Chapter 5

Fuzzy Singularity

The notion of singularity has been defined in chapter 2 using fuzzy ideals such that singular fuzzy ideal, denoted by $Z_f(R)$ is a subset of R. In this chapter the notion of fuzzy singularity is defined in terms of fuzzy subsets of R which is introduced by Kalita [28]. Using this definition we study various characteristics on singular fuzzy ideals of rings and singular submodules of modules

5.1 Basic Definitions and Results

In this section we present some basic definitions and results of singular ideals of rings, singular submodules and essential submodules of modules.

Given a subset S of a left R-module M, the annihilator $ann_R(S)$ of S in R is defined as $\{r \in R | rS = 0\}$. If $m \in M$ then $ann_R(\{m\})$ is denoted by $ann_R(m)$. The singular submodule $Z_R(M)$ of M is defined as $\{m \in M | ann_R(m) \text{is an essential left ideal of } R\}$. The singular submodule of the left R-module ${}_RR$ is called the singular ideal of the ring R and denoted as $Z(R)$. If the ring R is such that $Z(R) = R$ then it is called singular whereas those for which $Z(R) = 0$ are called non-singular.

Definition 5.1.1. [20] *A ring R is said to be strongly prime if every non-zero ideal I of R contains a finite subset F such that $ann(F) = 0$.*

Theorem 5.1.1. [18] *Every commutative nil ring R is singular.*

Theorem 5.1.2. [18] *Every strongly prime ring R is non-singular.*

Theorem 5.1.3. [18] *If I is a subring of R and I is semiprime, then $Z(I) = I \cap Z(R)$*

Theorem 5.1.4. [18] *If I is an ideal of R, then $Z(I)I \subseteq Z(R)$.*

Definition 5.1.2. [7] *A submodule N of an R-module M is said to be essentially closed submodule if whenever H is a submodule of M such that N is an essential submodule of H, we have $H = N$.*

Definition 5.1.3. [7] *A submodule N of an R-module M is said to be complement submodule of some submodule H of M if N is maximal in the set of all submodules K of M such that $K \cap H = 0$.*

Lemma 5.1.5. [7] *If N is a submodule of an R-module M and N^c is a complement submodule of N in M, then the following statements are equivalent:*

(a) N is essentially closed submodule of M;

(b) N is a complement submodule of N^c in M;

(c) N is a complement submodule of some submodule of M.

It is known that the set $\{m \in M : (0 : m)$ is an essential left ideal of $R\}$ forms a submodule of M, which is called the singular submodule of M and is denoted by

$Z(M)$. Also the $\{m \in M\ (0 : m) \neq 0\}$ of all torsion elements of M is denoted by $T_R(M)$. Clearly $Z(M) \subseteq T_R(M)$.

Lemma 5.1.6. [7] *If N is a submodule of an R-module M and N^c is a complement submodule of N in M, then the following statements are equivalent:*

(a) N is essentially closed submodule of M and $N \supseteq Z(M)$;

(b) N is a complement submodule of N^c in M and $N \supseteq Z(M)$;

(c) N is a complement submodule of some submodule of M and $N \supseteq Z(M)$.

(d) N is a closed submodule M.

For the following sections of this chapter unless and otherwise specified R is considered as a commutative ring.

5.2 Singular Fuzzy Ideal:

This section deals with the study of singular fuzzy ideals of rings.

Definition 5.2.1. *Let δ be a fuzzy ideal of R. We define fuzzy subset $Z(\delta)$ and $T(\delta)$ by:*

$Z(\delta) = \bigcup\{\gamma | \gamma \in [0,1]^R, \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\}$

$T(\delta) = \bigcup\{\gamma | \gamma \in [0,1]^R, \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0, \text{for some fuzzy ideal } \sigma\}$

Clearly we have $T(\delta) \supseteq Z(\delta)$.

Lemma 5.2.1. $Z(\delta) = \bigcup\{r_\alpha | r_\alpha \in \delta, r_\alpha\sigma \subseteq \chi_0, , \text{for some essential fuzzy ideal } \sigma\}$

Proof. Clearly $\{r_\alpha | r_\alpha \in \delta, r_\alpha\sigma \subseteq \chi_0, , \text{for some essential fuzzy ideal } \sigma\}$

$\subseteq \{\gamma | \gamma \in [0,1]^R, \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\}$

Therefore,$\bigcup\{r_\alpha | r_\alpha \in \delta, r_\alpha\sigma \subseteq \chi_0,$, for some essential fuzzy ideal $\sigma\}$

$\subseteq \bigcup\{\gamma|\gamma \in [0,1]^R, \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0,$ for some essential fuzzy ideal $\sigma\}$

$= Z(\delta)$

Let $\gamma \in [0,1]^R$ such that $\gamma\sigma \subseteq \chi_0$, for some essential fuzzy ideal σ.

Let $r \in R$ such that $\gamma(r) = \alpha$.

Now,$(r_\alpha\sigma)(x) = \vee\{r_\alpha(s) \wedge \sigma(y) | x = sy; s, y \in R\}$

$= \vee\{\gamma(r) \wedge \sigma(y) | x = ry; s, y \in R\}$

$\leq \{\gamma(s) \wedge \sigma(y) | x = sy; s, y \in R\}$

$= (\gamma\sigma)(x)$

$\subseteq \chi_0(x)$.

Thus $(r_\alpha\sigma) \subseteq \chi_0$.

So, $Z(\delta) \subseteq \bigcup\{r_\alpha | r_\alpha \in \delta, r_\alpha\sigma \subseteq \chi_0,$, for some essential fuzzy ideal $\sigma\}$

Hence the result follows. ∎

Lemma 5.2.2. $Z(\delta) = \bigcup\{\gamma|\gamma \in FI(R), \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0,$ *for some essential fuzzy ideal* $\sigma\}$

Proof. Clearly $\{\gamma|\gamma \in FI(R), \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0,$ for some essential fuzzy ideal $\sigma\}$

$\subseteq \{\gamma|\gamma \in [0,1]^R, \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0,$ for some essential fuzzy ideal $\sigma\}$

Therefore $\bigcup\{\gamma|\gamma \in FI(R), \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0,$ for some essential fuzzy ideal $\sigma\} \subseteq Z(\delta)$.

Let $r \in R, \alpha \in [0,1]$ and $(r_\alpha\sigma) \subseteq \chi_0$, for some essential fuzzy ideal σ.

Let $\gamma = \langle r_\alpha \rangle$.

Now, $\langle r_\alpha \rangle\sigma = (\chi_0 \cup \alpha_{\langle r \rangle})\sigma$

$= \chi_0\sigma \cup \alpha_{\langle r\rangle}\sigma$

$\subseteq \chi_0 \cup \alpha_{\langle r\rangle}\sigma$

Again,$(\alpha_{\langle r\rangle}\sigma) = \vee\{\alpha_{\langle r\rangle}(s) \wedge \sigma(y)|s, y \in R, sy = x\}$

$= \vee\{\alpha \wedge \sigma(y)|s \in \langle r\rangle, y \in R, sy = x\}$

$\leq \vee\{(r_\alpha\sigma)(ry)|t, r, y \in R, x = t(ry)\}$

$\leq \vee\{\chi_0(ry)t, r, y \in R, x = t(ry)\}$

$\leq \vee\{\chi_0(t(ry))|t, r, y \in R, x = t(ry)\}$

$= \chi_0(x), \forall x.$

So,$r_\alpha\sigma \subseteq \chi_0\chi_0 = \chi_0$.

Hence $\bigcup\{\gamma|\gamma \in FI(R), \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\} \supseteq$

$\supseteq \bigcup\{r_\alpha|r \in R, \alpha \in [0,1], r_\alpha\sigma \subseteq \chi_0, \text{for some essential fuzzy ideal} = Z(\delta)$

Therefore $Z(\delta) = \bigcup\{\gamma|\gamma \in FI(R), \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\}$

▪

Lemma 5.2.3. *$Z(\delta)$ is a fuzzy ideal of δ.*

Proof. Let $r_1, r_2 \in R$ be any two elements.

Now, $Z(\delta)(r_1) \wedge Z(\delta)(r_2)$

$= [\vee\{\gamma_1(r_1)|\gamma_1 \in FI(R), \gamma_1 \subseteq \delta, \gamma_1\sigma_1 \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma_1\}]$

$\bigwedge[\vee\{\gamma_2(r_2)|\gamma_2 \in FI(R), \gamma_2 \subseteq \delta, \gamma_2\sigma_2 \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma_2\}]$

$= \vee\{\gamma_1(r_1) \wedge \gamma_2(r_2)|\gamma_1, \gamma_2 \in FI(R), \gamma_1, \gamma_2 \subseteq \delta, \gamma_1\sigma_1 \subseteq \chi_0, \gamma_2\sigma_2 \subseteq \chi_0,$

$\text{for some essential fuzzy ideal } \sigma_1 \text{ and } \sigma_2\}$

$\leq \vee\{(\gamma_1 + \gamma_2)(r_1) \wedge (\gamma_1 + \gamma_2)(r_2)|\gamma_1, \gamma_2 \in FI(R), \gamma_1, \gamma_2 \subseteq \delta,$

$\gamma_1\sigma_1 \subseteq \chi_0, \gamma_2\sigma_2 \subseteq \chi_0\}$

We have, $\gamma_1\sigma_1 \subseteq \chi_0, \gamma_2\sigma_2 \subseteq \chi_0$

$\Rightarrow (\gamma_1 + \gamma_2)(\sigma_1 \cap \sigma_2)$

$\subseteq \gamma_1(\sigma_1 \cap \sigma_2) + \gamma_2(\sigma_1 \cap \sigma_2)$

$\subseteq \gamma_1\sigma_1 + \gamma_2\sigma_2$

$\subseteq \chi_0 + \chi_0 \subseteq \chi_0$

Thus we get,

$Z(\delta)(r_1) \wedge Z(\delta)(r_2)$

$\leq \vee\{(\gamma_1 + \gamma_2)(r_1 - r_2)|(\gamma_1 + \gamma_2)\sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\}$

$= Z(\delta)(r_1 - r_2)$

Again, $Z(\delta)(sr)$

$= \vee\{\gamma(sr)|\gamma \in FI(R), \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\}$

$\geq \{\gamma(r)|\gamma \in FI(R), \gamma \subseteq \delta, \gamma\sigma \subseteq \chi_0, \text{for some essential fuzzy ideal } \sigma\}$

$= Z(\delta)(r), \forall s, r \in R$

Therefore $Z(\delta)$ is a fuzzy ideal of R. ∎

Theorem 5.2.4. [26] *For $r_\alpha \in [0,1]^R, r_\alpha \in Z(\chi_R)$ if and only if $ann(r_\alpha)$ is essential.*

Proof. Let $r_\alpha \in Z(\chi_R)$.

Then $r_\alpha\sigma \subseteq \chi_0$, for some essential fuzzy ideal σ.

Now σ being essential , $ann(r_\alpha)$ is also essential.

Conversely, let $ann(r_\alpha)$ is also essential.

Now, $r_\alpha ann(r_\alpha) \subseteq \chi_0$.

By definition of $Z(\chi_R), r_\alpha \in Z(\chi_R)$. ∎

Lemma 5.2.5. *[28] $Z(\chi_R)$ is a fuzzy ideal of R.*

Lemma 5.2.6. *[26] Let $\mu \in [0,1]^R$. Then $ann(\mu) = \{r_\alpha | r \in R, \alpha \in [0,1], r_\alpha \mu \subseteq \chi_0\}$.*

Definition 5.2.2. *Let μ be a fuzzy ideal of R. μ is called a nil ideal if each of its fuzzy point is nilpotent. R is called a fuzzy nil ring if each of its fuzzy ideal is nilpotent.*

Definition 5.2.3. *A ring R is said to be strongly fuzzy prime if every non-zero fuzzy ideal μ of R contains a finite fuzzy subset $\nu \subseteq \mu$ of fuzzy points such that $ann(\nu) = \chi_0$.*

Theorem 5.2.7. *For every commutative nil fuzzy ring R, χ_R is singular.*

Proof. Clearly, $Z(\chi_R) \subseteq \chi_R$. Let $a_\alpha \in \chi_R$. Let σ be a fuzzy ideal and $b_{\alpha'} \in \sigma$ where $b_{\alpha'} \neq 0$.

Let n be the smallest natural number such that $a_\alpha^n . b_{\alpha'} \subseteq \chi_0$. (5.2.1)

Now,

$$\begin{aligned}(b_{\alpha'}.\chi_R)(y) &= \bigvee \{b_{\alpha'}(x) \wedge \chi_R(z) | y = xz\} \\ &= \begin{cases} \alpha', if \quad y = xz \text{ for some } z \\ 0, \quad otherwise \end{cases} \\ &= \langle b \rangle_{\alpha'}(y)\end{aligned}$$

Thus $(b_{\alpha'}.\chi_R) = \langle b \rangle_{\alpha'}$.

If $y \notin \langle b \rangle$, then $\langle b \rangle_{\alpha'}(y) \subseteq \sigma(y)$.

Also if $y \in \langle b \rangle$, then $y = br$, for some $r \in R$.

Then $\sigma(y) = \sigma(br) \geq \sigma(b) \geq \alpha' = \langle b \rangle_{\alpha'}(y)$.

$\Rightarrow \langle b \rangle_{\alpha'} \subseteq \sigma$

That is $b_{\alpha'}.\chi_R \subseteq \sigma$.

Now either $a_\alpha.b_{\alpha'} \subseteq \chi_0$ or $a_\alpha.b_{\alpha'} \nsubseteq \chi_0$.

So $a_\alpha.b_{\alpha'} \subseteq \chi_0 \Rightarrow b_{\alpha'} \in ann(a_\alpha)$

and $a_\alpha.b_{\alpha'} \nsubseteq \chi_0$

$\Rightarrow a_\alpha(a_\alpha^{n-1}.b_{\alpha'}) \subseteq \chi_0$ [by equation (5.2.1)]

$\Rightarrow a_\alpha^{n-1}.b_{\alpha'} \in ann(a_\alpha)$.

In both cases, $b_{\alpha'}.\chi_R \cap ann(a_\alpha) \neq \chi_0$, where $b_{\alpha'} \in \sigma$.

Therefore $\sigma \cap ann(a_\alpha) \neq \chi_0$.

$\Rightarrow ann(a_\alpha)$ is fuzzy essential.

$\Rightarrow a_\alpha \in Z(\chi_R)$

Therefore $\chi_R \subseteq Z(\chi_R)$.

Thus $Z(\chi_R) = \chi_R$. Hence χ_R is singular. ∎

Theorem 5.2.8. *If R is strongly fuzzy prime ring then χ_R is non-singular.*

Proof. Suppose $Z(\chi_R) \neq \chi_0$.

Since $Z(\chi_R)$ is a fuzzy ideal of R and R is strongly fuzzy prime,

$\Rightarrow$there exists $\mu = \{m_{\alpha_1}, m_{\alpha_2}, \ldots, m_{\alpha_n}\} \subseteq Z(\chi_R)$

such that $ann(m_{\alpha_1}) \cap ann(m_{\alpha_2}) \cap \ldots \cap ann(m_{\alpha_n}) = ann(\mu) = \chi_0$.

Again each $m_{\alpha_i} \in Z(\chi_R), \forall i = 1, 2, \ldots, n$.

This implies $ann(m_{\alpha_i})$ is fuzzy essential.

Consequently $ann(\mu)$ is fuzzy essential,which is a contradiction.

Hence $Z(\chi_R) = \chi_0$ i.e χ_R is non-singular. ∎

Theorem 5.2.9. *If μ is a fuzzy ideal of R then $Z(\mu) = \mu \cap Z(\chi_R)$.*

Proof. Let $a_\alpha \in Z(\mu)$.

Then $a_\alpha \in \mu$ and $a_\alpha.\sigma \subseteq \chi_0$, for some essential fuzzy ideal σ of R.

$\Rightarrow a_\alpha \in Z(\chi_R)$.

$\Rightarrow a_\alpha \in \mu \cap Z(\chi_R)$.

Therefore $Z(\mu) \subseteq \mu \cap Z(\chi_R)$.

Conversely, let $a_\alpha \in \mu \cap Z(\chi_R)$.

Then $a_\alpha \in \mu$ and $a_\alpha.\sigma \subseteq \chi_0$.

$\Rightarrow a_\alpha \in Z(\mu)$.

Thus $Z(\mu) \subseteq \mu \cap Z(\chi_R)$.

Hence $Z(\mu) = \mu \cap Z(\chi_R)$ ∎

Theorem 5.2.10. *If μ is a fuzzy ideal of R, then $Z(\mu)\mu \subseteq Z(\chi_R)$.*

Proof. Let $a_\alpha \in Z(\mu)$.

Then $a_\alpha.\sigma \subseteq \chi_0$, for some essential fuzzy ideal σ of R.

Now for any $b_p \in \mu$ we claim that $a_\alpha b_p \in [Z(\mu)]\mu$.

Let $a_\alpha \in Z(\mu) \Rightarrow a \in [Z(\mu)]_\alpha$.

Also $b_p \in \mu$ gives $b \in \mu_p$. Let $s = \alpha \wedge p$, then $s \leq \alpha, p$

$\Rightarrow [Z(\mu)]_\alpha \subseteq [\mu]_s$ and $\mu_p \subseteq \mu_s$.

$\Rightarrow [Z(\mu)]_\alpha \mu_p \subseteq [Z(\mu)]_s \mu_s \subseteq [Z(\mu)\mu]_s$.

$\Rightarrow ab \in [Z(\mu)]_\alpha \mu_p \subseteq [Z(\mu)\mu]_s$

$\Rightarrow (ab)_s = a_\alpha b_p \in [z(\mu)]\mu$.

Now if $b_p\sigma = \chi_0$, then $a_\alpha b_p \sigma \subseteq \chi_0 \Rightarrow \sigma \subseteq ann(a_\alpha b_p)$.

Also σ being essential, $ann(a_\alpha b_p)$ is also essential and therefore $a_\alpha b_p \in Z(\chi_R)$.

Again if $b_p\sigma \neq \chi_0$ then there exists $x(\neq 0) \in R$ such that $(b_p\sigma)(x) \neq 0$.

Thus $(b_p \cap \sigma)(x) \neq 0$i.e $b_p(x) \wedge \sigma(x) \neq 0$ and this implies $b = x$ and $\sigma(x) > 0$.

As $x \in R$ and σ is fuzzy essential, there exists fuzzy essential ideal δ of R such that $x_p\delta \subseteq \sigma$.

Thus $a_\alpha b_p \delta = a_\alpha x_p \delta \subseteq a_\alpha \sigma \subseteq \chi_0$ which implies $\delta \subseteq ann(a_\alpha b_p)$.

Now δ being essential $ann(a_\alpha b_p)$ is essential and thus $a_\alpha b_p \in Z(\chi_R)$.

Hence $Z(\mu)\mu \subseteq Z(\chi_R)$. ∎

For the following result R is considered as a non commutative ring.

Theorem 5.2.11. *Ifχ_R is a right fuzzy non-singular ring c_t is a fuzzy point of χ_R such that $c_t\chi_R$ is a fuzzy essential ideal of R then c_t is right regular.*

Proof. Let $x_t \in \chi_R$ and μ, σ be fuzzy right ideals of R such that $\mu \subseteq_e \sigma$.

Let $b_p \in \sigma$ with $x_t b_p \neq \chi_0$.

Then there is a fuzzy essential right ideal θ of R such that $b_p\theta \subseteq \mu$.

But χ_R is non-singular, so we have $x_t b_p \theta \neq \chi_0$.

Also $x_t b_p \theta \subseteq x_t \mu$.

Thus $x_t b_p \theta \neq \chi_0, x_t b_p \theta \subseteq x_t \mu \Rightarrow x_t b_p \theta \cap x_t \mu \neq \chi_0$.

This gives $x_t \mu \subseteq_e x_t \sigma$.

Taking $x_t = c_t$ with $\mu = c_t \chi_R$ and $\sigma = \chi_R$, we see that $c_t^2 \chi_R$ is fuzzy essential right ideal of R.

Similarly $c_t^k \chi_R$ is essential for each positive integer k.

Further χ_R is non-singular i.e.$Z(\chi_R) = \chi_0$ implies that χ_0 is fuzzy semiprime.

Let $\mu = c_t^n \chi_R \cap r(c_t)$ with $\mu(0) = 1$.

Now $(c_t^n \chi_R) r(c_t) \subseteq \chi_0$.

Therefore $\mu^2 = \mu\mu \subseteq (c_t^n \chi_R) r(c_t) \subseteq \chi_0$.

This implies $\mu = \chi_0$, as χ_0 is semiprime.

Therefore we have $(c_t^n \chi_R) \cap r(c_t) = \chi_0$.

But $c_t^n \chi_R$ is fuzzy essential, so this gives $r(c_t) = \chi_0$. Hence c_t is fuzzy right regular.

■

5.3 Singular Fuzzy Submodule

As in section 2 we can define singular fuzzy submodule $Z(\delta)$ where δ is a fuzzy submodule of M. Similar results as lemma 5.2.1, 5.2.2, 5.2.3 hold in case of fuzzy submodule as proved in [28].

Definition 5.3.1. *A fuzzy submodule μ of M is singular or non singular according as $Z(\mu) = \mu$ or χ_θ.*

Definition 5.3.2. *We have,* $cl(\sigma) = \bigcup\{m_\alpha | m_\alpha\delta \subseteq \sigma, \text{for some fuzzy essential}\, \delta \text{ of } R\}$. σ *is closed if* $\sigma = cl(\sigma)$.

Lemma 5.3.1. [27] *Let* $\mu, \sigma \in [0,1]^M$. *If* $\mu \subseteq \sigma$ *then* $ann(\sigma) \subseteq ann(\mu)$.

Lemma 5.3.2. *If* σ *is a fuzzy closed submodule then* σ_t *is a closed submodule of* M.

Proof. Clearly $cl(\sigma_t) \subseteq \sigma_t$. If $m_t \in cl(\sigma)$, then $m_t.\delta \subseteq \sigma$, for some fuzzy ideal δ of R. Then for any $w \in m.\delta_t, w = mz$, for some $z \in \delta_t$. So $\delta(z) \geq t$, and

$$t = t \wedge \delta(z) \leq sup_{w=mx}\{t \wedge \delta(x)\} = m_t\delta(w) \leq \sigma(w)$$

. Thus $t \leq \sigma(w)$, that is $w \in \sigma_t$. Therefore $m.\delta_t \subseteq \sigma_t$ and therefore $m \in cl(\sigma_t)$, which implies $[cl(\sigma)]_t \subseteq cl(\sigma_t)$. Now let $\in \sigma_t$ then $y_t \in \sigma = cl(\sigma)$, that is $y \in [cl(\sigma)]_t \subseteq cl(\sigma_t)$. Thus $\sigma_t \subseteq cl(\sigma_t)$. Hence σ_t is closed submodule of M. ∎

Definition 5.3.3. μ *is a fuzzy complement submodule of some fuzzy submodule* δ *of* σ, *if* μ *is maximal in the set of all submodules* ν *of* σ *such that* $\nu \cap \delta = \chi_\theta$.

Definition 5.3.4. μ *is a fuzzy essential closed submodule of* σ *if whenever* ν *is a fuzzy submodule of* σ *such that* μ *is fuzzy essential submodule of* ν,$\nu = \mu$.

Lemma 5.3.3. [28] *If* ν *is an essential fuzzy submodule of* M *and* $a \neq \theta$ *then there exists an essential fuzzy ideal* σ *such that* $a_p\sigma \neq \chi_\theta$ *and* $a_p\sigma \subseteq \nu$ *where* $p \in (0,1]$.

Theorem 5.3.4. *If* μ *is a fuzzy submodule of* σ *and* μ^c *is fuzzy complement submodule of* μ *in* σ, *then the following are equivalent:*

(a) μ *is fuzzy essential closed submodule of* σ.

(b) μ *is fuzzy complement submodule of* μ^c *in* σ.

(c) μ *is fuzzy complement submodule of some submodule of* σ.

Proof. $(a) \Rightarrow (b)$: Suppose μ is a fuzzy essential closed submodule of σ.

Suppose there exists a fuzzy submodule ν of σ such that $\mu \subsetneqq \nu$ and $\nu \cap \mu^c = \chi_\theta$.

Since μ is fuzzy essential closed in σ and $\mu \subsetneqq \nu$, by (a) we have μ is not fuzzy essential submodule of σ. So there exists a non-zero fuzzy submodule ν' of ν such that $\nu' \cap \mu = \chi_\theta$.

Let $x_t \in \mu \cap (\mu^c \oplus \nu')$

$\Rightarrow (\mu \cap (\mu^c \oplus \nu'))(x) \geq t, t \in (0, 1]$.

$\Rightarrow \mu(x) \geq t$ and $(\mu^c \oplus \nu')(x) \geq t$.

$\Rightarrow$ there exists some unique $y, z \in M$ such that $x = y + z$ and

$\mu^c(y) \wedge \nu'(z) \geq t$ with $\mu^c(y) > 0, \nu'(z) > 0$.

Thus x=y+z with $x \in \mu^*, y \in (\mu^c)^*, z \in (\nu')^*$.

$\Rightarrow x = y + z \in \mu^* \cap ((\mu^c)^* \oplus (\nu')^*)$.

Also $-z + x = y \in (\mu^c)^* \cap (\nu')^* = 0$.i.e -z+x=0,y=0 and this gives $z = x \in \mu^* \cap (\nu')^* = 0$.

Thus $\mu^* \cap ((\mu^c)^* \oplus (\nu')^*) = \chi_\theta^*$.

By Theorem 3.3.7 in [48] we get $\mu \cap (\mu^c \oplus \nu') = \chi_\theta$, this contradicts that μ^c is a fuzzy complement submodule of μ in σ.

$(b) \Rightarrow (c)$: Obviously follows.

$(c) \Rightarrow (a)$: Suppose μ is a fuzzy complement submodule of ν in σ and μ is not essentially closed in σ.

Then there exists a fuzzy submodule $\delta \subseteq \sigma$ such that $\mu \nsubseteq_e \delta$.

Also $\delta \cap \nu$ is a non-zero fuzzy submodule of δ.

But $(\delta \cap \nu) \cap \mu \subseteq \nu \cap \mu = \chi_\theta$.

This implies $(\delta \cap \nu) \cap \mu = \chi_\theta$, contrary to the fact that μ is an essential fuzzy submodule of δ. ∎

Theorem 5.3.5. *Let μ, σ are fuzzy submodules of an R-module M. If μ is a fuzzy submodule of σ such that $Z(\sigma) \subseteq \mu$, then μ is a fuzzy essential submodule of $cl(\sigma)$.*

Proof. $cl(\sigma) = \bigcup\{m_\alpha | m_\alpha \delta \subseteq \sigma, \text{for some essential fuzzy ideal } \delta \text{ of } R\}$.

Let ν be a fuzzy submodule of $cl(\sigma)$ such that $\nu \cap \sigma = \chi_\theta$. Let $m_\alpha \in \nu$, this gives $m_\alpha \in cl(\sigma)$. Then we have $m_\alpha \delta \subseteq \sigma$, for some fuzzy essential δ of R.

$$\begin{aligned} (m_\alpha.\delta)(x) &= \bigvee\{m_\alpha(y) \wedge \delta(r) | y \in M, r \in R, yr = x\} \\ &= \bigvee\{\alpha \wedge \delta(r) | m \in M, r \in R, mr = x\} \\ &\leq \bigvee\{\nu(m) \wedge \delta(r) | m \in M, r \in R, mr = x\} \\ &\leq (\nu.\delta)(x) \end{aligned}$$

Thus we have $m_\alpha.\delta \subseteq \nu.\delta$.

Also for any $x \in M$, we have

$$\begin{aligned} \nu(x) = \nu(mr) &\geq \nu(m), where \ rm = x, r \in R, m \in M \\ &\geq \nu(m) \wedge \delta(r) \\ &\geq (\nu.\delta)(mr) \\ &= (\nu.\delta)(x) \end{aligned}$$

From the above it follows that $\nu.\delta \subseteq \nu$. Thus $m_\alpha.\delta \subseteq \nu.\delta \subseteq \nu$.

This implies that $m_\alpha.\delta \subseteq \nu \cap \sigma = \chi_\theta$, which gives $m_\alpha \in Z(\sigma) \subseteq \mu$ i.e $m_\alpha \in \sigma \cap \nu = \chi_\theta$ and this implies that $\nu \subseteq \chi_\theta$. Therefore $\nu = \chi_\theta$. Hence σ is fuzzy essential closed in $cl(\sigma)$. ∎

Theorem 5.3.6. *If μ is a fuzzy submodule of σ and μ^c is fuzzy complement submodule of μ in σ, then the following are equivalent:*

(a) μ is fuzzy essentially closed submodule of σ and $\mu \supseteq Z(\sigma)$.

(b) μ is a fuzzy complement submodule μ^c in σ and $\mu \supseteq Z(\sigma)$.

(c) μ is a fuzzy complement submodule of some submodule of σ and $\mu \supseteq Z(\sigma)$.

(d) μ is a fuzzy closed submodule of σ

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$: Follows from Theorem 5.3.4

$(a) \Rightarrow (d)$: Since $\mu \supseteq Z(\sigma)$, by Theorem 5.3.5, we have σ is a fuzzy essential submodule of $cl(\sigma)$. But then by hypothesis we get $\sigma = cl(\sigma)$. Hence μ is a fuzzy closed submodule of σ.

$(d) \Rightarrow (a)$

$$\begin{aligned} cl(\chi_\theta)(x) &= \bigvee\{m_\alpha(x)|m_\alpha.\delta \subseteq \chi_\theta \text{ , for some essential fuzzy ideal } \delta \text{ of } R\} \\ &\leq \bigvee\{m_\alpha(x)|m_\alpha.\delta \subseteq \mu \text{ , for some essential fuzzy ideal } \delta \text{ of } R\} \\ &= cl(\mu)(x) \end{aligned}$$

Thus we have $cl(\chi_\theta) \subseteq cl(\mu)$ and so $\mu = cl(\mu) \supseteq cl(\chi_\theta) = Z(\sigma)$.

Let $\mu \subseteq_e \nu \subseteq \sigma$. Then for each $x_t \in \nu$, $(\mu_t : x)$ is an essential left ideal of R, by lemma5.2.3 [15] and so $x \in cl(\mu_t) = \mu_t$. Thus $x_t \in \nu$ implies $x_t \in \mu$. Hence $\nu = \mu$. Therefore μ is fuzzy essentially closed submodule of σ. ∎

Theorem 5.3.7. *Let μ, σ are fuzzy submodules of an R-module M. If μ is a fuzzy superhonest in σ then μ is fuzzy essentially closed submodule of σ and $\mu \supseteq Z(\sigma)$.*

Proof: Suppose there exists a fuzzy submodule ν of σ such that $\mu \not\subseteq_e \nu$.

Then there is some $a_t \in \nu$ with $a_t \notin \mu$.

Also $(\chi_R.a_t)(a) = (\chi_R.a_t)(1a) = t \leq \nu(a)$.

$(\chi_R.a_t) \subseteq \nu$, then we have $Ra \subseteq \nu_t$.

Thus Ra is a non-zero submodule of ν_t.

Also $\mu \not\subseteq_e \nu$ implies that $\mu_t \not\subseteq_e \nu_t$.

Since $Ra \cap \mu_t \neq 0$, which implies that $(\mu_t : a) \neq 0$.

Suppose, $x \in (\mu : a_t)_t$

$\Leftrightarrow (\mu : a_t)(x) \geq t$

$\Leftrightarrow \vee\{r_t(x)|r_t a_t \in \mu, r \in R\} \geq t$

$\Leftrightarrow \vee\{t|r_t a_t \in \mu, r \in R\} \geq t$

$\Leftrightarrow \mu(xa) \geq t$

$\Leftrightarrow xa \in \mu_t$

$\Leftrightarrow x \in (\mu_t : a)$.

Thus we have $(\mu : a_t)_t = (\mu_t : a)$.

Now if $(\mu : a_t) = \chi_\theta$ then we get $(\mu : a_t)_t = (\mu_t : a) = 0$, which is a contradiction.

Thus we have $(\mu : a_t) \neq \chi_\theta$.

Hence μ is fuzzy essentially closed in σ.

Next let $m_\alpha \in T(\sigma)$

$\Rightarrow m_\alpha \delta \subseteq \chi_\theta$ for some fuzzy ideal δ of R and this gives $m.\delta_\alpha = 0$.

If $m_\alpha \in \mu$, then we are done.

If $m_\alpha \notin \mu$, then $m_\alpha \in \sigma_\alpha \backslash \mu_\alpha$ and $ma = 0$ for any $a \in \delta_\alpha$.

This implies $a = 0$ and thus $\delta_\alpha = 0$ which is a contradiction.

Therefore $m_\alpha \in \mu$.

Thus $\mu \supseteq T(\sigma)$ and $Z(\sigma) \subseteq T(\sigma)$.

Hence $\mu \supseteq Z(\sigma)$. ▪