Introduction

The word "fuzzy" means "vagueness". Fuzzyness occurs when boundary of a piece of information is not clear-cut. In real world there exists much fuzzy knowledge: knowledge that is vague, imprecise, uncertain, ambiguous, inexact or probabilistic in nature. Human thinking and reasoning frequently involve fuzzy information, originating from inherently inexact human concepts. Human can give satisfactory answers which are probably true. However, our systems are unable to answer many questions. The reason is, most systems are designed based upon "crisp" or "classical set theory". In mathematics, the crisp set theory is a well defined theory. The crisp set theory is defined in such a way that it makes a division among the individuals in some given universe into two opposing groups- members (those that certainly belong in the set) and non-members (those that certainly do not). The crisp set theory provides a clear distinction between members and non-members of the set. Thus, "crisp" means dichotomous, which is "yes or no" type rather than "more or less" type. But many of accumulations we generally utilize, for instance, the "collection of young men" in the set of all men, "collection of good students" in the set of all students, "collection of costly vehicles" in the set of all vehicles etc do not display this characteristic. Thus words like young, tall, high or good are fuzzy. The crisp set can be exhibited by it's characteristic function. The characteristic function associates each element of a universe of discourse either 1 or 0 which means that a particular element either belongs to the set or does not, respectively. In fuzzy set theory, the characteristic function defined in a crisp set is generalized to a membership function that assigns every element of the set , a value from the unit interval $[0,1]$ instead from the two valued set $\{0,1\}$. Fuzzy set theory is a graded concept. A fuzzy set consists of elements and their grades of membership in the set. Thus, a fuzzy set is any set that allows its members to have different degrees of membership in the interval $[0,1]$. The nearer the value of an element to unity, the higher is the grade of its membership.

Over the centuries, lots of great mathematical theories have been introduced and the concept of fuzzy sets, coined by L.A.Zadeh in his seminal paper "Fuzzy Sets" (Info.Control 8, 338-353) in 1965 is one such theory. This concept has emerged as a fundamental and fresh idea and it was a new episode towards the development of science and engineering. As a discipline, fuzzy sets have roots in the set theory. The elimination of two valued (yes-no) dogma has paved the way of an interesting mathematical insight and investigation that can easily stand on their own. We are however, asserted that the term fuzzy mathematics goes a bit too far - since the mathematical language is universal and orthodox, from which the development of theoretical foundation benefits greatly. The real world applications and the emerging sound and powerful methodologies arising from the fuzzy sets enhances and promotes the fuzzy set as a technology.

Topology has provided a natural framework for the theory of fuzzy sets to flourish, resulting in a massive influx of literature on fuzzy topological spaces. The concept of topology is fuzzified by Zadeh's student C.L.Chang in his paper entitled

"Fuzzy topological Spaces" (J. Math. Appl.24 (1968) 182-190). In the trajectory of stupendous growth of fuzzy set theory, fuzzy algebra has become an important area of research. Fuzzification of the "group" concept into "fuzzy subgroup" was made by Rosenfeld [58] and opened up a new insight in the field of mathematical sciences. Since then, many researchers are working in extending the concepts and results of abstract algebra to broader framework of fuzzy setting. Anthony and Sherwood [4] have redefined the concept of fuzzy subgroup with respect to a general t-norm. Subsequently, among others, Sherwood [59], Abuosman [3] and P. Das [9] have studied fuzzy subgroups with respect to a general t-norm and showed that most of the results obtained by Rosenfeld [58] with respect to t-norm 'min' are valid for t-norms of a more general nature. W.J.Liu [43] defined the fuzzy normality of a fuzzy subgroup in 1982. A coherent study of the fuzzy normal subgroups was initiated by Mukherjee and Bhattacharya [50, 51]. Liu [43] also initiated the study of fuzzy subrings and fuzzy ideals of rings around 1982. Mukherjee and Sen [52] characterized regular rings in terms of fuzzy ideals of rings. Abou-Zaid [2], Rajesh Kumar [33, 34], Malik and Mordeson [44, 45], Swamy and Swamy [61], V.N.Dixit [11, 12], M.M.Zadehi [64], Kumbhojkar and Bapat [41] etc. have carried out extensive work on fuzzy ideal of rings. Mukherjee and Sen [52] also studied various aspects of fuzzy prime ideals and determined all fuzzy prime ideals of the ring of integers.

Rosenfeld [58] also introduced the concept of fuzzy homomorphism and established some properties of the string of definitions of homomorphic image and preimage of fuzzy subgroups. Subsequently, F.P. Choudhury, A.B.Chakraborty and S.S.khare [8], Fang Jin- Xuan [16], Zhu Nan-de [66] studied fuzzy homomorphism and obtained certain properties.

Fuzzy submodules of a module over a ring were first introduced by Negoita and Ralecsu [55], Pan [56] studied fuzzy finitely generated modules and quotient modules. Mukherjee et al [54], Kumar at al [35, 36] studied various aspects of fuzzy submodules. The notion of free fuzzy modules was introduced by Muganda [49] as an extension of free modules in the fuzzy context. The concept of a free L-module is available in [48]. Zadehi and Ameri [65] introduced the concept of fuzzy projective and injective modules. In [60] Sidky introduced the notion of radical of a fuzzy submodule and also defined primary fuzzy submodule.

The theory of honest subgroups was developed by Abian and Rinehart in [1]. The concepts of isolated submodules, honest submodules are studied by Fay and Joebert, Jara in [22, 23]. For a skew field, the notions of isolated submodules and honest submodules coincide. The honest submodules lead to a new characterization of ore domain. Moreover following the theory developed by Fay and Joebert, Jara obtained the characterizations of rings of quotients in terms of honest operator. Closure operators have played a fundamental role in a variety of areas of mathematics. In the category of groups isolated subgroups are useful in the study of torsion-free groups. In case of category of modules closure operators like T-closed, T-honest have been studied by Fay and Joubert . The concept of super honest submodules was introduced by Joubert and Schoeman [23]. Super honest submodules of quasi injective modules are studied by Cheng [7].

The Objective of the Work:

Our objective is to study the fuzzy aspects of some algebraic structures like rings and modules. Though the study of fuzzy aspects of rings and modules is well developed but there are many scopes for further studies.

One type of largeness is embodied in essential ideals of rings as well as essential submodules of modules. The large character motivates us to fuzzify the notion of essentiality in rings. We attempt to study various characteristics of such fuzzy ideals.

Different chain conditions imposed on rings and modules lead to some illuminating and interesting results. For example, rings (modules) with ascending chain conditions on annihilators leads to the development of Goldie rings (modules). Rings with descending chain condition on essential ideals need not be Artinian, but some rings such as self-injective rings with descending chain condition on essential ideals satisfy the Noetherian condition. These studies have left scopes to impose different chain conditions and study some interesting results. Our objective is to study fuzzy aspects of rings and modules satisfying different chain conditions on the substructures.

Our aim is to study the fuzzy aspects of honest and super honest submodules and to obtain some structure theorems. We attempt to establish the analogous results that hold for modules and rings.

The concept of essentiality leads to the notion of singularity in rings and modules. So it is natural to study fuzzy singularity using the concept of fuzzy essentiality.

The Outcome of the Work: This volume consists of five chapters. Chapter 1, presents the concepts of fuzzy subsets, fuzzy subgroup, fuzzy subrings, fuzzy ideals and fuzzy submodules. All the definitions and the results presented in this chapter are needed for the development of the subsequent chapters and these materials are taken from the standard literatures [35, 48]. In the results that are taken from [Mordeson], we replace L by $[0,1]$ and make the necessary changes, where L is a complete Heyting algebra. A complete Heyting algebra L is a complete lattice such that for all $A \subseteq L$ and for all $b \in L, \vee\{a \wedge b | a \in A\} = (\vee\{a|a \in A\}) \wedge b$ and $\wedge\{a \vee b | a \in A\} = (\wedge\{a|a \in A\}) \vee b$ and the closed interval $[0,1]$ together with the operations min, max and $\leq$ form a complete Heyting algebra.

In chapter 2, we present various results that deals with fuzzy essential ideals and T-essential fuzzy ideals. This chapter is divided into five sections. In the first section of this chapter, some basic results are presented which will be used in the sequel. The second section deals with annihilators of fuzzy subsets of rings and various results related to fuzzy annihilators are established. We prove that the left (right) annihilator of a fuzzy subset of a ring is a fuzzy left (right) ideal of the ring. For a regular fuzzy subring μ of R and $a_t \in \mu$, every fuzzy ideal of the form $a_t\chi_R$ is the left fuzzy annihilator of some fuzzy point of R. Third section is devoted to the study of essential fuzzy ideals of rings. In this section we define essential fuzzy ideal and exhibit some examples. We prove various results which establish necessary and sufficient condition between essential fuzzy ideal and essential ideals in crisp set theory. Some of the results proved in this section are:

(i) Let μ be a fuzzy left ideal of R. Then μ is a essential fuzzy left ideal of R if and only if μ_t, for some $t \in (0,1]$ is an essential left ideal of R.

(ii) A left ideal A of R is essential in R if and only if χ_A is essential fuzzy left ideal of R.

(iii) If A and B are two nonzero left ideals of R, then A is essential left ideal of B if and only if χ_A is essential fuzzy left ideal in χ_B.

(iv) Let μ and σ be two nonzero fuzzy left ideals of R such that $\mu \subseteq \sigma$. Then $\mu \subseteq_e \sigma$ if and only if $\mu_t \subseteq_e \sigma_t$, for some $t \in (0,1]$.

(v) If A is an essential left ideal of R, then the fuzzy subset μ of R defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in A, \text{ where } t \in (0,1] \\ 0 & \text{if } x \notin A \end{cases}$$

is a essential fuzzy left ideal of R.

In the fourth section some characteristics of essential fuzzy ideals are studied. Some interesting results proved in this section are:

(i) Let μ and σ be two nonzero fuzzy left ideals of R with $\mu \subseteq_e \sigma$. Then for some $a_t \in \sigma, t \in (0,1], a \in R$ there exists a fuzzy left ideal ν of R with $\nu \subseteq_e R$ such that νa_t is nonzero and $\nu a_t \subseteq \mu$.

(ii) Let μ be a nonzero fuzzy left ideal of R with $\mu \subseteq_e R$. Then for any nonzero $a \in R$, there exists a fuzzy left ideal ν of R with $\nu \subseteq_e R$ such that νa is nonzero and $\nu a \subseteq \mu$.

(iii) Let μ, ν and σ be nonzero fuzzy left ideals of R such that $\mu \subseteq \nu \subseteq \sigma$. Then $\mu \subseteq_e \sigma$ if and only if $\mu \subseteq_e \nu \subseteq_e \sigma$.

(iv) If the essential fuzzy left ideal of R satisfy the supremum condition, then $Z_f(R)$ is a left ideal of R.

In the fifth section introducing the notion of T-fuzzy essential ideal various results are proved which are generalization of some of the results discussed in the previous section. Some parts of this chapter form the paper entitled "**On essential fuzzy ideals of rings**" which is published in the journal "**Advances in fuzzy sets and systems**".

In chapter 3, we study rings and modules with chain condition on the fuzzy substructures. This chapter is divided into five sections. In the first section some existing results of ring theory are stated which we attempt to fuzzify. In the second section some elementary results of fuzzy annihilators are presented for the development of the subsequent sections. In the third section we study some characteristics of rings with ascending (descending) chain condition on fuzzy ideals. Some of the results proved in this section are:

(i)If a ring R with unity is fuzzy left Artinian then every fuzzy left ideal on R has finite number of values.

(ii) Let R and S be two fuzzy left Artinian rings with unity 1. Then $R \times S$ is also fuzzy left Artinian.

(iii) Let R be a ring. Then if R is fuzzy left Noetherian then the set of fuzzy left ideals on R is a well ordered subset of $[0, 1]$.

(iv) If R is fuzzy left Noetherian, $R[x]$ is also fuzzy left Noetherian.

In the fourth section we study some characteristics of modules with descending chain condition on fuzzy submodules. Finally, in the fifth section we study fuzzy aspects of rings chain conditions. The concept of fuzzy Goldie ring is introduced. Various characteristics of fuzzy ideals of rings with chain condition of fuzzy annihilators are studied. With the idea of singular ideal denoted by $Z_f(R)$ and essential fuzzy ideal which are defined in chapter 2 we establish the following results:

(i) If R is a fuzzy semi-prime commutative ring then, $Z_f(R) = 0$.

(ii) Let R be a commutative ring with unity. If $Z_f(R) = 0$, then R is a semi-prime ring.

(iii) Let R be a semi-prime left fuzzy Goldie ring and μ be a essential fuzzy left ideal of R, then μ contains a left fuzzy regular point.

Some parts of this chapter form the paper entitled "**Fuzzy aspects of rings with chain conditions**" which is published in the"**International journal of fuzzy mathematics and systems**". Also some parts of this chapter are presented in "**National conference on recent trends in mathematics and its application", organized by Gauhati University, 2009**.

In chapter 4, we attempt to study the fuzzy aspects of honest and superhonest modules. This chapter is divided into three sections. In this chapter we define the concepts like fuzzy honest submodules, fuzzy closure, fuzzy torsion and fuzzy superhonest submodules. Various characteristics of honest and superhonest submodules are fuzzified in this chapter. Some of main results established in this chapter are stated as follows:

(i)Let M be a left R-module.

(a)If $\mathfrak{F}$ is weak closed under intersection, then for any $\mu \in F(M)$ we have that $Cl_{\mathfrak{F}}^{M}(\mu)$ is a fuzzy submodule of M.

(b) If $\mathfrak{F}$ is weak closed under intersection if and only if $Cl_{\mathfrak{F}}^{M}(\sigma_1) \cap Cl_{\mathfrak{F}}^{M}(\sigma_2) = Cl_{\mathfrak{F}}^{M}(\sigma_1 \cap \sigma_2)$, for any two $\sigma_1, \sigma_2 \in F(M)$.

(c) If $\mathfrak{F}$ is weak closed under intersection, then $\mathfrak{F}$ is left closed if and only if $Cl_{\mathfrak{F}}^{M}(\sigma)$ is a fuzzy submodule of M for any $\sigma \in F(M)$.

(ii) Let $\mathfrak{F}$ be an inductive set of fuzzy ideals, then the following statements are equivalent:

(a) $\mathfrak{F}$ is a topological filter.

(b) $Cl_{\mathfrak{F}}^{M}(\sigma)$ is a fuzzy submodule for any $\sigma \in F(M)$.

(iii) Let $\sigma \in F(M)$ and $\mathfrak{F}$ be inductive, then the following statements are equivalent:

(a) σ is $\mathfrak{F}$-honest in 1_M.

(b) For any $m_\alpha \in Cl_{\mathfrak{F}}^{M}(\sigma)\backslash\sigma$, we have $(\sigma : m_\alpha) = Ann(m_\alpha)$.

(c) For any $m_\alpha \in Cl_{\mathfrak{F}}^{M}(\sigma)\backslash\sigma$, we have $1_R \cap m_\alpha = \chi_\theta$.

(iv) If σ is a fuzzy submodule of an R-module A, then the following statements are equivalent:

(a) σ is fuzzy superhonest in μ.

(b) For each $x_t \in \mu, (\sigma : x_t)$ is a fuzzy superhonest left ideal of 1_R.

(v) If μ and σ are fuzzy submodules of an R-module M with $\mu \subseteq \sigma$ and μ is fuzzy superhonest in σ then $T(\mu) = T(\sigma)$.

Some results of this chapter are presented in "**Technical Session of North East Regional Conference of Mathematics**", **organized by AAM, Gauhati Uni-**

versity, 2011. and the chapter forms the paper entitled **"Honest and superhonest fuzzy submodules"** and is being communicated to **"International Journal of Algebra"**.

In chapter 5 we study fuzzy aspects of rings and modules. Defining the notion of singular fuzzy ideal and singular submodule as in [28] we study various characteristics of these concepts. For a nil fuzzy ring R it is proved that χ_R is singular and for a strongly fuzzy prime ring χ_R is non singular. The equivalence of the concepts like essentially closed, closed and complement submodule are established. If μ is a fuzzy submodule of σ such that $Z(\sigma) \subseteq \mu$, then μ is a fuzzy essential submodule of $cl(\sigma)$. If μ, σ are fuzzy submodules of an R-module M such that μ is a fuzzy superhonest in σ then μ is fuzzy essentially closed submodule of σ and $\mu \supseteq Z(\sigma)$. Some results of this chapter are presented in **"National conference on frontiers in mathematics, organized by Gauhati University, 2011."**