, of the form represented by the equation $<G_q> \propto M^{-\tau_q}$ indicates self similarity in the emission pattern [24,37,56]. The exponent $\tau_q$, called the fractal index, can be obtained from the asymptotic behavior.

It is worthwhile to mention here that up to this stage of analysis of multifractality, no technique has so far been adopted to filter out the statistical part of the fluctuation. Since $<G_q>$ of Eq.(4.19) contains contribution from both statistical as well as dynamical components, it is therefore necessary to extract the dynamical information from the mixture of the two.
To calculate the statistical contribution to $\langle G_q \rangle$, $n$ particles are distributed randomly in the specified phase space. For each event, fractal moment is calculated with redistributed particles and the $\langle G_q^* \rangle$ is obtained by averaging the statistical $G_q$s. The dynamical component of $\langle G_q \rangle$ is then estimated by using the following formula given by Chiu [57]:

$$\langle G_q^{\text{dyn}} \rangle = \left( \frac{\langle G_q \rangle}{\langle G_q^* \rangle} \right) M^{1-q} \quad (4.20)$$

If $\langle G_q \rangle$ contains purely statistical information, then

$$\langle G_q^{\text{dyn}} \rangle = M^{1-q} \quad (4.21)$$

Under such condition:

$$\tau_q^{\text{dyn}} = q - 1 \quad (4.22)$$

Thus, any deviation of $\tau_q^{\text{dyn}}$ from $q - 1$ indicates that $\langle G_q \rangle$ contains dynamical information.

The self-similarity of a fractal object is characterized by the generalized dimension $D_q$ which is now defined by the relation:

$$D_q = \frac{\tau_q^{\text{dyn}}}{q - 1} \quad (4.23)$$

### 4.4 Non-thermal phase transition for spatial and size distribution of projectile fragments

In heavy ion collision a type of transition in which the new phase may not exhibit thermodynamical behavior often referred to as non-thermal phase transition. It is already shown in ref. [58-59] the intermittent behavior in the final state of multiparticle production in a heavy-ion collision may be a projection of non-thermal phase transition believed to occur during the evolution of the collision which in turn would be responsible for the occurrence of anomalous events. It is convenient and
better to find a suitable observable which can be measured experimentally and can provide information about phase transition (thermal or nonthermal). It has been assumed that a self similar cascade of multiparticle system is not consistent with the creation of particle during one phase, but instead requires a non-thermal phase transition [13, 60-61]. The intermittency exponent $\phi_q$ is related to a parameter $\lambda_q$ which provides the signature of non-thermal phase transition. We can study the non-thermal phase transition with the help of the parameter,

$$\lambda_q = (\phi_q + 1)/q, \quad (4.24)$$

where $\phi_q$ is the intermittency index. The condition that such non-thermal phase transition may occur is that the function $\lambda_q$, is predicted to have a minimum value at $q = q_\alpha$, where $q_\alpha$ need not necessarily be an integer. Among the two different regions $q < q_\alpha$ and $q > q_\alpha$, the numerous small fluctuation dominates the region $q < q_\alpha$ but in the region $q > q_\alpha$ dominance of small number of very large fluctuations occurs. There is a co-existence of the liquid phase of the many small fluctuation and the dust phase of a few grains of very high density fluctuation, depending on whether we probe the system by a moment of order $q < q_\alpha$ or $q > q_\alpha$ respectively.

4.5 Results and discussion

Part A: On the spatial distribution of PFs

4.5.1 Angular distribution of charged projectile fragments

The measurements on angular distribution of various charged secondaries emitted from each interaction were carried out according to the procedure already mentioned in section 2.7.3.

In the study of the relativistic heavy ion collision, the spatial distribution of emitted charged secondaries are often defined in terms of what is called ‘pseudorapidity’ $\eta$ that is related with the emission angle $\theta$ through the relation:

$$\eta = -\ln \{\tan (\theta/2)\} \quad (4.25)$$
The angular distribution of various charge projectile fragments in the pseudorapidity space is plotted in Fig. 4.1. It can readily be seen from this plot that, as expected, the PFs are emitted in the extreme forward angle following a Gaussian distribution.

As is evident from the very definition, the shape of the single particle density distribution spectrum in $\chi(\eta)$ space should be flat in nature. In Fig. 4.2, we present the frequency distribution of emitted projectile fragments of the work in $\chi(\eta)$ space. As expected, the best fitted lines for the data points are found to be flat in nature, with slope $m = 0$ and correlation coefficient $R = 1$. 

Fig. 4.1 Angular distribution of projectile fragments in $\eta(\cos \theta)$ space.
4.5.2 Scaled Factorial Moment Analysis for spatial distribution of projectile fragments

4.5.2.1 Variation of $\ln <F_q>$ with $\ln M$

Fig. 4.3 shows a plot of $\ln <F_q>$ against $\ln M$ in $\chi(\eta)$ space for different order of moments $q = 2$–$5$ for all the projectile fragments with $Z_{pf} \geq 1$ for $4.5$ AGeV $^{24}$Mg projectile with emulsion targets. The straight lines drawn are the least square fit to experimental data points with Pearson correlation coefficients $R$ equal to $0.875$, $0.917$, $0.952$ and $0.954$ for $q = 2,3,4$, and $5$ respectively. From this figure, the SFM is observed to increase linearly with decreasing bin widths, indicating thereby the presence of intermittent behavior in the emission spectrum.
4.5.2.2 Variation of $\phi_q$ with $q$

The intermittency indices $\phi_q$ which characterize the strength of intermittency effect have been obtained for the linear dependence of $\ln \langle F_q \rangle$ on $\ln M$. The values of slope parameter $\phi_q$ obtained for the interactions of 4.5 AGeV $^{24}$Mg nuclei with emulsion target are listed in table 4.1. Fig. 4.4 shows $\phi_q$ versus $q$ plot. The errors indicated in the plot are the statistical errors only. The slopes are observed to increase with the order of the moments.
4.5.2.3 Variation of anomalous dimension $d_q$ with $q$

As stated earlier, Lipa and Buschbeck [35] for the first time had correlated the scaling behavior of factorial moments to the physics of fractal and multifractal objects. They pointed out that the intermittency index $\phi_q$ has a special significance from the point of view that the anomalous dimension, $d_q$, which is used for the description of the fractal objects, can be directly computed from the intermittency index $\phi_q$ using the Eq. (4.10).

Fig. 4.4 Variation of the exponent $\phi_q$ with the order of the moments $q$ for Mg -Em interactions at 4.5 AGeV.
As mentioned earlier, the order independence of $d_q$ is associated with the monofractal behavior of multiparticle spectra whereas an increase will indicate multifractality. In Fig. 4.5, the variations of anomalous dimension $d_q$ with the orders of the moment $q$ are shown for the emission spectra of PFs. From this figure it can be readily seen that $d_q$ increases linearly with $q$ thereby indicating multifractal pattern of the emission spectra of the projectile fragments.

Fig. 4.5 Variation of anomalous dimension $d_q$ with $q$ for Mg-Em interactions at 4.5 AGeV.
Table 4.1 Values of the slope parameters $\phi_q$ for the collisions of 4.5 AGeV Mg nuclei with emulsion targets.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\phi_q$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$0.088 \pm 0.07$</td>
<td>0.986</td>
</tr>
<tr>
<td>3</td>
<td>$0.349 \pm 0.074$</td>
<td>0.977</td>
</tr>
<tr>
<td>4</td>
<td>$0.550 \pm 0.05$</td>
<td>0.966</td>
</tr>
<tr>
<td>5</td>
<td>$0.792 \pm 0.078$</td>
<td>0.967</td>
</tr>
</tbody>
</table>

4.5.2.4 Variation of $\lambda_q$ with $q$

The variation of $\lambda_q$, estimated from the values of $\phi_q$ of the table 4.1, with $q$ is shown in Fig. 4.6. From this plot no observed minimum at $q = q_c$ could be seen indicating no evidence of non thermal phase transition in the spatial distribution of projectile fragments.

![Graph showing variation of $\lambda_q$ with order $q$](image_url)

**Fig. 4.6** Variation of $\lambda_q$ with the order of the moment $q$ for Mg-Em interactions at 4.5 AGeV.
4.5.3 Fractal Analysis in the spatial distribution of projectile fragments

4.5.3.1 Variation of $\ln <G_q>$ with $\ln M$

To calculate the statistical contribution to $<G_q>$, equal number of events are generated in $\cos \theta$ space by a random number generator with $\cos \theta$ values lying between -1 and +1. The $\cos \theta$ values are then converted into $\chi(\eta)$ values using Eq. (4.1). $<G_q^\sigma>$ is then estimated using Eqs. (4.18) and (4.19). The variation of $\ln <G_q>$ with $\ln M$ for $q = 2, 3$ and 4 for experimental data as well as for random number generated events are shown in Fig. 4.7. The solid lines are for the experimental data points and the dotted lines are for the equal number of generated events.

Fig. 4.7 Variation of $\ln <G_q>$ with $\ln M$ for experimental and random number generated events.
4.5.3.2 Variation of $\tau_q^{\text{exp}}$ with $q$

The exponents $\tau_q^{\text{exp}}$, obtained from the portion of linear dependence of $\ln < G_q >$ with $\ln M$ as a function of the order of the moment $q$, for experimental data are plotted in the Fig. 4.8. The values of the exponents are found to increase with $q$.

![Graph showing variation of $\tau_q^{\text{exp}}$ with $q$ for Mg-Em interactions at 4.5 AGeV.](graph.png)

**Fig. 4.8** Variation of $\tau_q^{\text{exp}}$ with $q$ for Mg-Em interactions at 4.5 AGeV.

4.5.3.3 Deviation of exponents from $q-1$

Since the exponent $\tau_q^{\text{exp}}$ as obtained above, are estimated from the experimental data set, it contains statistical component also. The dynamical component of exponent $\tau_q$ is then estimated using the relation:

$$
\tau_q^{\text{dyn}} = \tau_q^{\text{exp}} - \tau_q^{\text{st}} + q-1
$$

(4.26)
where $\tau_{st}^q$ represents the slopes of randomly generated data obtained from $\ln < G_q >$ vs $\ln M$ plot. Fig. 4.9 shows the deviation of $\tau_{q}^{dyn}$ from $q-1$.

Fig. 4.9 Deviation of exponents from $q-1$.

4.5.3.4 Variation of $D_q^{dyn}$ with $q$

The value of generalized dimensions $D_q$ for various values of $q$ have been estimated using Eq. (4.23) for the emission spectrum of projectile fragments with $\chi(\eta)$ as phase space variable in $^{24}$Mg-Em interactions at 4.5 AGeV and their variations with different order of moments $q = 2 - 4$ are shown in Fig. 4.10. It is observed from the figure that the generalized dimensions $D_q$ decreases linearly with $q$, indicating multifractality in the emission spectra of projectile fragments.
Fig. 4.10 Variation of $D_q^{\text{dyn}}$ with $q$ for Mg-Em interactions at 4.5 AGeV.

The values of $\tau_q^{\text{exp}}$ for experimental data points, $\tau_q^{\text{st}}$ for generated events, obtained from the slopes of the best fitted lines, $\tau_q^{\text{dyn}}$, $q - 1 - \tau_q^{\text{dyn}}$ and $D_q^{\text{dyn}}$ for different orders of moment are listed in table 4.2.
Table 4.2 Values of $\tau_q$ for spatial distribution of projectile fragments.

<table>
<thead>
<tr>
<th>q</th>
<th>$\tau_q^{\exp}$ (R)</th>
<th>$\tau_q^{\st}$ (R)</th>
<th>$\tau_q^{\dyn}$</th>
<th>$q-1-\tau_q^{\dyn}$</th>
<th>D_q</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.711 ± 0.018</td>
<td>0.905 ± 0.017</td>
<td>0.805 ± 0.035</td>
<td>0.19415</td>
<td>0.805</td>
</tr>
<tr>
<td></td>
<td>(0.991)</td>
<td>(0.995)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.207 ± 0.072</td>
<td>1.87 ± 0.175</td>
<td>1.337 ± 0.247</td>
<td>0.6625</td>
<td>0.668</td>
</tr>
<tr>
<td></td>
<td>(0.985)</td>
<td>(0.974)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.77 ± 0.073</td>
<td>2.97 ± 0.174</td>
<td>1.8 ± 0.247</td>
<td>1.2</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>(0.989)</td>
<td>(0.986)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The deviation of $\tau_q^{\dyn}$ from $q-1$ is clear from this table and this deviation is more as we go to the higher order of moments. This indicates that $G_q$ contains information about dynamical contribution to the fluctuations.

Part B: On the charge distribution of Projectile Fragments

4.5.4 The Fragments multiplicity distribution

The multiplicity distribution of all the charged fragments with $Z_{PF} = 1-12$ is shown in the Fig. 4.11.
4.5.5 Scaled Factorial Moment Analysis for fragments charge (mass) distribution.

Following Eq.(4.6) the scaled factorial moment analysis has been performed for searching intermittency signal for fragments charge distribution with the experimental data of this work. Here \( M \) is the total number of bins as the fragment charge interval \( \Delta s \) (1-12) is divided into bins of equal width \( \delta s = \Delta s / M \). \( n \) is the fragment multiplicity in the interval \( \Delta s \). For non flat fragment multiplicity distribution varying within a finite bin of width \( \delta s \) introduces an extra \( M \)-dependent correction factor \( R_q \) which is given by:

\[
R_q = \frac{1}{M} \sum_{m} M^q \frac{<n_m>}{<n>^q} \tag{4.27}
\]

Thus, \( <F_q>/R_q = <F_q>_c \) measures the contribution of dynamical fluctuations. In doing so, one must be careful in selecting the smallest bin, which must not be
smaller than the charge resolution of the detector \[62\]. If self-similar fluctuations exist at all scales of \(\delta s\), the corrected factorial moment of the order \(q\) is given by \(\langle F_q \rangle_c = (\Delta s/\delta s)^\phi_q\). The exponent \(\phi_q\) is the slope characterizing a linear rise of \(\ln \langle F_q \rangle_c\) with \(-\ln \delta s\) for all bins of \(\delta s\), which increases with the increasing order \(q\) of the moment.

4.5.5.1 Variation of \(\ln \langle F_q \rangle_c\) with \(-\ln \delta s\)

Plots of \(\ln \langle F_q \rangle_c\) against \(-\ln \delta s\) for different orders of moments are shown in Fig. 4.12. It can be readily seen from this plot that the moment for the fragment multiplicity distribution continue to increase according to power law with the decreasing bin width \(\delta s\) variable, thereby indicating the intermittent pattern. The errors shown in this plot are standard deviation and the straight lines drawn are the best fitted lines for the respective data points. The different values of intermittency indices, \(\phi_q\) along with the correlation coefficient \(R\) values for the present work and the values reported earlier are shown in table 4.3.

![Fig. 4.12 Variation of \(\ln \langle F_q \rangle_c\) with \(-\ln \delta s\) for charge distribution in the interactions of Mg nuclei with emulsion targets at 4.5 AGeV.](image-url)
Table 4.3 Values of the slope parameters $\phi_q$ for charge distribution of projectile fragments in Mg-Em interaction at 4.5 AGeV.

<table>
<thead>
<tr>
<th>System</th>
<th>Energy</th>
<th>$\phi_q$ for $q=2$ (R)</th>
<th>$\phi_q$ for $q=3$ (R)</th>
<th>$\phi_q$ for $q=4$ (R)</th>
<th>$\phi_q$ for $q=5$ (R)</th>
<th>$\phi_q$ for $q=6$ (R)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mg-Em</td>
<td>4.5</td>
<td>0.080±0.017 (0.940)</td>
<td>0.216±0.035 (0.963)</td>
<td>0.571±0.137 (0.924)</td>
<td>1.121±0.334 (0.888)</td>
<td>-</td>
<td>PW</td>
</tr>
<tr>
<td>Kr-Em</td>
<td>0.95</td>
<td>0.011±0.002 (0.892)</td>
<td>0.038±0.004 (0.949)</td>
<td>0.081±0.007 (0.968)</td>
<td>0.134±0.010 (0.974)</td>
<td>0.196±0.014 (0.976)</td>
<td>63</td>
</tr>
<tr>
<td>Au-Em</td>
<td>0.1-1</td>
<td>0.010±0.011</td>
<td>0.027±0.017</td>
<td>0.049±0.023</td>
<td>0.073±0.030</td>
<td>-</td>
<td>64</td>
</tr>
<tr>
<td>Au-Em</td>
<td>10.6</td>
<td>0.005±0.004</td>
<td>0.015±0.005</td>
<td>0.026±0.007</td>
<td>0.039±0.009</td>
<td>-</td>
<td>64</td>
</tr>
<tr>
<td>U-Em</td>
<td>0.96</td>
<td>0.0068±0.0002</td>
<td>0.010±0.0003</td>
<td>0.013±0.0004</td>
<td>0.0163±0.0006</td>
<td>0.0191±0.0006</td>
<td>4</td>
</tr>
</tbody>
</table>

4.5.5.2 Variation of $\phi_q$ with $q$

Variation of the intermittency exponent $\phi_q$ with $q$ is plotted in the Fig. 4.13 and is found to increase exponentially with the order of the moment $q$ following the relation:

$$y = A_1 \exp(-x/t_1) + y_0$$  \hspace{1cm} (4.28)
The solid line represents the best fit to the data points with the values of the different parameters as, $A_1 = 0.27 \pm 0.013$, $y_0 = -0.046 \pm 0.039$, $t_1 = -1.31 \pm 0.191$ and $R^2 = 0.985$ respectively. Such an exponential increase of $\phi_q$ clearly indicates strong intermittency in the fragments charge distribution of Mg-Em interaction.

### 4.5.5.3 Variation of anomalous dimension $d_q$ with $q$

Following equation (4.10) the variation of anomalous dimension $d_q$ with $q$ is plotted in Fig. 4.14. for fragments charge distribution for the present set of experimental data. From this figure it can be readily seen that $d_q$ increases exponentially with $q$ thereby indicating multifractal pattern in the size (charge) of the projectile fragments as well. The $R^2$ value for the fitted parameter is found to be 0.978.
Fig. 4.14 Variation of anomalous dimension $d_q$ with $q$ for Mg-Em interactions at 4.5 AGeV.

4.5.5.4 Variation $\lambda_q$ with $q$

Fig. 4.15 shows the variation of $\lambda_q$ with $q$ for the size (charge) distribution of projectile fragments in the interactions of Mg nuclei with emulsion targets at 4.5 AGeV. It is interesting to note that a distinct minimum at $q \approx 3.5$ is observed; the observed minimum in the variation of $\lambda_q$ with $q$ may be an indication for the occurrence of non-thermal phase transition in the size distribution of projectile fragments.
4.5.6 Fractal Analysis of the charge distribution of projectile fragments

4.5.6.1 Variation of $\ln <G_q>$ with $-\ln \delta s$

The data set for the various charges of projectile fragments has been analyzed further, applying the same concept of section 4.3 for evaluating $G_q$ moments. Here also if the charge distributions have the fractal structure, then the $G_q$ moment should follow a power law i.e.,

$$<G_q> \propto (\delta s)^{-\tau_q}$$  \hspace{1cm} (4.29)

where $\tau_q$ is fractal index or mass (charge) exponent. From the linear dependence of $\ln <G_q>$ on $\ln \delta s$, $\tau_q$ can be calculated as:
To calculate the statistical contribution to \( \langle G_q \rangle \), equal number of events are also generated by random number generator with charge of the PFs lying between 1 to 12 for Mg projectile beams. \( \langle G_q'' \rangle \) is then calculated for uncorrelated projectile fragments in randomly generated events. From Fig. 4.16, a significant difference could easily be seen between the experimental and random data for the fragments of Mg projectile. The dynamical part of \( \langle G_q \rangle \) can be determined from [57]:

\[
\langle G_q'' \rangle = \frac{\langle G_q \rangle}{\langle G_q'' \rangle} (\Delta \delta)_{q-1}^{-1}
\]

(4.31)

Fig. 4.16 Variation of \( \ln \langle G_q \rangle \) with \( -\ln \delta_s \) for fragments charge distribution in 4.5 AGeV Mg nuclei with emulsion target.
4.5.6.2 Variation of $\tau_q^{\exp}$ with $q$

The exponents $\tau_q^{\exp}$, obtained for the portion of linear dependence of $\ln <G_q>$ with $-\ln \delta s$ is plotted in Fig. 4.17 as a function of the order of the moment $q$. The values of the exponents are found to increase linearly with $q$.

![Graph showing variation of $\tau_q^{\exp}$ with $q$]

Fig. 4.17 Variation of $\tau_q^{\exp}$ with $q$ for fragments charge distribution in 4.5 AGeV Mg nuclei with emulsion target.

4.5.6.3 Deviation of mass exponents from $q^{-1}$

The dynamical component of exponent $\tau_q^{\text{dyn}}$ is calculated in a similar way as done in section 4.5.3.3. In the present calculation, $\tau_q^{\text{st}}$ represents the slopes of
randomly generated data obtained from $\ln <G_q>$ vs $-\ln \delta s$ plot. Fig. 4.18 shows the deviation of $\tau_q^{\text{dyn}}$ from $q-1$.

4.5.6.4 Variation of $D_q^{\text{dyn}}$ with $q$

The value of generalized dimensions $D_q$ for various values of $q$ have been estimated using Eq. (4.23) for the charge distribution of projectile nuclei with emulsion target, and their variations with different order of moments $q = 2 - 4$ are shown in Fig. 4.19. It is observed from the figure that the generalized dimensions $D_q$ decreases linearly with $q$ for the present set of data. Thus it is evident that there is a signature of multifractality in the charge distribution of projectile fragments.
Fig. 4.19 Variation of $D_{q}^{\text{dyn}}$ with $q$ for fragments charge distribution in 4.5 AGeV Mg nuclei with emulsion target.
The values of $\tau_q^{\exp}$, $\tau_q^{st}$, $\tau_q^{dy}$, $q - 1 - \tau_q^{dy}$ and $D_q^{dy}$ for different orders of moment as obtained from the charge distribution of various projectile fragments are listed in table 4.4. It is observed from this table that the value of $\tau_q^{dy}$ deviates from $q - 1$ and this deviation is more as we go to the higher order of moments. This indicates that $G_q$ contains information about dynamical contribution to the fluctuations in the charge (mass) distribution of the fragments of projectile spectator.
References

28. V. Bradnova et al., JINR-LHE-0983-2.


