Chapter 1

Preliminaries

1.1 Fuzzy Subsets

In 1965 Zadeh [63] mathematically formulated the fuzzy subset concept. He defined fuzzy subset of a non-empty set as a collection of objects with grade of membership in a continuum, with each object being assigned a value between 0 and 1 by a membership function. Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate or useful notions in describing the real life problems, because every object encountered in this real physical world carries some degree of fuzziness. Further the concept of grade of membership is not a probabilistic concept.

Definition 1.1.1. *By a fuzzy subset of a set X we mean a function from X into $[0,1]$. The set $[0,1]^X$, of all fuzzy subsets of X, is called the fuzzy power set of X.*

Definition 1.1.2. *Let μ be a fuzzy subset of a set X. For $t \in [0,1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a t-level subset of the fuzzy subset μ .*

Definition 1.1.3. *A fuzzy set of a set X is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is $t, (0 < t \leq 1)$*

then we denote this fuzzy point by x_t .i.e

When $Y \subseteq X$ and $\alpha \in [0,1]$, we define $\alpha_Y \in [0,1]^X$ as follows:

$$\alpha_Y(x) = \begin{cases} \alpha \; if \; x \in Y \\ 0 \; if \; x \in X - Y \end{cases}$$

If $Y = \{y\}$, then $\alpha_{\{y\}}$ is called a fuzzy point denoted by y_α.

Definition 1.1.4. *Let μ be a fuzzy subset of X. Its complement is denoted by μ^c and defined as the fuzzy subset $\mu^c : X \longrightarrow [0,1]$ such that $\mu^c(x) = 1 - \mu(x)$ for every $x \in X$.*

Example 1.1.1. *Let us consider the set $X = \{0,1,2,3,4\}$ and $\mu \in [0,1]^X$ be defined as follows:*

$\mu(0) = 1, \mu(1) = 0.3, \mu(2) = 0.1, \mu(3) = 0.6, \mu(4) = 0.5.$

Then $\mu^c(0) = 0, \mu^c(1) = 0.7, \mu^c(2) = 0.9, \mu^c(3) = 0.4, \mu^c(4) = 0.5$

Definition 1.1.5. *Let μ be a fuzzy subset of X.*

$\mu(X) or Im(\mu) = \{\mu(x) | x \in X\}$ is called the image of μ.

$\mu^* = \{x \in X | \mu(x) > 0\}$ is called the support of μ. We note that μ^* is a subset of the crisp set X.

Definition 1.1.6. *Let $\mu, \sigma \in [0,1]^X$. Union of μ and σ, denoted by $\mu \cup \sigma$, is a fuzzy subset of the set X defined as follows:*

$(\mu \cup \sigma)(x) = \mu(x) \vee \sigma(x), \forall x \in X$ *and*

the intersection of μ and σ denoted by $\mu \cap \sigma$ is defined as follows:

$(\mu \cap \sigma)(x) = \mu(x) \wedge \sigma(x), \forall x \in X.$

For an arbitrary family $\{\mu_i | i \in I\}$ of fuzzy subsets of X, where I is any non empty index set, we define union and intersection by:

$$(\bigcup \mu_i)(x) = \bigvee_{i \in I} \mu_i(x)$$

and

$$(\bigcap \mu_i)(x) = \wedge_{i \in I} \mu_i(x)$$

Definition 1.1.7. *Let μ and σ be two fuzzy subsets of X i.e.$\mu, \sigma \in [0,1]^X$. Then μ is said to be contained in σ, written as $\mu \subseteq \sigma$ if $\mu(x) \leq \sigma(x), \forall x \in X$. If $\mu(x) = \sigma(x)$ for every $x \in X$ then we say μ and σ are equal and write $\mu = \sigma$.*

Theorem 1.1.1. *For $\mu, \sigma \in [0,1]^X$*

(i) $\mu \subseteq \sigma, t \in [0,1]$ implies $\mu_t \subseteq \sigma_t$

(ii) $t \leq s; t, s \in [0,1]$ implies $\mu_s \subseteq \mu_t$

(iii) $\mu = \sigma \Leftrightarrow \mu_t = \sigma_t, \forall t \in [0,1]$.

Theorem 1.1.2. *Suppose that $\{\mu_i | i \in I\} \subseteq [0,1]^X$. Then for any $t \in [0,1]$*

(i) $\bigcup (\mu_i)_t \subseteq (\bigcup_{i \in I} \mu_i)_t$

(ii)$\bigcap (\mu_i)_t = (\bigcap_{i \in I} \mu_i)_t$.

Definition 1.1.8. *Let $f : X \to Y$ and $\mu \in [0,1]^X, \sigma \in [0,1]^Y$ then $f(\mu) \in [0,1]^Y$ and $f^{-1}(\sigma) \in [0,1]^X$ are defined as follows:*

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) | x \in X, f(x) = y\}, \ if \ f^{-1}(y) \neq \phi \\ 0, \ otherwise \end{cases}$$

and $f^{-1}(\sigma)(x) = \sigma(f(x))$

$f(\mu)$ *is the image of* μ *under* f *and* $f^{-1}(\sigma)$ *is the pre image of* σ *under* f.

Definition 1.1.9. *Let* f *be any function from a set* S *to a set* T *and let* μ *be any fuzzy subset of* S. *Then* μ *is called f-invariant if* $f(x) = f(y)$ *implies* $\mu(x) = \mu(y)$ *where* $x, y \in S$.

Example 1.1.2. *Let* μ *be a fuzzy subset of* Z_+ *defined by*

$$\mu(x) = \begin{cases} 1 \;\; if \;\; x \in \{2,4,6,8\ldots\} \\ 0, \; otherwise \end{cases}$$

Define $f : Z_+ \longrightarrow \{0,1,2,3\}$ *as follows:*

$f(x) = x \bmod 4$ *for all* $x \in Z_+$.

Then, μ *is f-invariant.*

Definition 1.1.10. *Let* μ *be a fuzzy subset of* R. *Then* μ *is said to have supremum property if for any subset* A *of* R, *there exists* $x \in A$ *such that* $\mu(x) = \vee\{\mu(a) : a \in A\}$.

Definition 1.1.11. *A mapping* $T : [0,1] \times [0,1] \longrightarrow [0,1]$ *is called a triangular norm [T-norm] if and only if it satisfies the following conditions:*

T1) $T(x,1) = T(1,x), \forall x \in [0,1]$.

T2) If $x \geq x^*, y \geq y^*$ *then* $T(x,y) \geq T(x^*,y^*)$.

T3) $T(x,y) = T(y,x), \forall x,y \in [0,1]$.

T4) $T(x,T(y,z)) = T(T(x,y),z), \forall x,y,z \in [0,1]$.

Note: The T-norm minimum (min T-norm) is defined by $T(x,y) = \min(x,y)$. Some other T-norms are $T_p(x,y) = xy, T_n(x,y) = \max(x+y-1,0)$ and

$$T_w(x,y) = \begin{cases} x, & \text{if } y=1 \\ y, & \text{if } x=1 \\ o, & \text{otherwise} \end{cases}$$

Every T-norm T^* satisfies the inequality:

$T_w(x,y) \leq T^*(x,y) \leq \min(x,y)$ and $T^*(0,0) = 0$

A triangular norm is called an Archimedian t-norm if it satisfies the following conditions:

T5) T is a continuous function (continuity)

T6) $T(x,x) < x, \forall x \in [0,1]$ (subidempotency)

Moreover, if T also satisfies (T7), where

T7) $x_1 < x_2,\ y_1 < y_2 \Rightarrow T(x_1,y_1) < T(x_2,y_2)$(strict monotnicity)

is called a strict Archimedian t-norm.

1.2 Fuzzy Subgroups

Rosenfeld [58] introduced the notion of fuzzy group and showed that many group theory results can be extended in an elementary manner to develop the theory of fuzzy group. The underlying logic of the theory of fuzzy group is to provide a strict fuzzy algebraic structure where level subset of a fuzzy group of a group G is a subgroup of the group. [4, 5] reduced fuzzy subgroup of a group using the general t-norm. However, [58] used the t-norm min in his definition of fuzzy subgroup of a group.

Here G always denotes an arbitrary group with a multiplicative binary operation and identity e.

Definition 1.2.1. *A fuzzy subset μ of G is called a fuzzy subgroup of G if*

(G1) $\mu(xy) \geq \mu(x) \wedge \mu(y), \forall x, y \in G$

(G2) $\mu(x^{-1}) \geq \mu(x), \forall x \in G.$

$F(G)$ is the set of all fuzzy subgroups of G.

If $\mu \in F(G)$ we let

$\mu_* = \{x \in G | \mu(x) = \mu(e)\}$ *and* $\mu^* = \{x \in G | \mu(x) > 0\}$

If $\mu \in [0,1]^G$ satisfies $(G1)$, then $\mu(x^n) \geq \mu(x)$, $\forall x \in G, n \in N$.

Also μ satisfies $(G1)$ and $(G2)$ if and only if $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y), \forall x, y \in G$

Example 1.2.1. *Let $G = \{1, -1, i, -i\}$ be the group, with respect to the usual multiplication.*

$$\mu(x) = \begin{cases} 1, if \quad x = 1 \\ t, if \quad x = -1 \quad , t \in (0,1) \\ 0, \quad if \quad x = i, -i \end{cases}$$

Then μ is a fuzzy subgroup of the group G.

Lemma 1.2.1. *Let $\mu \in F(G)$. Then $\forall x \in G$*

(i) $\mu(e) \geq \mu(x)$

(ii)$\mu(x) = \mu(x^{-1})$

In G2 of definition 1.2.1, *if we put x^{-1} in place of x then we get)$\mu(x) \geq \mu(x^{-1})$.*

Hence $\mu(x) = \mu(x^{-1})$. Also if we put $y = x^{-1}$ in $G1$ then $\mu(xx^{-1}) \geq \mu(x) \wedge \mu(x^{-1})$

i.e. $\mu(e) \geq \mu(x)$.

Theorem 1.2.2. *Let μ be a fuzzy subset of a group G. Then μ is a fuzzy subgroup of G if and only if μ_t is a subgroup (called level subgroup) of the group G for every $t \in [0, \mu(e)]$, where e is the identity element of the group G.*

Corollary 1.2.3. *If $\mu \in F(G)$, then μ_* is a subgroup of G.*

Theorem 1.2.4. *If $\mu \in F(G)$, then μ^* is a subgroup of G.*

Definition 1.2.2. *Let $\mu, \sigma \in [0,1]^G, x \in G$.*

We define the product of μ and σ written as $\mu\sigma$ as follows:

$(\mu\sigma)(x) = \vee\{\mu(y) \wedge \sigma(z) | y, z \in G, yz = x\}$.

Also, the inverse of μ denoted by μ^{-1} is defined as :

$\mu^{-1}(x) = \mu(x^{-1})$.

Theorem 1.2.5. *Let $\mu \in [0,1]^G$. Then $\mu \in F(G)$ iff μ satisfies the following conditions:*

(i) $\mu\mu \subseteq \mu$

(ii) $\mu^{-1} \subseteq \mu$ *(or* $\mu \subseteq \mu^{-1}$ *or* $\mu^{-1} = \mu$*)*

Theorem 1.2.6. *Let $\mu, \sigma \in F(G)$. Then $\mu\sigma \in F(G)$ iff $\mu\sigma = \sigma\mu$.*

Theorem 1.2.7. *A fuzzy subset μ of a group G is a fuzzy subgroup of the group G if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ for every $x, y \in G$.*

Definition 1.2.3. *A fuzzy subgroup μ of G is called improper, if μ is constant on G; otherwise μ is termed as proper.*

Theorem 1.2.8. *Let $\mu \in F(G)$ and H be a subgroup of G. Suppose that f is an epimorphism of G onto H. Then $f(\mu) \in F(H)$.*

Theorem 1.2.9. *Let $\sigma \in F(H)$ where H be a subgroup of G. Suppose that f is a homomorphism of G into H. Then $f^{-1}(\sigma) \in F(G)$.*

Theorem 1.2.10. *Let $\{\mu_i | i \in I\} \subseteq F(G)$. Then $\bigcap_{i\in I} \mu_i \in F(G)$.*

Theorem 1.2.11. *Let $\nu \in F(G)$ such that $Im\nu = \{0, t\}$ where $t \in (0, 1]$. If $\nu = \mu \cup \sigma$ for some $\mu, \sigma \in F(G)$ then either $\mu \subseteq \sigma$ or $\sigma \subseteq \mu$.*

Definition 1.2.4. *Let $\mu \in [0, 1]^G$. The fuzzy subgroup generated by μ in G is defined as the smallest fuzzy subgroup of G containing μ and is denoted by $\langle\mu\rangle$.*
Thus, $\langle\mu\rangle = \cap\{\nu | \mu \subseteq \nu, \nu \in F(G)\}$.

Notation: $\mu^1 = \mu, \mu^n = \mu^{n-1}\mu, \forall n \in N, n \geq 2$.

Definition 1.2.5. *A fuzzy subgroup μ of G is called a normal fuzzy(fully invariant) subgroup of G if it fulfills the following condition:*
$\mu(xy) = \mu(yx)$ for all $x, y \in G$.
This is just an equivalent formation of the normal fuzzy subgroup. Let μ be a fuzzy normal subgroup of a group G. For $t \in [0, 1]$, the set $\mu_t = \{(x, y) \in G\times G/\mu(xy^{-1}) \geq t\}$ is called the t-level relation of μ. For the fuzzy normal subgroup μ of G and for $t \in [0, 1]$, μ_t is a congruence relation on the group G.

Theorem 1.2.12. *Let $\mu \in [0,1]^G$. Then the following assertions are equivalent:*

(N1) $\mu(yx) = \mu(xy), \forall x, y \in G$.

In this case μ is called an abelian fuzzy subset of G.

(N2) $\mu(yxy^{-1}) = \mu(x), \forall x, y \in G$.

(N3) $\mu(yxy^{-1}) \geq \mu(x), \forall x, y \in G$.

(N4) $\mu(yxy^{-1}) \leq \mu(x), \forall x, y \in G$.

(N5) $\mu\sigma = \sigma\mu, \forall \sigma \in [0,1]^G$.

$NF(G)$ denotes the class of normal fuzzy subgroups of G. If $\mu, \sigma \in F(G)$ and $\exists u \in G$ such that $\mu(x) = \sigma(uxu^{-1}), \forall x \in G$, then μ and σ are called conjugate fuzzy subgroups and we write $\mu = \sigma_u$.

Theorem 1.2.13. *Let $\mu \in [0,1]^G$. Then $\mu \in NF(G)$ if and only if μ_t is a normal subgroup of $G, \forall t \in \{b \in [0,1] | b \leq \mu(e)\}$.*

Theorem 1.2.14. *Let $\mu \in NF(G)$. Then μ_* and μ^* are normal subgroups of G.*

Theorem 1.2.15. *If $\mu \in NF(G)$ and $\sigma \in F(G)$, then $\mu\sigma \in F(G)$.*

Theorem 1.2.16. *If $\mu, \sigma \in NF(G)$ then $\mu \cap \sigma \in NF(G)$.*

Definition 1.2.6. *Let $\mu, \nu \in F(G)$ and $\mu \subseteq \nu$. Then μ is called a normal fuzzy subgroup of the fuzzy subgroup ν written as $\mu \lhd \nu$ if $\mu(xyx^{-1}) \geq \mu(x) \wedge \mu(y), \forall x, y \in G$.*

Theorem 1.2.17. *(i) If G_1 and G_2 are two subgroups of G then $G_1 \lhd G_2$ if and only if $\chi_{G_1} \lhd \chi_{G_2}$.*

(ii) If $\mu \in NF(G), \nu \in F(G)$ *and* $\mu \subseteq \nu$ *then* μ *is a normal subgroup of* ν.

(iii) Every fuzzy subgroup is a normal subgroup of itself.

(iv) $\mu \in [0,1]^G$ *is a normal fuzzy subgroup of* G *if and only if* μ *is a normal fuzzy subgroup of the fuzzy subgroup* χ_G.

Theorem 1.2.18. *Let* $\mu, \nu \in F(G)$ *and* $\mu \subseteq \nu$. *Then the following assertions are equivalent:*

(I) μ *is a normal fuzzy subgroup of* ν.

(II) $\mu(yx) \geq \mu(xy) \wedge \nu(y), \forall x, y \in G$.

(III) $\mu(e)_{\{x\}}\mu \supseteq (\mu\mu(e)_{\{x\}}) \cap \nu, \forall x \in G$.

Theorem 1.2.19. *Let* $\mu, \nu \in F(G)$. *Then* $\mu \lhd \nu$ *if and only if* $\mu_t \lhd \nu_t, \forall t \in \{b \in [0,1] | b \leq \mu(e)\}$.

Theorem 1.2.20. *Let* $\mu, \nu \in F(G)$ *and* $\mu \lhd \nu$. *Then* $\mu_* \lhd \nu_*$ *and* $\mu^* \lhd \nu^*$.

Theorem 1.2.21. *If* $\mu \in NF(G)$ *and* $\nu \in F(G)$ *then* $\mu \cap \nu$ *is a normal fuzzy subgroup of* ν.

Theorem 1.2.22. *Let* $\mu, \nu, \xi \in F(G)$ *be such that* μ *and* ν *are normal fuzzy subgroups of* ξ. *Then* $\mu \cap \nu$ *is a normal fuzzy subgroups of* ξ.

1.3 Fuzzy Subrings and Fuzzy Ideals

Throughout our discussion in this section unless and otherwise stated, R denotes a commutative ring with unity. In this section we give definitions of fuzzy subring , fuzzy ideal and also define a level ideal of a fuzzy ideal. We also present some basic results related with fuzzy ideals and fuzzy subrings.

Definition 1.3.1. *Let $\mu \in [0,1]^R$. Then μ is called a fuzzy subring of R if it satisfies the following properties:*

(R1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$, *for all* $x, y \in R$

(R2) $\mu(xy) \geq \mu(x) \wedge \mu(y)$, *for all* $x, y \in R$

The set of all fuzzy subrings is denoted by $F(R)$.

Definition 1.3.2. *Let R be a non commutative ring. A fuzzy subset μ of R is called a fuzzy left ideal if it satisfies the following properties:*

(i) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$, *for all* $x, y \in R$

(ii) $\mu(xy) \geq \mu(y)$, *for all* $x, y \in R$

Definition 1.3.3. *Let R be a non commutative ring. A fuzzy subset μ of R is called a fuzzy right ideal if it satisfies the following properties:*

(i) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$, *for all* $x, y \in R$

(ii) $\mu(xy) \geq \mu(x)$, *for all* $x, y \in R$

Definition 1.3.4. *Let $\mu \in [0,1]^R$. Then μ of R is called a fuzzy ideal if it satisfies the following properties:*

(i) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$, *for all* $x, y \in R$

(ii) $\mu(xy) \geq \mu(x) \vee \mu(y)$, *for all* $x, y \in R$

The set of all fuzzy ideals is denoted by $FI(R)$.

If $\mu \in FI(R)$, *then* $\mu_* = \{x \in R | \mu(x) = \mu(0)\}$.

Definition 1.3.5. *Let* μ *be any fuzzy subring (fuzzy ideal) of a ring* R; $t \in [0, 1]$; *and* $t \leq \mu(0)$. *Then subring (ideal)* μ_t *is called a level subring (level ideal) of* μ.

Theorem 1.3.1. *If* A *is any subring (ideal) of a ring* R, $A \neq R$, *then the fuzzy subset* μ *of* R *defined by*

$$\mu(x) = \begin{cases} s, \; if \; x \in A \\ t \; if \; x \in R - A \end{cases}$$

where $s, t \in [0, 1], s > t$ *is a fuzzy subring (ideal) of* R.

Example 1.3.1. *Let* R *be the ring of real numbers under usual operations of addition and multiplication. We define fuzzy subset* μ *by:*

$$\mu(x) = \begin{cases} 1, \; if \; x \; is \; rational \\ 0.6 \quad otherwise \end{cases}$$

Then μ *is a fuzzy subring but not a fuzzy ideal of* R.

Definition 1.3.6. *Let* $\mu \in [0, 1]^R$ *and* $\nu \in F(R)$ *with* $\mu \subseteq \nu$. *Then* μ *is called a fuzzy ideal of* ν *if*

(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$.

(ii) $\mu(xy) \geq (\nu(x) \wedge \mu(y)) \vee (\mu(x) \wedge \nu(y))$, *for all* $x, y \in R$.

Definition 1.3.7. *Let* $\mu, \nu \in [0, 1]^R$. *Then we define*

$$(\mu + \nu) = \vee\{\mu(y) \wedge \nu(z) | y, z \in R, x = y + z\}$$

$$(-\mu)(x) = \mu(-x)$$

$$(\mu - \nu) = \vee\{\mu(y) \wedge \nu(z) | y, z \in R, x = y - z\}$$

$$(\mu\nu) = \vee\{\mu(y) \wedge \nu(z) | y, z \in R, x = yz\}$$

$(\mu + \nu), (\mu - \nu), \mu\nu$ are called sum, difference and product of μ and ν.

Also we have $\mu + \nu = \nu + \mu, \mu - \nu = \mu + (-\nu)$ and since R is commutative $\mu\nu = \nu\mu$

Theorem 1.3.2. *Let $\mu, \nu, \xi \in [0,1]^R$. Then $\mu(\nu + \xi) \subseteq \mu\nu + \mu\xi$.*

Definition 1.3.8. *Let $\mu, \nu \in [0,1]^R$. Then $\mu.\nu \in [0,1]^R$ is defined by*

$$(\mu.\nu)(x) = \bigvee\{\bigwedge_{i=1}^{n}(\mu(y_i) \wedge \nu(z_i)) | y_i, z_i \in R, 1 \leq i \leq n, n \in N, \sum_{i=1}^{n} y_i z_i = x\}$$

Since R is commutative $\mu.\nu = \nu.\mu$.

Theorem 1.3.3. *Let $\mu, \nu, \xi \in [0,1]^R$. Then the following assertions hold:*

(i) $\mu\nu \subseteq \mu.\nu$.

(ii) $\nu \subseteq \xi$ implies $\mu.\nu \subseteq \mu.\xi$

(iii) $(\mu.\nu).\xi = \mu.(\nu.\xi)$

(iv) $(\mu.\nu)(x + y) \geq (\mu.\nu)(x) + (\mu.\nu)(y), \forall x, y \in R$

(v) If R has identity 1 and $\chi_{\{1\}} \subseteq \nu$ then $\mu \subseteq \mu.\nu$.

Definition 1.3.9. *Let $\mu \in [0,1]^R$. Then for $n \in N, n > 1$ we have*

$\mu^1 = \mu$

$\mu^n = \mu^1 \mu^{n-1}$

Also, $\mu^{(1)} = \mu$

$\mu^{(n)} = \mu^{(1)}.\mu^{(n-1)}$

Theorem 1.3.4. *If* $\{\mu_n | n \in Z_+\}$ *is a collection of fuzzy ideals of a ring* R *such that* $\mu_1 \subseteq \mu_2 \subseteq \ldots \mu_n \subseteq \ldots$, *then* $\bigcup_{n \in Z_+} \mu_n$ *is a fuzzy ideal of* R.

Theorem 1.3.5. *Let* A *be a subset of* R. *Then* A *is an ideal of* R *if and only if* χ_A *is a fuzzy ideal of* R.

Theorem 1.3.6. *Let* $\mu \in [0,1]^R$, *then* μ *is a fuzzy ideal of the ring* R *if and only if* μ *is a fuzzy ideal of the fuzzy subring* χ_R.

Theorem 1.3.7. *Intersection of any family of fuzzy subrings (fuzzy ideals) of a ring is a fuzzy subring (fuzzy ideal) of* R.

Theorem 1.3.8. *If* μ *is any fuzzy ideal of a ring* R, *then* $\mu + \mu = \mu$.

Theorem 1.3.9. *Let* $\mu \in [0,1]^R$. *Then* $\mu \in FI(R)$ *if and only if* μ_t *is an ideal of* R, $\forall\ t \in \{b \in [0,1] | b \leq \mu(0)\}$.

Definition 1.3.10. *Let* $\mu \in [0,1]^R$ *and* $\nu \in F(R)$. *Then* μ *is a fuzzy ideal of* ν *if and only if* μ_t *is an ideal of* ν_t , $\forall t \in \{s \in [0,1] | s \leq \mu(0)\}$.

Theorem 1.3.10. *Let* $\mu, \nu \in FI(R)$. *Then*

(1) $\mu(0) \geq \mu(x), \forall x \in R$.

(2) If R *is with unity then* $\mu(1) \leq \mu(x), \forall x \in R$.

(3) Let $x, y \in R$. *If* $\mu(x-y) = \mu(0)$ *then* $\mu(x) = \mu(y)$.

(4) μ_* *and* μ^* *are ideals of* R.

(5) $\mu_* \cap \nu_* \subseteq (\mu \cap \nu)_*$

Theorem 1.3.11. *Let* $\nu \in F(R)$ *and* μ *be a fuzzy ideal of* ν. *Then* μ_* *(and* μ^**) is a fuzzy ideal of* ν_* *(and* ν^**).*

Example 1.3.2. *Let* $\mu, \nu \in [0,1]^R$ *be defined by* $\mu(x) = \frac{1}{2}, \forall\ x \in R$ *and* $\nu(x) = \frac{1}{2}$ *if* $x \neq 0$ *and* $\nu(0) = 1$. *Then* $\mu, \nu \in FI(R)$. *Also* $\mu_* \cap \nu_* = R \cap \{0\} = \{0\}$ *and* $(\mu \cap \nu)_* = \mu_* = R$

Lemma 1.3.12. *Let* $\mu, \nu \in FI(R)$ *and* $\mu(0) = \nu(0)$. *Then* $\mu_* \cap \nu_* = (\mu \cap \nu)_*$.

Theorem 1.3.13. *Let* $\mu, \nu, \eta \in FI(R)$. *Then* $\mu\nu \subseteq \eta$ *if and only if* $\mu.\nu \subseteq \eta$

Corollary 1.3.14. *Let* $\mu, \nu, \eta \in FI(R)$. *Then* $\mu\nu \subseteq \eta \Rightarrow \mu \subseteq \eta$ *or* $\nu \subseteq \eta$ *if and only if* $\mu.\nu \subseteq \eta \Rightarrow \mu \subseteq \eta$ *or* $\nu \subseteq \eta$

Lemma 1.3.15. *Let* μ, ν *be fuzzy subrings of* R. *Then*

(I) For all $t \in [0,1], \mu_t + \nu_t \subseteq (\mu + \nu)_t$.

(II) If μ *and* ν *are finite valued then for all* $t \in [0,1], \mu_t + \nu_t = (\mu + \nu)_t$.

Theorem 1.3.16. *Let* $\mu \in FI(R)$. *Suppose* R *has an identity.*

(I) Let $x \in R$ *be a unit. Then* $\mu(x) = \mu(x^{-1}) = \mu(1)$.

(II) If x *and* y *are associates then* $\mu(x) = \mu(y)$,

Theorem 1.3.17. *Let* $\mu_i \in FI(R), i \in I$. *Then* $\bigcap_{i \in I} \mu_i \in FI(R)$.

Definition 1.3.11. *Let μ be any fuzzy subset of a ring R. The smallest fuzzy ideal (fuzzy subring) of R containing μ is called the fuzzy ideal (fuzzy subring) generated by μ in R and is denoted by $\langle\mu\rangle$.*

i.e. $\langle\mu\rangle = \cap\{\nu|\mu \subseteq \nu, \nu \in FI(R)\}$ $(\langle\mu\rangle = \cap\{\nu|\mu \subseteq \nu, \nu \in F(R)\})$.

Theorem 1.3.18. *Let $\mu, \nu \in [0,1]^R$. then*

(I) $\mu \in FI(R) \Leftrightarrow \langle\mu\rangle = \mu$.

(II) $\mu \subseteq \nu \Rightarrow \langle\mu\rangle \subseteq \langle\nu\rangle$.

(III) $\langle\mu|_S\rangle \subseteq \langle\mu\rangle|_S \in F(S)$, *where S is a subring of R,$\mu|_S$ and $\langle\mu\rangle|_S$ are the restrictions of μ and $\langle\mu\rangle$ to S respectively.*

Theorem 1.3.19. *Let A be a non empty subset of R. Then $\langle\alpha_A\rangle = \alpha_{\langle A\rangle}$, where $\langle A\rangle$ is the ideal of R generated by A and $\alpha \in [0,1]$.*

Theorem 1.3.20. *Let $\mu \in [0,1]^R$ and define $\nu \in [0,1]^R$ as follows:*

$$\nu(x) = \bigvee\{\bigwedge_{y\in A} \mu(y)|A \subseteq R, 1 \leq |A| < \infty, x \in \langle A\rangle\}$$

, $\forall x \in R$, *then* $\langle\mu\rangle = \nu$.

Theorem 1.3.21. *Let $x \in R$ and $\alpha \in [0,1]$. Then $\langle x_\alpha\rangle = \alpha_{\langle x\rangle}$*

Theorem 1.3.22. *If $\mu \in F(R), \sigma \in FI(R)$ then, $\mu \cap \sigma$ is a ideal of the ring* $\{x \in R|\mu(x) = \mu(0)\}$

Theorem 1.3.23. *If $\mu, \nu \in FI(R)$ be such that $\mu(0) = \nu(0)$. Then $\mu + \mu \in FI(R)$ and $\mu + \nu = \langle\mu \cup \nu\rangle$.*

Theorem 1.3.24. *Let* $\mu, \nu \in [0,1]^R$. *Then*

(I) If $\mu \in FI(R)$ *then* $\mu\nu \subseteq \mu$.

(II) If $\mu \in FI(R)$ *and* $\nu \in FI(R)$ *then* $\mu\nu \subseteq \mu \cap \nu$.

Theorem 1.3.25. *Let* $\mu, \nu \in F(R)$. *Then* $\mu.\nu \in F(R)$.

Theorem 1.3.26. *Let* $\mu \in FI(R), \nu \in F(R)$. *Then* $\mu.\nu \in FI(R), \mu.\nu = \langle \mu\nu \rangle$.

Theorem 1.3.27. *If* $\mu, \nu \in FI(R)$ *then* $\mu.\nu \subseteq \mu \cap \nu$.

Theorem 1.3.28. *If* $\mu, \nu, \xi \in FI(R)$ *and* $\nu(0) = \xi(0)$. *Then* $\mu.(\nu + \xi) = \mu.\nu + \mu.\xi$.

Theorem 1.3.29. *Let* $\mu \in FI(R)$ *and* $\nu \in F(R)$. *Then* $\mu \cap \nu$ *is a fuzzy ideal of* ν.

Theorem 1.3.30. *Let* $\gamma \in F(R)$ *and* μ, ν *be two fuzzy ideals of* γ, *then* $\mu \cap \nu$ *is also a fuzzy of* γ.

Definition 1.3.12. *Let* $\delta \in FI(R)$. *Then* δ *is called a prime fuzzy ideal of* R *if* δ *is non constant and for* $\mu, \nu \in FI(R)$, $\mu\nu \subseteq \delta$ *implies* $\mu \subseteq \delta$ *or* $\nu \subseteq \delta$.

Theorem 1.3.31. *If* δ *is a prime fuzzy ideal of* R, *then* δ_* *is a prime ideal of* R.

Theorem 1.3.32. *Let* $\mu \in [0,1]^R$. *Then* μ *is a prime fuzzy ideal of* R *if and only if* $\mu(0) = 1$, μ_* *is a prime ideal of* R, $Im\mu = \{1, \alpha\}$.

Theorem 1.3.33. *Let* $\mu \in [0,1]^R$. *Then* μ *is a prime fuzzy ideal of* R *if and only if* μ *is non constant and* $x_a y_b \subseteq \mu, a, b \in [0,1]$ *then either* $x_a \subseteq \mu$ *or* $y_b \subseteq \mu$.

Definition 1.3.13. *Let* $\xi \in [0,1]^R$. *Then* ξ *is called semi prime if* ξ *is non constant and if* $\mu^2 \subseteq \xi \Rightarrow \mu \subseteq \xi$, *where* $\mu \in FI(R)$.

Theorem 1.3.34. *Let ν be a semi prime fuzzy ideal of R. Then ν_* is a semi prime ideal of R.*

Theorem 1.3.35. *Let ν be a semi prime fuzzy ideal of R. Then $\nu(x^2) = \nu(x)$, for all $x \in R$.*

1.4 Fuzzy Submodules

The concept of fuzzy submodules and some basic properties of fuzzy submodules are presented in this section. In this section, unless and otherwise stated, R is a commutative ring with unity, $1 \neq 0$, M is a module over R and θ is the zero element of M.

Definition 1.4.1. *A fuzzy subset μ of M is called fuzzy submodules of M if the following conditions are satisfied:*

(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$, *for all* $x, y \in M$.

(ii) $\mu(rx) \geq \mu(x)$, *for all* $r \in R, x \in M$.

(iii) $\mu(\theta) = 1$

Let μ and σ be two fuzzy submodules of M. If $\mu \subseteq \sigma$ then σ is called a fuzzy submodule of μ.

$F(M)$ denotes the set of all fuzzy submodules of M.

Since R is a module over itself, it follows that μ is a fuzzy submodule of the module R if and only if μ is a fuzzy ideal of the ring R.

Since $-1x = -x, \forall x \in M$, condition (ii) of definition of fuzzy submodule implies that $\mu(-x) \geq \mu(x)$, for all $x \in M$. Hence $\mu \in F(M)$ if and only if μ is a fuzzy subgroup of the additive group of M and satisfies condition (ii) of definition of fuzzy submodule.

An example of a fuzzy submodule of an R-module M with $R = \mathbb{Z}, M = \mathbb{Z}_6$ is

$$\mu(x) = \begin{cases} 1, & if\ x = 0 \\ \frac{1}{3} & if\ x = 2, 4 \\ \frac{1}{4} & if\ x = 1, 3, 5 \end{cases}$$

Definition 1.4.2. *Let $\mu, \nu \in [0,1]^M$. We define $\mu + \nu, -\mu \in [0,1]^M$ as follows: For all $x \in M$,*

$$(\mu + \nu)(x) = \vee\{\mu(y) \wedge \nu(z) | y, z \in M, y + z = x\},$$

$$(-\mu)(x) = \mu(-x).$$

Then $\mu + \nu$ is called the sum of μ and ν, and $-\mu$ the negative of μ.

Definition 1.4.3. *Let $r \in R$ and $\mu \in [0,1]^M$. Define $r\mu \in [0,1]^M$ as follows:*

$$(r\mu)(x) = \{\mu(y) | y \in M, ry = x\}, \forall x \in M.$$

Then $r\mu$ is called the product of r and μ.

Definition 1.4.4. *Let $\zeta \in F(R)$ and $\mu \in F(M)$. We define $\zeta\mu \in [0,1]^M$ as follows:*

$$(\zeta\mu)(x) = \vee\{\zeta(r) \wedge \mu(y) | r \in R, y \in M, ry = x\}, \forall x \in M.$$

To give an example of the product of $\zeta \in F(R)$ and $\mu \in F(M)$ with $R = \mathbb{Z}$ and $M = \mathbb{Z}_6$, let

$$\zeta(r) = \begin{cases} \frac{1}{2} & if\ r \in 2\mathbb{Z} \\ \frac{1}{5} & otherwise \end{cases}$$

and

$$\mu(x) = \begin{cases} 1, & if\ x = 0 \\ \frac{1}{3} & if\ x = 2, 4 \\ \frac{1}{4} & if\ x = 1, 3, 5 \end{cases}$$

Then,

$$(\zeta\mu)(0) = \vee\{\underbrace{\zeta(0) \wedge \mu(1)}_{0=0.1}, \underbrace{\zeta(2) \wedge \mu(3)}_{0=2.3}, \underbrace{\zeta(2) \wedge \mu(1) \wedge \zeta(-1) \wedge \mu(2)}_{0=2.1-1.2} \ldots\}$$
$$= \vee\{\frac{1}{2} \wedge 1, \frac{1}{2} \wedge \frac{1}{4}, \frac{1}{2} \wedge \frac{1}{4} \wedge \frac{1}{5}, \ldots\} = \frac{1}{2}$$

$$(\zeta\mu)(1) = \vee\{\underbrace{\zeta(1) \wedge \mu(1)}_{1=1.1}, \underbrace{\zeta(7) \wedge \mu(1)}_{1=7.1}, \underbrace{\zeta(2) \wedge \mu(2) \wedge \zeta(-1) \wedge \mu(3)}_{1=2.2-1.3} \ldots\}$$
$$= \vee\{\frac{1}{4} \wedge \frac{1}{5}, \frac{1}{5} \wedge \frac{1}{4}, \frac{1}{3} \wedge \frac{1}{2} \wedge \frac{1}{5} \wedge \frac{1}{4}, \ldots\} = \frac{1}{5}$$

$$(\zeta\mu)(2) = \vee\{\underbrace{\zeta(2) \wedge \mu(1)}_{2=2.1}, \underbrace{\zeta(2) \wedge \mu(4)}_{2=2.4}, \underbrace{\zeta(2) \wedge \mu(2) \wedge \zeta(1) \wedge \mu(2)}_{0=2.2-1.2} \ldots\}$$

$$= \vee\{\frac{1}{2} \wedge 1, \frac{1}{2} \wedge \frac{1}{3}, \frac{1}{2} \wedge \frac{1}{4} \wedge \frac{1}{5}, \ldots\} = \frac{1}{3}$$

Continuing this way we obtain,

$$(\zeta\mu)(x) = \begin{cases} \frac{1}{2}, & if\ x = 0 \\ \frac{1}{3} & if\ x = 2, 4 \\ \frac{1}{5} & if\ x = 1, 3, 5 \end{cases}$$

Theorem 1.4.1. *Let $r, s \in R$ and $\mu, \nu, \zeta, \mu_i, i \in I$, where I is a non empty index set. Then the following assertions hold:*

(1) $1\mu = \mu, (-1)\mu = -\mu$;

(2) $r\chi_\theta = \chi_\theta$;

(3) $\mu \subseteq \nu \Rightarrow r\mu \subseteq r\nu$;

(4) $r(s\mu) = (rs)\mu$;

(5) $r(\mu + \nu) = r\mu + r\nu$;

(6) $r(\cup_{i\in I}\mu_i) = \cup_{i\in I} r\mu_i$;

(7) $(r\mu)(rx) \geq \mu(x), \forall x \in M$;

(8) $\zeta(rx) \geq \mu(x), \forall x \in M \Leftrightarrow r\mu \subseteq \zeta$;

(9) $(r\mu + s\nu)(rx + sy) \geq \mu(x) \wedge \nu(y) \forall x, y \in M$;

(10) $\zeta((rx + sy) \geq \mu(x) \wedge \nu(y) \forall x, y \in M \Leftrightarrow r\mu + s\nu \subseteq \zeta$.

Definition 1.4.5. *Let μ be a fuzzy subset of a non empty set X. Then the support of μ, denoted by μ^* is defined as* $\mu^* = \{x \in X : \mu(x) > 0\}$. *If μ is a fuzzy submodule of M then μ^* is a submodule of M.*

Theorem 1.4.2. *Let $\mu, \nu \in F(M)$. Then $\mu + \nu \in F(M)$.*

Theorem 1.4.3. *Let $r_i \in R$ and $\mu_i \in F(M), 1 \leq i \leq n, n \in \mathbb{N}$. Then $\sum_{i=1}^{n} r_i\mu_i \in F(M)$.*

Theorem 1.4.4. *Let $\nu \in F(M)$ and N be a submodule of M. Define $\xi \in [0,1]^{\frac{M}{N}}$ as follows:*

$$\xi([x]) = \bigvee\{\nu(u) \mid u \in [x]\}, for\ all\ \ x \in M$$

where $\frac{M}{N}$ *denotes the quotient module of* M *with respect to* N *and* $[x]$ *represents the coset* $x + N$. *Then* $\xi \in F(\frac{M}{N})$.

We now consider a special case of the above theorem where $N = \nu_t$ for some $t \in [0, 1]$. Then N is a submodule of M. Then it follows that $\xi \in F(\frac{M}{N})$. Let $x \in M$ and think about $\xi([x])$.

If $[x] = N$, then $\xi([x]) = 1$.

Again if $[x] \neq N$, then $x \notin N$ and so $\nu(x) < t$.

Thus for any $y \in N, \exists z \in N$ such that $y = x + z$.

Hence $\nu(y) = \nu(x + z) \geq \nu(x) \wedge \nu(z) = \nu(x)$.

Similarly, $\nu(x) = \nu(y + (-z)) \geq \nu(y) \wedge \nu(-z) = \nu(y)$.

Thus $\xi([x])$ can be given by

$$\xi([x]) = \begin{cases} 1 \ \ if\ \nu(x) \geq t \\ \nu(x) \ \ otherwise \end{cases}$$

Let $\mu, \nu \in F(M)$ be such that $\mu \subseteq \nu$. Then μ^* and ν^* are submodules of M such that $\mu^* \subseteq \nu^*$. Then μ^* is a submodule of ν^*. Also we have $\nu|_{\nu^*} \in F(\nu^*)$. Therefore it follows from the above theorem that if we define $\xi \in [0, 1]^{\frac{\nu^*}{\mu^*}}$ as follows:

$\xi([x]) = \vee\{\nu(z) | z \in [x]\}$, for all $x \in \nu^*$,

where $[x]$ denotes the coset $x + \mu^*$, then $\xi \in F(\frac{\nu^*}{\mu^*})$. The fuzzy submodule ξ is called the quotient of ν with respect to μ and written as $\frac{\nu}{\mu}$.