Chapter 4

Honest and superhonest fuzzy submodules

In this chapter we attempt to fuzzify the concepts of honest and superhonest submodules of a module. This chapter has three sections. In the first section the existing results on honest and superhonest submodules are presented. The second section deals with fuzzy honest submodules and the third section is devoted to the study of superhonest fuzzy submodules.

4.1 Preliminaries

Throughout this chapter R is a non commutative ring with unity.

In this section we give the basic definitions and results which we attempt to fuzzify.

By $\mathfrak{X}$ we mean a non empty set of left ideals of R.

Definition 4.1.1. [22] *Let M be a left R-module, we define the $\mathfrak{X}$-torsion of M as the subset*

$$T_{\mathfrak{X}}(M) = \{m \in M : \text{there is } I \in \mathfrak{X} \text{ such that } Im = 0\}$$

A left R-module is $\mathfrak{X}$-torsion if $T_{\mathfrak{X}}(M) = M$ and $\mathfrak{X}$-torsionfree if $T_{\mathfrak{X}}(M) = 0$.

Definition 4.1.2. [22] *Let M be a left R-module and $N \subseteq M$. We define the $\mathfrak{X}$-closure of N in M as*

$$Cl^M_{\mathfrak{X}}(N) = \{m \in M : \text{there is } I \in \mathfrak{X} \text{ such that } Im \subseteq N\}$$

Clearly $Cl^M_{\mathfrak{X}}(0) = T_{\mathfrak{X}}(M)$

Definition 4.1.3. [22] *Let M be a left R-module and $N \subseteq M$. We call N is $\mathfrak{X}$-closed if $Cl^M_{\mathfrak{X}}(N) = N$*

Definition 4.1.4. [22] *$\mathfrak{X}$ is called weak closed under intersection if for any $I_1, I_2 \in \mathfrak{X}$ there exists $J \in \mathfrak{X}$ such that $J \subseteq I_1 \cap I_2$.*

Definition 4.1.5. [22] *$\mathfrak{X}$ is called inductive if for any $I \in \mathfrak{X}$ and any left ideal $J \supseteq I$, we have $J \in \mathfrak{X}$.*

Definition 4.1.6. [22] *$\mathfrak{X}$ is called left closed if for any $r \in R$ and any $I \in \mathfrak{X}$, there is $J \in \mathfrak{X}$ such that $Jr \subseteq I$.*

Definition 4.1.7. [22] *$\mathfrak{X}$ is called a topological filter if it is closed under intersection, inductive and left closed.*

Lemma 4.1.1. [22] *(1)If $\mathfrak{X}$ is weak closed under intersection, then for any left R-module M and any submodule $N \subseteq M$ we have that $Cl^M_{\mathfrak{X}}$ is a subgroup of M.*
(2) $\mathfrak{X}$ is weak closed under intersection if and only if $Cl^M_{\mathfrak{X}}(N_1) \cap Cl^M_{\mathfrak{X}}(N_2) = Cl^M_{\mathfrak{X}}(N_1 \cap N_2)$ for any submodules $N_1, N_2 \subseteq M$ and any left R-module M.
(3) If $\mathfrak{X}$ is weak closed under intersection, then $\mathfrak{X}$ is left closed if and only if $Cl^M_{\mathfrak{X}}(N)$ is a submodule of M for any submodule $N \subseteq M$ and any left R-module M.

Lemma 4.1.2. [22] *Let $\mathfrak{X}$ be an inductive set of left ideals, then the following statements are equivalent:*

(a) $\mathfrak{X}$ is a topological filter.

(b) $Cl_{\mathfrak{X}}^{M}(N)$ is a submodule for any submodule $N \subseteq M$.

Definition 4.1.8. [22] *If $\mathfrak{X}$ is a nonempty set of ideals such that $0 \notin \mathfrak{X}$, a submodule $N \subseteq M$ of a left R- module M is said to be $\mathfrak{X}$-honest, or N is $\mathfrak{X}$ honest in M, if for any $I \in \mathfrak{X}$ and any $m \in M$, if $0 \neq Im \subseteq N$, then $m \in N$, and we write $N \subseteq_{\mathfrak{X}}^{h} M$.*

Lemma 4.1.3. [22] *Let $H \subseteq N \subseteq M$ be submodules. If $H \subseteq N$ and $N \subseteq M$ are $\mathfrak{X}$ honest submodules, then $H \subseteq M$ is $\mathfrak{X}$-honest.*

Lemma 4.1.4. [22] *Let $\{N_\lambda : \lambda \in \Lambda\}$ be a family of $\mathfrak{X}$-honest submodules of M, then $\bigcap_\lambda N_\lambda$ is $\mathfrak{X}$-honest.*

Lemma 4.1.5. [22] *Let $N \subseteq M$ be a submodule, then the following statements are equivalent:*

(a) N is $\mathfrak{X}$-honest in M, inductive.

(b) For any $m \in Cl_{\mathfrak{X}}^{M}(N)\backslash N$ we have $(N : m) = Annm$.

(c) For any $m \in Cl_{\mathfrak{X}}^{M}(N)\backslash N$ we have $Rm \cap N = 0$

Definition 4.1.9. [7] *A submodule N of an R-module M is said to be superhonest if for all $x \in M, x \notin N$ with $rx \in N$ for some $r \in R$, it follows that $r = 0$. If I is a left ideal of R, we say I is superhonest if I is superhonest in the R-module R.*

It is known that for a submodule N of an R-module M and an element $m \in$

$M, (N : m) = \{r \in R : rm \in N\}$ is a left ideal of R. And it follows from the definition, the N is superhonest in M if and only if $(N : m) = 0$ for each $m \in M - N$.

Lemma 4.1.6. [7] *If N is a submodule of an R-module M, then the following statements are equivalent:*

(a) N is superhonest in M

(b) For each $m \in M, (N : m)$ is a superhonest left ideal of R.

Lemma 4.1.7. [7] *0 is a superhonest left ideal of R if and only if R is a domain.*

Lemma 4.1.8. [7] *If for each $i \in I, N_i$, is a superhonest submodule of an R-module M, then $\bigcap_{i \in I} N_i$ is also a superhonest submodule of M*

Lemma 4.1.9. [7] *If M and M_1 are R-modules, f is an R-homomorphism from M into M_1, then, for each superhonest submodule N_1 of M_1, $f^{-1}(N_1)$ is a superhonest submodule of M.*

Lemma 4.1.10. [7] *If N and N_1 are R-modules, f is an R-homomorphism from N into N_1, then for each superhonest submodule M_1 of N_1, $f^{-1}(M_1)$ is a superhonet submodule of N.*

Lemma 4.1.11. [22] *Let $N \subseteq M$ be an $\mathfrak{X}$-honest submodule, then $Cl^M_{\mathfrak{X}}(N) = N \cup T_{\mathfrak{X}}(M)$.*

Definition 4.1.10. [26] *Let μ be a fuzzy subset of an R-module M. Then the fuzzy subset $ann(\mu)$ of R is defined as follows:*

$ann(\mu) = \bigcup\{\eta | \eta \in [0,1]^R, \eta\mu \subseteq \chi_\theta\}$.

Lemma 4.1.12. [26] *Let* $\mu \in [0,1]^R$ *Then* $ann(\mu) = \bigcup\{r_\alpha | r \in R, \alpha \in [0,1], r_\alpha\mu \subseteq \chi_0\}$.

Definition 4.1.11. [48] *Let* μ *be a fuzzy submodule of* M *and* ν *be any fuzzy subset of* M.

Then $(\mu : \nu) = \bigcup\{\eta | \eta \in [0,1]^R, \eta\nu \subseteq \mu\}$.

Lemma 4.1.13. [48] *Let* μ, ν *be any two fuzzy subsets of* M.

Then $\mu : \nu = \bigcup\{r_\alpha | r \in R, \alpha \in [0,1], r_\alpha\nu \subseteq \mu\}$

4.2 Fuzzy honest submodules

Throughout this section let $\mathfrak{F}$ be a non empty set of fuzzy left ideals of R. In this section we define $\mathfrak{F}$-honest submodules, $\mathfrak{F}$-torsion and $\mathfrak{F}$-closure of a fuzzy submodule and various related results are established.

Definition 4.2.1. *Let* M *be a left* R*-module and* μ *be a fuzzy submodule of* M. *Then we define,*

$$T(\mu) = \bigcup\{\gamma | \gamma \in [0,1]^M, \gamma \subseteq \mu, \gamma\sigma \subseteq \chi_\theta, \textit{for some fuzzy ideal}\ \sigma\}$$

. *Also we define the* $\mathfrak{F}$*-torsion of* μ *as follows:*

$$T^M_{\mathfrak{F}}(\mu) = \bigcup\{\gamma | \gamma \in [0,1]^M, \gamma \subseteq \mu \ \textit{there is}\ \ \sigma \in \mathfrak{F}\ \textit{such that}\ \gamma\sigma \subseteq \chi_\theta\}$$

1_M *is* $\mathfrak{F}$*-torsion if* $T^M_{\mathfrak{F}}(1_M) = 1_M$ *and* $\mathfrak{F}$*-torsionfree if* $T^M_{\mathfrak{F}}(1_M) = \chi_\theta$.

Definition 4.2.2. *Let M be a left R-module and μ be a fuzzy submodule of M. Then we define,*

$Cl(\mu) = \bigcup\{\sigma | \sigma \in [0,1]^M, \sigma \subseteq \mu, \gamma\sigma \subseteq \mu, \textit{for some fuzzy ideal}\, \gamma\}$

Also we define the $\mathfrak{F}$-closure of μ as follows:

$Cl^M_{\mathfrak{F}}(\mu) = \bigcup\{\sigma | \sigma \in [0,1]^M, \sigma \subseteq \mu, \textit{ there is } \gamma \in \mathfrak{F} \textit{ such that}\, \gamma\sigma \subseteq \mu\}.$

Lemma 4.2.1. $T(\mu) = \bigcup\{m_\alpha | m_\alpha \in \mu, m_\alpha\sigma \subseteq \chi_\theta, \textit{for some fuzzy ideal}\, \sigma\}$

Proof. Clearly,$\{m_\alpha | m_\alpha \in \mu, m_\alpha\sigma \subseteq \chi_\theta, \text{for some fuzzy ideal}\, \sigma\}$

$\subseteq \{\gamma | \gamma \in [0,1]^M, \gamma \subseteq \mu, \gamma\sigma \subseteq \chi_\theta, \text{for some fuzzy ideal}\, \sigma\}.$

Therefore, $\bigcup\{m_\alpha \in \mu, m_\alpha\sigma \subseteq \chi_\theta, \text{for some fuzzy ideal}\, \sigma\}$

$\subseteq \bigcup\{\gamma | \gamma \in [0,1]^M, \gamma \subseteq \mu, \gamma\sigma \subseteq \chi_\theta, \text{for some fuzzy ideal}\, \sigma\}.$

$= T(\mu)$

Let $\gamma \in [0,1]^M$ such that $\gamma\sigma \subseteq \chi_0$, for some fuzzy ideal σ.

Let $m \in m$ such that $\gamma(m) = \alpha$.

Now,$(m_\alpha\sigma)(x) = \vee\{m_\alpha(s) \wedge \sigma(y) | x = sy; s \in M, y \in R\}$

$= \vee\{\gamma(m) \wedge \sigma(y) | x = my; s \in M, y \in R\}$

$\leq \{\gamma(s) \wedge \sigma(y) | x = sy; s \in M, y \in R\}$

$= (\gamma\sigma)(x)$

$\subseteq \chi_\theta(x).$

Thus $(m_\alpha\sigma) \subseteq \chi_\theta.$

So, $T(\mu) \subseteq \bigcup\{m_\alpha | m_\alpha \in \mu, m_\alpha\sigma \subseteq \chi_\theta, \text{for some fuzzy ideal}\, \sigma\}.$

Hence the result follows. ▪

The proofs of the following lemmas are similar.

Lemma 4.2.2. $T_{\mathfrak{F}}^{M}(\mu) = \bigcup\{m_\alpha | m_\alpha \dot{\in} \mu$ *there is* $\gamma \in \mathfrak{F}$ *such that* $m_\alpha \gamma \subseteq \chi_\theta\}$

Lemma 4.2.3. $Cl(\mu) = \bigcup\{m_\alpha | m_\alpha \in \mu, \gamma m_\alpha \subseteq \mu,$ *for some fuzzy ideal* $\gamma\}$

Lemma 4.2.4. $Cl_{\mathfrak{F}}^{M}(\mu) = \bigcup\{m_\alpha | m_\alpha \in \mu,$ *there is* $\gamma \in \mathfrak{F}$ *such that* $\gamma m_\alpha \subseteq \mu\}$

Definition 4.2.3. *Let M be a left R-module and $\mu \in F(M)$. We call μ is $\mathfrak{F}$-closed if $Cl_{\mathfrak{F}}^{M}(\mu) = \mu$*

Definition 4.2.4. *$\mathfrak{F}$ is called weak closed under intersection if for any $\mu_1, \mu_2 \in \mathfrak{F}$ there exists $\sigma \in \mathfrak{F}$ such that $\sigma \subseteq \mu_1 \cap \mu_2$.*

Definition 4.2.5. *$\mathfrak{F}$ is called inductive if for any $\mu \in \mathfrak{F}$ and any left ideal $\sigma \supseteq \mu$, we have $\sigma \in \mathfrak{F}$.*

Definition 4.2.6. *$\mathfrak{F}$ is called left closed if for any $r_t \in 1_R$ and any $\mu \in \mathfrak{F}$, there is $\sigma \in \mathfrak{F}$ such that $\sigma r_t \subseteq \mu$ i.e. $(\mu : r_t) \subseteq \sigma$.*

Definition 4.2.7. *$\mathfrak{F}$ is called a topological filter if it is closed under intersection, inductive and left closed.*

Theorem 4.2.5. *If $\mathfrak{F}$ is the set of all fuzzy essential ideals of R, then $\mathfrak{F}$ is inductive.*

Proof. Let $\mu \in \mathfrak{F}$.

If σ is any fuzzy left ideal of R such that $\sigma \supseteq \mu$.

Then $\mu \subseteq_e R$ and so $\mu \subseteq_e 1_R$.

Thus from $\mu \subseteq \sigma \subseteq 1_R$ it follows that $\sigma \subseteq_e R$ and hence $\sigma \in \mathfrak{F}$. ∎

Theorem 4.2.6. *Let M be a left R-module.*

(a)If $\mathfrak{F}$ is weak closed under intersection, then for any $\mu \in F(M)$ we have that $Cl_{\mathfrak{F}}^M(\mu)$ is a fuzzy submodule of M.

(b) If $\mathfrak{F}$ is weak closed under intersection if and only if $Cl_{\mathfrak{F}}^M(\sigma_1) \cap Cl_{\mathfrak{F}}^M(\sigma_2) = Cl_{\mathfrak{F}}^M(\sigma_1 \cap \sigma_2)$, for any two $\sigma_1, \sigma_2 \in F(M)$.

(c) If $\mathfrak{F}$ is weak closed under intersection, then $\mathfrak{F}$ is left closed if and only if $Cl_{\mathfrak{F}}^M(\sigma)$ is a fuzzy submodule of M for any $\sigma \in F(M)$.

Proof. (a) Let $m_1, m_2 \in M$.

Now, $Cl_{\mathfrak{F}}^M(\mu)(m_1) \wedge Cl_{\mathfrak{F}}^M(\mu)(m_2)$

$= (\bigvee\{\gamma_1(m_1) | \gamma_1 \in [0,1]^M, \gamma_1 \subseteq \mu,$ there is $\mu_1 \in \mathfrak{F}$ such that $\mu_1\gamma_1 \subseteq \mu\})$

$\bigwedge(\bigvee\{\gamma_2(m_2) | \gamma_2 \in [0,1]^M, \gamma_1 \subseteq \mu,$ there is $\mu_2 \in \mathfrak{F}$ such that $\mu_2\gamma_2 \subseteq \mu\})$

$= \bigvee\{\gamma_1(m_1) \wedge \gamma_2(m_2) | \gamma_i \in [0,1]^M, \gamma_i \subseteq \mu,$ there is

$\mu_i \in \mathfrak{F}$ such that $\mu_i\gamma_i \subseteq \mu; i = 1, 2\}$

$\leq \{(\gamma_1 + \gamma_2)(m_1) \wedge (\gamma_1 + \gamma_2)(m_2) | \gamma_i \in [0,1]^M, \gamma_i \subseteq \mu,$ there is

$\mu_i \in \mathfrak{F}$ such that $\mu_i\gamma_i \subseteq \mu; i = 1, 2\}$

$\leq \{(\gamma_1 + \gamma_2)(m_1 - m_2) | \mu(\gamma_1 + \gamma_2)\}$

Since $\mathfrak{F}$ is weak closed, so $\mu_1, \mu_2 \in \mathfrak{F}$ implies there is $\sigma \in \mathfrak{F}$ such that $\sigma \subseteq \mu_1 \cap \mu_2$.

Therefore we have,

$$\begin{aligned}\sigma(\gamma_1+\gamma_2) &\subseteq (\mu_1\cap\mu_2)(\gamma_1+\gamma_2)\\ &\subseteq (\mu_1\cap\mu_2)\gamma_1+(\mu_1\cap\mu_2)\gamma_2\\ &\subseteq \mu_1\gamma_1+\mu_2\gamma_2\\ &\subseteq \mu+\mu=\mu\end{aligned}$$

Thus, $Cl_{\mathfrak{F}}^M(\mu)(m_1)\wedge Cl_{\mathfrak{F}}^M(\mu)(m_2)$

$\leq \{(\gamma_1+\gamma_2)(m_1-m_2)|\sigma(\gamma_1+\gamma_2)(m_1-m_2)\subseteq\mu,$ for some $\sigma\in\mathfrak{F}\}$

$=Cl_{\mathfrak{F}}^M(\mu)(m_1-m_2)$

Also $Cl_{\mathfrak{F}}^M(\mu)(rm)$

$=\bigvee\{\gamma(rm)|\gamma\in[0,1]^M,\gamma\subseteq\mu,$ there is $\sigma\in\mathfrak{F}$ such that $\sigma\gamma\subseteq\mu\}$

$\geq\bigvee\{\gamma(m)|\gamma\in[0,1]^M,\gamma\subseteq\mu,$ there is $\sigma\in\mathfrak{F}$ such that $\sigma\gamma\subseteq\mu\}$

$=Cl_{\mathfrak{F}}^M(\mu)(m)$

Therefore $Cl_{\mathfrak{F}}^M(\mu)\in F(M)$.

(b) $(\Rightarrow)$. For any $m\in M$,

$Cl_{\mathfrak{F}}^M(\sigma_1\cap\sigma_2)(m)$

$=\bigvee\{\gamma(m)|\gamma\in[0,1]^M,\gamma\subseteq(\sigma_1\cap\sigma_2),$ there is $\sigma\in\mathfrak{F}$ such that $\sigma\gamma\subseteq\sigma_1\cap\sigma_2\}$

$=\bigvee\{\gamma(m)\wedge\gamma(m)|\gamma\in[0,1]^M,\gamma\subseteq(\sigma_1\cap\sigma_2),$ there is $\sigma\in\mathfrak{F}$ such that

$\sigma\gamma\subseteq\sigma_1,\sigma\gamma\subseteq\sigma_2\}$

$=(\bigvee\{\gamma(m)|\gamma\in[0,1]^M,\gamma\subseteq\sigma_1,$ there is $\sigma\in\mathfrak{F}$ such that $\sigma\gamma\subseteq\sigma_1\})$

$\bigwedge(\bigvee\{\gamma(m)|\gamma\in[0,1]^M,\gamma\subseteq\sigma_2,$ there is $\sigma\in\mathfrak{F}$ such that $\sigma\gamma\subseteq\sigma_2\})$

$\leq Cl_{\mathfrak{F}}^{M}(\sigma_1)(m) \bigwedge Cl_{\mathfrak{F}}^{M}(\sigma_2)(m)$

$= [Cl_{\mathfrak{F}}^{M}(\sigma_1) \cap Cl_{\mathfrak{F}}^{M}(\sigma_2)](m).$

Thus $Cl_{\mathfrak{F}}^{M}(\sigma_1 \cap \sigma_2) \subseteq Cl_{\mathfrak{F}}^{M}(\sigma_1) \cap Cl_{\mathfrak{F}}^{M}(\sigma_2).$

Also, $[Cl_{\mathfrak{F}}^{M}(\sigma_1) \cap Cl_{\mathfrak{F}}^{M}(\sigma_2)](m)$

$= Cl_{\mathfrak{F}}^{M}(\sigma_1)(m) \bigwedge Cl_{\mathfrak{F}}^{M}(\sigma_2)(m)$

$= (\bigvee\{\gamma_1(m) | \gamma_1 \in [0,1]^M, \gamma_1 \subseteq \sigma_1,$ there is $\mu_1 \in \mathfrak{F}$ such that $\mu_1\gamma_1 \subseteq \sigma_1\})$

$\bigwedge(\bigvee\{\gamma_2(m) | \gamma_2 \in [0,1]^M, \gamma_2 \subseteq \sigma_2,$ there is $\mu_2 \in \mathfrak{F}$ such that $\mu_2\gamma_2 \subseteq \sigma_2\})$

$= \bigvee\{\gamma_1(m) \wedge \gamma_2(m) | \gamma_i \in [0,1]^M, \gamma_i \subseteq \sigma_i,$ there is $\mu_i \in \mathfrak{F}$ such that

$\mu_i\gamma_i \subseteq \sigma_i;\ i = 1,2\}$

$= \bigvee\{(\gamma_1 \cap \gamma_2)(m) | \gamma_i \in [0,1]^M, \gamma_i \subseteq \sigma_i,$ there is $\mu_i \in \mathfrak{F}$ such that

$\mu_1\gamma_1 \cap \mu_2\gamma_2 \subseteq \sigma_1 \cap \sigma_2;\ i = 1,2\}$

Now,

$$\begin{aligned}(\mu_1 \cap \mu_2)(\gamma_1 \cap \gamma_2) &\subseteq (\mu_1 \cap \mu_2)\gamma_1 \cap (\mu_1 \cap \mu_2)(\gamma_2 \\ &\subseteq \gamma_1\mu_1 \cap \gamma_2\mu_2 \\ &\subseteq \sigma_1 \cap \sigma_2\end{aligned}$$

Since $\mathfrak{F}$ is weak closed, for $\mu_1, \mu_2 \in \mathfrak{F}$ there exists $\mu \in \mathfrak{F}$ such that $\mu \subseteq \mu_1 \cap \mu_2$.

Therefore, $\mu(\gamma_1 \cap \gamma_2) \subseteq (\mu_1 \cap \mu_2)(\gamma_1 \cap \gamma_2) \subseteq \sigma_1 \cap \sigma_2$.

Thus we have, $[Cl_{\mathfrak{F}}^{M}(\sigma_1) \cap Cl_{\mathfrak{F}}^{M}(\sigma_2)](m)$

$\leq \bigvee\{\gamma(m) | \gamma \in [0,1]^M, \gamma \subseteq (\sigma_1 \cap \sigma_2),$ there is $\mu \in \mathfrak{F}$

such that $\mu\gamma \subseteq \sigma_1 \cap \sigma_2\}$

$= Cl_{\mathfrak{F}}^{M}(\sigma_1 \cap \sigma_2)$.

Hence the result follows.

($\Leftarrow$). Let $\mu_1, \mu_2 \in \mathfrak{F}$.

We have, $(\mu_i 1_\alpha)(x) = \vee\{\mu_i(y) \wedge 1_\alpha(z) | yz = x, y \in R\} \leq \mu_i(x) \wedge \alpha \leq \mu_i(x)$.

Thus $\mu_i 1_\alpha \subseteq \mu_i, \forall i = 1,2$ and therefore $1_\alpha \in [Cl_{\mathfrak{F}}^{R}(\sigma_1) \cap Cl_{\mathfrak{F}}^{R}(\sigma_2)] = Cl_{\mathfrak{F}}^{R}(\sigma_1 \cap \sigma_2)$, hence there exists $\mu \in \mathfrak{F}$ such that $\mu \subseteq \sigma_1 \cap \sigma_2$.

(c) ($\Rightarrow$). Follows from part (a).

($\Leftarrow$). Let $\mu \in \mathfrak{F}$ and $r_t \in 1_R$, then $Cl_{\mathfrak{F}}^{R}(\mu) = 1_R$, hence $r_t \in Cl_{\mathfrak{F}}^{R}(\mu)$ and therefore there is $\nu \in \mathfrak{F}$ such that $\nu r_t \subseteq \mu$. ∎

Theorem 4.2.7. *Let $\mathfrak{F}$ be an inductive set of fuzzy ideals, then the following statements are equivalent:*

(a) $\mathfrak{F}$ is a topological filter.

(b) $Cl_{\mathfrak{F}}^{M}(\sigma)$ is a fuzzy submodule for any $\sigma \in F(M)$.

Proof. *(a) $\Rightarrow$ (b).* $\mathfrak{F}$ closed under intersection, so weak closed under intersection. It is given that $\mathfrak{F}$ is left closed, hence by part (c) of the above theorem the result follows.

(b) $\Rightarrow$ (a). $\mathfrak{F}$ is weak closed under intersection and left closed as $Cl_{\mathfrak{F}}^{M}(\sigma)$ is a fuzzy submodule for any $\sigma \in F(M)$. Since $\mathfrak{F}$ is inductive, therefore it is closed under intersection. Hence $\mathfrak{F}$ is a topological filter. ∎

Definition 4.2.8. *Let M be an R-module. Then a fuzzy submodule μ of M is said* *to $\mathfrak{F}$-honest in 1_M if for any fuzzy ideal $\sigma \in \mathfrak{F}$ and any $m_\alpha \in 1_M, \chi_\theta \neq \sigma m_\alpha \subseteq \mu$ implies $m_\alpha \in \mu, \alpha \in (0,1)$.*

Theorem 4.2.8. *Let $\mu \subseteq \sigma \subseteq 1_M$. If μ is $\mathfrak{F}$-honest in σ, σ is $\mathfrak{F}$-honest in 1_M then μ is $\mathfrak{F}$-honest in 1_M.*

Proof. Let $\delta \in \mathfrak{F}$ and $m_\alpha \in 1_M$ such that $\chi_\theta \neq m_\alpha\delta \subseteq \mu$.

Thus $m_\alpha\delta \subseteq \mu \subseteq \sigma$.

Since σ is $\mathfrak{F}$-honest in $1_M \Rightarrow m_\alpha \in \sigma$.

Now, $m_\alpha \in \sigma$ and $m_\alpha\delta \subseteq \mu$.

Since μ is $\mathfrak{F}$-honest in σ, so it gives $m_\alpha \in \mu$.

Hence μ is $\mathfrak{F}$-honest in 1_M. ∎

Theorem 4.2.9. *If $\{\mu_i | i \in I\}$ be a family of $\mathfrak{F}$-honest submodules, then $\bigcap_{i\in I} \mu_i$ is $\mathfrak{F}$-honest.*

Proof. Let $\delta \in \mathfrak{F}$ and $m_\alpha \in 1_M$ such that $\chi_\theta \neq m_\alpha\delta \subseteq \bigcap_{i\in I} \mu_i$.

This implies $m_\alpha\delta \subseteq \mu_i, \forall i \in I$. But each μ_i is $\mathfrak{F}$-honest in 1_M, which gives $m_\alpha \in \mu_i, \forall i \in I$ and thus $m_\alpha \in \bigcap \mu_i$. Hence $\bigcap_{i\in I} \mu_i$ is $\mathfrak{F}$-honest. ∎

Theorem 4.2.10. *Let $\sigma \in F(M)$. If σ is $\mathfrak{F}$-honest in 1_M and inductive then, for any $m_\alpha \in Cl^M_{\mathfrak{F}}(\sigma)\backslash\sigma$, we have $(\sigma : m_\alpha) = Ann(m_\alpha)$.*

Proof. Suppose σ is $\mathfrak{F}$-honest in 1_M and inductive. Clearly $Ann(m_\alpha) \subseteq (\sigma : m_\alpha)$. Let $m_\alpha \in Cl^M_{\mathfrak{F}}(\sigma)\backslash\sigma$. Then there exists $\mu \in \mathfrak{F}$ such that $\mu m_\alpha \subseteq \chi_\theta$ and this implies

$\mu \subseteq Annm_\alpha$. Since $\mathfrak{F}$ is inductive, so we have $Ann(m_\alpha) \in \mathfrak{F}$. Now $Ann(m_\alpha) \subseteq (\sigma : m_\alpha)$ implies $(\sigma : m_\alpha) \in \mathfrak{F}$. Next let $r_t \in (\sigma : m_\alpha)$, then $r_t m_\alpha \subseteq \sigma$, therefore $(\sigma : m_\alpha)m_\alpha \subseteq \sigma$ and thus $(\sigma : m_\alpha)m_\alpha \subseteq \chi_\theta$. This implies $(\sigma : m_\alpha) \subseteq Ann(m_\alpha)$. Hence $(\sigma : m_\alpha) = Ann(m_\alpha)$. ∎

Theorem 4.2.11. *Let $\sigma \in F(M)$. For any $m_\alpha \in Cl_{\mathfrak{F}}^M(\sigma)\backslash\sigma$, we have $(\sigma : m_\alpha) = Ann(m_\alpha)$, then $1_R m_\alpha \cap \sigma = \chi_\theta$,*

Proof. $1_R m_\alpha(x) = \vee\{1_R(y) \wedge m_\alpha(z) | y \in R, z \in M, x = yz\}$

$$\begin{aligned} &= \vee\{m_\alpha(z) | y \in R, z \in M, x = yz\} \\ &= \begin{cases} 0 & \text{if } x \notin Rm \\ \alpha & \text{if } x \in Rm \end{cases} \\ &= (Rm)_\alpha \end{aligned}$$

We claim that $(Rm)_\alpha$ is a fuzzy submodule.

Let $x, y \in Rm$. Then $x = r_1 m, y = r_2 m$ where $r_1, r_2 \in R$ and $x - y \in Rm$.

Now, $(Rm)_\alpha(x) \wedge (Rm)_\alpha(y) = \alpha = (Rm)_\alpha(x - y)$

So, $(Rm)_\alpha(x) \wedge (Rm)_\alpha(y) \leq (Rm)_\alpha(x - y)$

Also, $(Rm)_\alpha(x) = 0 \Rightarrow (Rm)_\alpha(x) \leq (Rm)_\alpha(rx)$

If $(Rm)_\alpha(x) \neq 0$, then $x = r_1 m \Rightarrow\Rightarrow rx = r(r_1 m) = (rr_1)m$.

Thus, $(Rm)_\alpha(rx) \neq 0$ and hence $(Rm)_\alpha(x) \leq (Rm)_\alpha(rx)$.

Therefore $(Rm)_\alpha$ is a fuzzy submodule.

Let $\mu = (\sigma : m_\alpha)$ and $L = \{r \in R | rm \in \sigma^*\}$

Let $x \in \mu^*$

$\Rightarrow \exists \alpha \in [0,1]$ such that $(xm)_{\alpha \wedge p} \in \sigma$

$\Rightarrow \sigma(xm) \geq \alpha \wedge p > 0$

$\Rightarrow xm \in \sigma^*$

$\Rightarrow x \in L$

This gives $\mu^* \subseteq L$

Again let $x \in L \Rightarrow x \in \sigma^* \Rightarrow \sigma(xm) > 0$.

Let $\sigma(xm) = \alpha$

$\Rightarrow \sigma(xm) > \alpha \wedge p$

$\Rightarrow (xm)_{\alpha \wedge p} \in \sigma \Rightarrow x_\alpha \in \mu$

$\Rightarrow \mu(x) \geq \alpha > 0 \Rightarrow x \in \mu^*$, thus $L \subseteq \mu^*$.

Hence $\mu^* = L$.

Thus $(\sigma : m_\alpha)^* = L = \{r | rm \in \sigma^*\} = (\sigma^* : m)$.

Also $[Ann(m_\alpha)]^* = Ann(m)$. Now, $[(Rm)_\alpha \cap \sigma]^* = \{x | (Rm)_\alpha \wedge \sigma(x) > 0\} = \{x | x \in Rm$ and $x \in \sigma^*\} = Rm \cap \sigma^*$.By hypothesis we have, $(\sigma : m_\alpha) = Ann(m_\alpha)$, this implies $(\sigma^* : m) = [Ann(m_\alpha)]^* = Ann(m)$ and therefore $Rm \cap \sigma^* = 0$. So, $[(Rm)_\alpha \cap \sigma]^* = 0$. Hence $(Rm)_\alpha \cap \sigma = 1_R m_\alpha \cap \sigma = \chi_\theta$. ∎

Theorem 4.2.12. *Let $\sigma \in F(M)$. If for any $m_\alpha \in Cl^M_{\mathfrak{F}}(\sigma) \backslash \sigma$, we have $1_R m_\alpha \cap \sigma = \chi_\theta$, then$\mu$ is $\mathfrak{F}$-honest.*

Proof. Let $\mu \in \mathfrak{F}, m_\alpha \in 1_M$ such that $\chi_\theta \neq \mu m_\alpha \subseteq \sigma$. If $m_\alpha \notin \sigma$ then we have, by hypothesis $1_R m_\alpha \cap \sigma = \chi_\theta$. Now $\mu m_\alpha \subseteq \sigma$ implies $\mu m_\alpha \cap \sigma = \mu m_\alpha \subseteq 1_R m_\alpha$. Therefore $\mu m_\alpha = \mu m_\alpha \cap \sigma \subseteq 1_R m_\alpha \cap \sigma = \chi_\theta$, a contradiction. Hence the result follows. ∎

As the consequences of the theorems 4.2.10, 4.2.11, 4.2.12 we obtain the following:

Theorem 4.2.13. *Let $\sigma \in F(M)$ and $\mathfrak{F}$ be inductive, then the following statements are equivalent:*

(a) σ is $\mathfrak{F}$-honest in 1_M.

(b) For any $m_\alpha \in Cl_{\mathfrak{F}}^M(\sigma)\backslash\sigma$, we have $(\sigma : m_\alpha) = Ann(m_\alpha)$.

(c) For any $m_\alpha \in Cl_{\mathfrak{F}}^M(\sigma)\backslash\sigma$, we have $1_R \cap m_\alpha = \chi_\theta$.

Theorem 4.2.14. *Let $\sigma \in F(M)$ be an $\mathfrak{F}$-honest, then $Cl_{\mathfrak{F}}^M(\sigma) = \sigma \cup T_{\mathfrak{F}}^M(1_M)$.*

Proof. Clearly $\sigma \cup T_{\mathfrak{F}}^M(1_M) \subseteq Cl_{\mathfrak{F}}^M(\sigma)$.
Now let $m_\alpha \in Cl_{\mathfrak{F}}^M(\sigma)\backslash\sigma$, there exists $\mu \in \mathfrak{F}$ such that $\mu m_\alpha = \chi_\theta$, thus $m_\alpha \in T_{\mathfrak{F}}^M(1_M)$. ∎

4.3 Fuzzy Superhonest Submodules

In this section we define fuzzy superhonest submodules and various properties on such submodules are obtained. Equivalence of superhonesty of submodules and ideals are established.

Definition 4.3.1. *Let M be an R-module. Then a fuzzy submodule μ of M is said to be fuzzy superhonest in 1_M if for all $x_t \in 1_M, x_t \notin \mu$ with $r_t x_t \subseteq \mu$ for some $r_t \in \chi_R$, it follows that $r_t \subseteq \chi_0$.*

Definition 4.3.2. *Let μ, σ be fuzzy submodules of an R module M with $\sigma \subseteq \mu$, then σ is said to be fuzzy superhonest in μ if for all $x_t \in \mu, x_t \notin \sigma$ with $r_t x_t \in \sigma$ for some $r_t \in \chi_R$, it follows that $r_t \subseteq \chi_0$.*

Lemma 4.3.1. *If μ is fuzzy superhonest in 1_M, then μ_t is superhonest in M.*

Proof. Let μ be fuzzy superhonest in 1_M. Let $x \in M, x \notin \mu_t$ with $rx \in \mu_t$, for some $r \in R$. Then $x_t \in 1_M, x_t \notin \mu$ with $(rx)_t \in \mu$. By hypothesis, $r_t \subseteq \chi_0$, $\forall t \in (0, 1]$, which implies $r = 0$, for if $r \neq 0$ then $r_t \nsubseteq \chi_0$ a contradiction. ∎

Lemma 4.3.2. *Let N be a superhonest submodule of an R-module M. If μ is fuzzy subset of M defined by*

$$\mu(x) = \begin{cases} t, if \quad x \in N - \{0\}, where \quad t \in (0, 1] \\ 1, if \quad x = 0 \\ 0, \quad otherwise \end{cases}$$

Then μ is a fuzzy superhonest of 1_M.

Proof. Let $x_t \in 1_M, x_t \notin \mu$ with $(rx)_t \in \mu$, for some $r \in R$. Then $x \in M, x \notin \mu_t$ with $rx \in \mu_t$. Since $\mu_t = N$ and N is superhonest, we have $r = 0$. Now for any $x \in M$, if $x = r$, then $r_t(x) = t$ and $\chi_0(x) = \chi_0(r) = 1$ i.e $r_t \subseteq \chi_0$. Again if $x \neq r$, then $r_t(x) = 0$ and $\chi_0(x) = 0$ i.e $r_t = \chi_0$. Thus in either case we have $r_t \subseteq \chi_0$. Hence μ is fuzzy superhonest in 1_M. ∎

Lemma 4.3.3. *B is superhonest in A if and only if χ_B is fuzzy superhonest in χ_A.*

Proof. Let B be superhonest in A. Let $x_t \in \chi_A, x_t \notin \chi_B$, with $(rx)_t \in \chi_B$. Then $x \in A, x \notin B$, with $rx \in B$. But then by hypothesis $r = 0$, this implies $r_t \subseteq \chi_0$.

Conversely let χ_B is fuzzy superhonest in χ_A. Let $x \in A, x \notin B$, with $rx \in B$, then $x_t \in \chi_A, x_t \notin \chi_B$, with $(rx)_t \in \chi_B$ and this gives $r_t \subseteq \chi_0$ i.e $r = 0$ ∎

Theorem 4.3.4. *If σ is a fuzzy submodule of an R-module A, then the following statements are equivalent:*

(a) σ is fuzzy superhonest in μ.

(b) For each $x_t \in \mu, (\sigma : x_t)$ is a fuzzy superhonest left ideal of 1_R.

Proof. (a) ⇒ (b) Let $r_t \in 1_R, r_t \notin (\sigma : x_t)$ with $(r'r)_t \subseteq (\sigma : x_t)$. Then $r'_t r_t x_t \subseteq \sigma$. Since $r_t x_t \nsubseteq \sigma$ and $(r_t x_t)(rx) = t \leq \mu(x) \leq \mu(rx)$. Thus $(rx)_t \notin \sigma, (rx)_t \in \mu$ with $r'_t r_t x_t \in \sigma$ and σ is fuzzy superhonest in μ, therefore $r'_t \subseteq \chi_0$. Hence $(\sigma) : x_t$ is fuzzy superhonest in 1_R.

(b) ⇒ (a) Let $r_t \notin \sigma, r_t \in \mu$ with $(r'r)_t \subseteq \sigma$. Then $1_t r_t = r_t \notin \sigma$ i.e$1_t \notin (\sigma : r_t)$ and $r'_t 1_t = r'_t \in (\sigma) : r_t$ with $1_t \in 1_R$. Then by hypothesis we have $r'_t \subseteq \chi_0$. Hence σ is fuzzy superhonest in μ. ∎

Theorem 4.3.5. *χ_0 is a fuzzy superhonest left ideal of χ_R if and only if R is a domain.*

Proof. Let R be a domain. Let $r_t \in \chi_R, r_t \in \chi_0$ with $r'_t r_t \subseteq \chi_0$, for some $r' \in R$. Then $r_t \in \chi_R, r \neq 0$ with $r'r = 0$. Since R is a domain we must have $r' = 0$, therefore $r'_t \subseteq \chi_0$. Hence χ_0 is fuzzy superhonest in χ_R.

Conversely χ_0 is fuzzy superhonest in χ_R. Then by lemma 0.3 $\{0\}$ is superhonest in R. Let $r(\neq 0) \in R$. If $r'r = 0$ for some $r' \in R$, then $r' = 0$, since $r \notin \{0\}$ and $r'r \in \{0\}$. Hence R is a domain. ∎

Theorem 4.3.6. *If μ and σ are fuzzy submodules of an R-module M with $\mu \subseteq \sigma$ and μ is fuzzy superhonest in σ then $T(\mu) = T(\sigma)$.*

Proof. We have $T(\mu) = \bigcup\{m_\alpha | m_\alpha \in \mu, m_\alpha\delta \subseteq \chi_\theta,$ for some fuzzy ideal δ of $R\}$.
Let $m_\alpha \in T(\mu)$. Then $m_\alpha \in \mu$ and $m_\alpha\delta \subseteq \chi_\theta$, for some fuzzy ideal δ of R. So $m_\alpha \in \sigma$ and $m_\alpha\delta \subseteq \chi_\theta$ imply $m_\alpha \in T(\sigma)$. Therefore $m_\alpha \in \mu \cap T(\sigma)$.
Conversely let $m_\alpha \in \mu \cap T(\sigma)$. Then $m_\alpha \in \mu$ and $m_\alpha\delta \subseteq \chi_\theta$, for some fuzzy ideal δ of R. So $m_\alpha \in T(\mu)$. Thus $T(\mu) = \mu \cap T(\sigma)$. Next let $m_\alpha \in T(\sigma)$, then $m_\alpha.\delta \subseteq \chi_\theta$ for some fuzzy ideal δ of R and this gives $m.\delta_\alpha = 0$.
If $m_\alpha \in \mu$, then we are done.
If $m_\alpha \notin \mu$, then $m_\alpha \in \sigma_\alpha \backslash \mu_\alpha$ and $ma = 0$ for any $a \in \delta_\alpha$.
This implies $a = 0$ and thus $\delta_\alpha = 0$ which is a contradiction.
Therefore $m_\alpha \in \mu$. Thus $\mu \supseteq T(\sigma)$. Hence $T(\mu) = \mu \cap T(\sigma) = T(\sigma)$. ∎

Theorem 4.3.7. *If for each $i \in I$, μ_i and σ are fuzzy submodules of an R-module M and each μ_i is fuzzy superhonest in σ, then $\bigcap_{i\in I} \mu_i$ is also fuzzy superhonest in σ.*

Proof. Let $a_t \in \sigma, a_t \notin \bigcap_{i\in I} \mu_i$ with $(ra)_t \in \bigcap_{i\in I} \mu_i$, for some $r \in R$. Then for some $i \in I$, we have $a_t \in \sigma, a_t \notin \mu_i$ with $(ra)_t \in \mu_i$, for some $r \in R$. Since each μ_i is fuzzy superhonest in σ, we get $r_t \subseteq \chi_0$. Hence $\bigcap_{i\in I} \mu_i$ is also fuzzy superhonest in σ. ∎

Theorem 4.3.8. *Let M and M_1 be R-modules, f be an R-homomorphism from M to M_1. If μ is fuzzy submodules of M and μ_1 is fuzzy submodule of M_1, then for each fuzzy superhonest submodule σ of μ_1, $f^{-1}(\sigma)$ is fuzzy superhonest in μ.*

Proof. Let $x \in [f^{-1}(\sigma)]^*$, then $[f^{-1}(\sigma)](x) > 0$ and this gives $\sigma(f(x)) > 0$ imply $f(x) \in \sigma^*$i.e $x \in f^{-1}(\sigma^*)$. Thus $[f^{-1}(\sigma)]^* = f^{-1}(\sigma^*)$. Now σ is fuzzy superhonest in μ_1 implies σ^* superhonest in μ_1^*. By lemma 4.1.10, $f^{-1}(\sigma^*) = [f^{-1}(\sigma)]^*$ is superhonest in μ^*. Hence $f^{-1}(\sigma)$ is fuzzy superhonest in μ. ∎

Note: Let M and M_1 be R-modules, f be an R-homomorphism from M to M_1. If μ is fuzzy submodules of M and μ_1 is fuzzy submodule of M_1, then for each fuzzy superhonest submodule σ of μ, $f(\sigma)$ need not be fuzzy superhonest in μ_1.