Chapter 3

Rings and Modules with Chain Conditions on the Fuzzy Substructures

In this chapter our attempt is to study fuzzy aspects of rings with chain conditions. Using the notion of fuzzy annihilators fuzzy Goldie ring is defined.

3.1 Basic Definitions and Results

In the first section some existing results of ring theory are stated which we attempt to fuzzify in this chapter.

Definition 3.1.1. [6] *Let M be a left R-module. A submodule K of M is said to be essential in M or to be an essential submodule of M if $K \cap A \neq 0$ whenever A is nonzero submodule of M. In these circumstances M is also called the essential extension of K.*

Lemma 3.1.1. [6] *Let M be a left R-module. The intersection of a finite number of essential submodules of M is essential in M.*

Lemma 3.1.2. [6] *Let M be a left R-module. Any submodule of M which contains* *an essential submodule of M is itself essential in M.*

Lemma 3.1.3. [6] *Let M be a left R-module. If K is an essential submodule of L and L is an essential submodule of M then K is essential in M.*

Lemma 3.1.4. [6] *Let M be a left R-module. Let a be a non-zero element of M and let K be an essential submodule of M, then there is an essential left ideal L of R such that $aL \neq 0$ and $aL \subseteq K$.*

Definition 3.1.2. [6] *The intersection of all prime ideals of a ring R is called the prime radical of R. A commutative ring R is called semi-prime if its prime radical is 0, i.e. if it has no nonzero nilpotent elements.*

Lemma 3.1.5. [6] *Let R be a commutative ring. Then the singular ideal $Z(R)$ of R is zero if and only if R is semi-prime.*

Definition 3.1.3. [6] *A ring R is called a prime ring if $\{0\}$ is a prime ideal of R. Thus a commutative ring is non-singular if and only if it is semi-prime.*

Theorem 3.1.6. [6] *Let R be a ring with the ascending chain condition for right annihilators, then the right singular ideal of R is nilpotent.*

An element c of a ring R is called right regular if $r(c) = 0$, left regular if $l(c) = 0$, and regular if $l(c) = r(c) = 0$. For example, every nonzero element of an integral domain is regular. If I is an ideal of R we set $C(I) = \{c \in R : c + I$ is a regular element of $\frac{R}{I}\}$ and $C(0)$ is the set of regular elements of R.

We next define the concept of Goldie dimension (also known as uniform dimension). A nonzero U is said to be uniform if any two nonzero submodules of U have nonzero intersection i.e. if each nonzero submodule of U is essential in U. Let M be an R-module. We say that M has finite Goldie dimension if M does not contain a direct sum of an infinite number of nonzero submodules. It is easy to show that M has finite Goldie dimension if M is Noetherian or Artinian. A ring R is said to have finite Goldie dimension if R has finite Goldie dimension as a right R-module. We call R a right Goldie ring if it has finite right Goldie dimension and satisfies the a.c.c. for right annihilators. A right Noetherian ring is right Goldie, but the converse is not true because any commutative integral domain is trivially a Goldie ring.

Lemma 3.1.7. [6] *Let M be a nonzero right R-module.*

(a) If M has finite Goldie dimension then each nonzero submodule of M contains a uniform submodule, and there is a finite number of uniform submodules of M whose sum is direct and is an essential submodule of M.

(b) Suppose that M has uniform submodules $U_1, U_2, \ldots, U_n$ such that $U_1+U_2+\ldots+U_n$ is direct and is an essential submodule of M, then M has finite Goldie dimension and the positive integer n is independent of the choice of U_i's. We call n the Goldie dimension of M.

Theorem 3.1.8. [6] *(Goldie) Let R be a semi-prime right Goldie ring and let I be an essential right ideal of R, then I contains a regular element of R.*

Theorem 3.1.9. [6] *Let R be a right non-singular ring with finite right Goldie dimension then R satisfies the a.c.c. and d.c.c. for right annihilators.*

Note: Let R be a semi-prime ring and let A and B be right ideals of R with $AB = 0$, then $(BA)^2 = 0$ and $(A \cap B)^2 = 0$ so that $BA = 0$ and $A \cap B = 0$. Thus if I is an ideal of R then $Ir(I) = 0$ so that $r(I)r = 0$. Similarly $Il(I) = 0$. Therefore $l(I) = r(I)$. If I is a right annihilator then $I = r(l(I)) = l(r(I))$ so that I is also a left annihilator, and in these circumstances we call I an annihilator ideal.

A prime ideal P of a ring R is called a minimal prime if P does not properly contain any prime ideal of R.

Lemma 3.1.10. [6] *Let R be a semi-prime ring with the a.c.c. (equivalently d.c.c.) for annihilator ideals, then R has only a finite number of minimal prime ideals. If $P_1, P_2, \ldots, P_n$ are the minimal prime ideals of R then $P_1 \cap P_2 \cap \ldots \cap P_n = 0$. Also a prime ideal of R is minimal if and only if it is an annihilator ideal.*

3.2 Some Elementary Properties on fuzzy substructures of rings

In this section we present some basic properties of fuzzy annihilators, which will be used repeatedly in the results of section 3.5.

Here $S(R)$ denotes the set of all fuzzy left ideals of R.

Lemma 3.2.1. *Let $\mu \in S(R)$ and σ be fuzzy ideal of R with $\sigma(0) = 1$. If $\sigma \subseteq l(\mu)$, then $\sigma\mu = \chi_0$.*

Proof. Let $x \in R$. Then

$$\begin{aligned}(\sigma\mu)(x) &= \bigvee_{x=yz}\{\sigma(y) \wedge \mu(z) | y, z \in R\} \\ &= \sigma(a) \wedge \mu(b), \text{ for some } x = ab.\end{aligned}$$

$$\begin{aligned}\text{Also } x = ab \neq 0 &\Rightarrow a \notin l(\mu_t) \text{ where } \mu(b) = t \\ &\Rightarrow a \notin l(\mu_s) \text{ for all } s \in [0,1] \\ &\Rightarrow l(\mu)(a) = 0 \\ &\Rightarrow \sigma(a) = 0 \text{ since } \sigma \subseteq l(\mu)\end{aligned}$$

Hence $(\sigma\mu)(x) = 0$.

Again, if $x = 0$ then $(\sigma\mu)(x) = 1$.

Thus $(\sigma\mu) = \chi_0$. ∎

Lemma 3.2.2. *Let $\mu \in S(R)$. If μ satisfies supremum condition or finite , then $l(\mu)\mu = \chi_0$.*

Proof. Let $x(\neq 0) \in R$

$$\begin{aligned}(l(\mu)\mu)(x) &= \bigvee_{x=yz}\{l(\mu)(y) \wedge \mu(z)\} \\ &= l(\mu)(a) \wedge \mu(b), \text{ for some } a, b \in R \text{ such that } x = ab\end{aligned}$$

Let $l(\mu)(a) = s(\neq 0),\ \mu(b) = t$.

Then $a \in l(\mu_s) = l(\mu_t) \Rightarrow ab = 0$, i.e. $x = 0$, a contradiction.

So $l(\mu)(a) = 0$. Hence

$$l(\mu)\mu(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Therefore $l(\mu)\mu = \chi_0$. ■

Similarly the following result follows:

Lemma 3.2.3. *Let $\mu \in S(R)$. If μ satisfies supremum condition or finite , then $\mu r(\mu) = \chi_0$.*

Lemma 3.2.4. *If μ is a fuzzy subset of R with the supremum condition, then*

(i) $\mu \subseteq l(r(\mu)) \cap r(l(\mu))$

(ii) $r\big(l(r(\mu))\big) = r(\mu)$ *and* $l\big(r(l(\mu))\big) = l(\mu)$

Proof. (i) We have,

$$\mu r(\mu) = \chi_0 \Rightarrow \mu \subseteq l(r(\mu)) \tag{3.2.1}$$

and

$$l(\mu)\mu = \chi_0 \Rightarrow \mu \subseteq r(l(\mu)) \tag{3.2.2}$$

Thus $\mu \subseteq l(r(\mu)) \cap r(l(\mu))$

(ii) By (i) we have

$\mu \subseteq l(r(\mu)) \Rightarrow r\big(l(r(\mu))\big) \subseteq r(\mu)$

and $\mu \subseteq r(l(\mu)) \Rightarrow l\big(r(l(\mu))\big) \subseteq l(\mu)$.

Now replacing μ by $r(\mu)$ and $l(\mu)$ in equation [3.2.1] and [3.2.2] of (i), we get

$l(\mu) \subseteq l(r(l(\mu)))$ and $r(\mu) \subseteq r(l(r(\mu)))$.

Hence, $l(\mu) = l(r(l(\mu)))$ and $r(\mu) = r(l(r(\mu)))$.

∎

Lemma 3.2.5. *μ be fuzzy subset of R with $\mu(0) = 1$, σ fuzzy ideal of R. If $\sigma\mu = \chi_0$ then, $\sigma \subseteq l(\mu)$*

Proof. Let $x(\neq 0) \in R$ such that $\sigma(x) = t(\neq 0)$. We consider $y \in R$ such that $y \in \mu_t$.

If $xy \neq 0$ then $(\sigma\mu)(xy) = 0$,

$\Rightarrow 0 = (\sigma\mu)(xy) \geq \sigma(x)$, which is not possible.

$\Rightarrow xy = 0, \forall y \in \mu_t$.

Thus $x \in l(\mu_t) \Rightarrow l(\mu)(x) \geq t = \sigma(x)$.

If $t = 0$, then $l(\mu)(x) \geq 0 = \sigma(x)$.

Thus for $x(\neq 0), \sigma(x) \subseteq l(\mu)(x)$

If $x = 0$, then $l(\mu)(x) = 1$, therefore $l(\mu)(0) \geq \sigma(0)$

Therefore we have, $l(\mu)(x) \geq \sigma(x), \forall x \in R$.

Hence $\sigma \subseteq l(\mu)$. ∎

Lemma 3.2.6. *μ be fuzzy subset of R with $\mu(0) = 1$, σ fuzzy ideal of R. If $\mu\sigma = \chi_0$ then $\sigma \subseteq r(\mu)$*

Proof. Proof is similar. ∎

3.3 Artinian and Noetherian Fuzzy Ring

This section is devoted to the study of rings with ascending (descending) chain conditions on their fuzzy substructures and various results are established.

Definition 3.3.1. *A ring in which every strictly descending chain of fuzzy left ideals is finite is called a fuzzy left Artinian ring. Also a fuzzy subset μ of a ring R is called fuzzy left Artinan if every strictly descending chain of fuzzy left ideals of μ is finite.*

Definition 3.3.2. *A ring in which every strictly ascending chain of fuzzy left ideals is finite is called a fuzzy left Noetherian ring. Also a fuzzy subset μ of a ring R is called fuzzy left Noetherian if every strictly ascending chain of fuzzy left ideals of μ is finite.*

Theorem 3.3.1. *If a ring R with unity is fuzzy left Artinian then every fuzzy left ideal on R has finite number of values.*

Proof. Let R be fuzzy left Artinian.Suppose there exists a fuzzy left ideal μ of R such that $\text{Im}\mu = \{\mu(x) : x \in R\}$ is infinite. Since $\text{Im}\mu \subseteq [0, 1]$, $\text{Im}\mu$ is a bounded set. Hence there is an infinite sequence $\{t_n\}$ of elements of $\text{Im}\mu$ such that either $t_1 < t_2 < t_3 \ldots$ or $t_1 > t_2 > t_3 \ldots$.

CaseI: Suppose $t_1 < t_2 < t_3 \ldots$.

Define,

$$\mu_r(x) = \begin{cases} 1 - t_r, if \quad \mu(x) \geq t_r \\ 0, \quad otherwise \end{cases}$$

Then μ_r is a fuzzy left ideal of R.

As $t_r < t_{r-1}$, we have for any $x \in R$, $\mu_r(x) < \mu_{r-1}(x)$i.e. $\mu_r \subset \mu_{r-1}$.

Thus we obtain a strictly descending sequence

$$\mu_0 \supset \mu_1 \supset \mu_2 \supset \dots$$

of fuzzy left ideals of R.

This contradicts that R is fuzzy left Artinian.

CaseII: Suppose $t_0 > t_1 > t_2 \dots$.

Define,

$$\mu_r(x) = \begin{cases} t_r, if \quad \mu(x) \geq 1 - t_r \\ 0, \quad otherwise \end{cases}$$

Then μ_r is a fuzzy left ideal of R.

Also as $t_r > t_{r-1}$,for any $x \in R$,we have $\mu_{r-1}(x) \geq \mu_r(x)$ i.e. $\mu_{r-1} \supset \mu_r$.

Thus in this case also we get a strictly descending chain

$$\mu_0 \supset \mu_1 \supset \mu_2 \supset \dots$$

of fuzzy left ideals of R, which is also a contradiction.

Thus in either case, every fuzzy left ideal of R has finite number of values. ∎

Using theorem 2 of Sen at al [53] and theorem 3.3.1 we get the following result.

Theorem 3.3.2. *A ring R with unity is fuzzy left Artinian if and only if every fuzzy left ideal on R has finite number of values.*

Theorem 3.3.3. *Let R and S be two fuzzy left Artinian rings with unity* 1. *Then $R \times S$ is also fuzzy left Artinian.*

Proof. Let μ be a fuzzy left ideal of $R \times S$.

Define fuzzy left ideals μ_1 and μ_2 of R and S respectively by

$\mu_1(x) = \mu(x, 0)$ for all $x \in R$ and

$\mu_2(y) = \mu(0, y)$ for all $y \in S$.

Now by hypothesis,

$\mu(x, 0) = \mu\{(1, 0)(x, y)\} \geq \mu(x, y) \quad , \forall y \in S$

and $\mu(0, y) = \mu\{(0, 1)(x, y)\} \geq \mu(x, y) \quad , \forall x \in R.$

Let $(x, y) \in R \times S$.

Then we have, $\mu(x, y) \leq min\{\mu_1(x), \mu_2(y)\}$.

Again $\mu(x, y) = \mu((x, 0) + (0, y)) \geq min\{\mu(x, 0), \mu(0, y)\} = min\{\mu_1(x), \mu_2(y)\}$.

Hence $\mu(x, y) = min\{\mu_1(x), \mu_2(y)\}$.

Now R and S are fuzzy Artinian rings with 1.

So, $Im\mu_1$ and $Im\mu_2$ are finite subsets of $[0, 1]$.

Hence $\mu(x, y) = min\{\mu_1(x), \mu_2(y)\}$ implies that $Im\mu$ is also a finite subset of $[0, 1]$ and consequently $R \times S$ is fuzzy left Artinian. ∎

Theorem 3.3.4. *Let R be a ring. Then if R is fuzzy left Noetherian then the set of fuzzy left ideals on R is a well ordered subset of* $[0, 1]$.

Proof. Suppose μ is fuzzy left ideal whose set of values is not a well ordered subset of $[0, 1]$. Then there exists a strictly descending sequence $\{t_n\}$ such that $\mu(x_n) = t_n$.

We define the fuzzy subset σ_n on R as

$$\sigma_n(x) = \begin{cases} t_n, if & \mu(x) \geq t_n \\ 0, & otherwise \end{cases}$$

Then σ_n is a fuzzy left ideal of R.

Also $t_{n-1} > t_n$ gives, $\sigma_{n-1} \subsetneq \sigma_n$.

Thus we get a strictly ascending chain

$$\sigma_0 \subsetneq \sigma_1 \subsetneq \sigma_2 \ldots$$

, of fuzzy left ideals of R, but this contradicts that R is fuzzy left Noetherian. ▪

Using theorem 3 of Sen at al [53] and theorem 3.3.4 we get the following result.

Theorem 3.3.5. *Let R be a ring. Then R is fuzzy left Noetherian if and only if the set of fuzzy left ideals on R is a well ordered subset of* $[0,1]$.

Lemma 3.3.6. *Let $S = \{\lambda_1, \lambda_2 \ldots \lambda_n \ldots\} \cup \{0\}$ where $\{\lambda_n\}$ is a fixed sequence, strictly descending to 0 and $0 < \lambda_n < 1$. Then a ring R is fuzzy left Noetherian if and only if for each fuzzy left ideal μ of $R, Im\mu \subset S$ implies that there exist a positive integer n_0 such that $Im\mu \subset S = \{\lambda_1, \lambda_2 \ldots \lambda_n \ldots\} \cup \{0\}$.*

Proof. If R is fuzzy left Noetherian, then from theorem 3.3.5, we have $Im\mu$ is a well ordered subset of $[0,1]$ and so the condition is necessary.

Conversely, let the condition is satisfied. If possible let, R be not fuzzy left Noetherian. Then there exists a strictly ascending chain of fuzzy left ideals $\mu_1 \subsetneq \mu_2 \subsetneq \ldots \subsetneq \mu_n \subsetneq \ldots$ in R.

Now define the fuzzy subset σ of R by

$$\sigma(x) = \begin{cases} \lambda_1, \ if \quad x \in (\mu_1)_t \\ \lambda_n, \ if \quad x \in (\mu_n)_t \backslash (\mu_{n-1})_t; n = 2, 3, \ldots \\ 0, \ if \quad x \in R \backslash \cup (\mu_n)_t \end{cases}$$

Then σ is a fuzzy ideal. This contradicts our assumption. Hence the result follows.

■

Theorem 3.3.7. *If R is fuzzy left Noetherian, $R[x]$ is also fuzzy left Noetherian.*

Proof. Let δ be a fuzzy left ideal of $R[x]$ such that $Im\delta \subseteq S$ where S is defined as in lemma 3.3.6.

Define for each positive $n > 0$, fuzzy subset μ_n of R by

$$\mu_n(x) = \begin{cases} \sup\{\delta(f(x)) : f(x) \in R[x] \\ \text{a polynmial of deg} \leq n \text{ with leading coefficient a }\}, \text{ when } a \neq 0 \\ \lambda_1, \quad \text{if} \quad a = 0 \end{cases}$$

Then μ_n is a fuzzy ideal of R.

From the definition of μ_n, we find that $Im\mu_n \subseteq S$.

Since R is fuzzy left Noetherian, $Im\mu_n$ is a finite subset of S .

Let

$$\inf_a \{\mu_n(a) : \mu_n(a) > 0\} = \lambda_{k(n)}.$$

Suppose if possible $\sup_n k(n)$ is infinite. Then there exists a strictly increasing sequence $\{n_r\}$ of positive integers such that $k(n_r)$ also strictly increases. We define

$$\sigma_r(a) = \begin{cases} 1 - \lambda_{k(n_r)}, if \quad \mu_{n_r} \geq \lambda_{k(n_r)} \\ 0, \quad otherwise \end{cases}$$

Then σ_r is a fuzzy left ideal of R.

For any $a \in R$,

CaseI: If $\mu_{n_{r+1}}(a) \geq \lambda_{k(n_{r+1})}$ and $\mu_{n_r}(a) \geq \lambda_{k(n_r)}$, then we have

$\sigma_{r+1}(a) = 1 - \lambda_{k(n_{r+1})}$ and $\sigma_r(a) = 1 - \lambda_{k(n_r)}$.

Since $\lambda_{k(n_{r+1})} < \lambda_{k(n_r)}$, so we have $\sigma_{r+1}(a) > \sigma_r(a)$.

CaseII: If $\mu_{n_{r+1}}(a) \geq \lambda_{k(n_{r+1})}$ but $\mu_{n_r}(a) = 0$,then we have

$\sigma_{r+1}(a) = 1 - \lambda_{k(n_{r+1})}$ and $\sigma_r(a) = 0$,this gives $\sigma_{r+1}(a) > \sigma_r(a)$.

CaseIII: If $\mu_{n_{r+1}}(a) = 0$ and $\mu_{n_r}(a) = 0$,then we have

$\sigma_{r+1}(a) = \sigma_r(a)$.

Thus in either case for any $a \in R$ we have, $\sigma_r(a) \leq \sigma_{r+1}(a) \Rightarrow \sigma_r \subseteq \sigma_{r+1}$.

Also for every ϵ, there exists $x \in R$ such that $\mu_{n_r}(x) < \lambda_{k(n_r)} + \epsilon$ gives $\mu_{n_r}(x) - \epsilon <$ $\lambda_{k(n_r)}$. Making $\epsilon \to 0$, we can find x such that $\mu_{n_r}(x) = \lambda_{k(n_r)}$.

Also from the definition of infimum given above we have $a_{r+1} \in R$ such that $\mu_{n_{r+1}}(a_{r+1}) = \lambda_{k(n_{r+1})} < \lambda_{k(n_r)}$.

As $\mu_{n_r}(a_{r+1}) < \mu_{n_{r+1}}(a_{r+1}) = \lambda_{k(n_{r+1})} < \lambda_{k(n_r)}$.

So, we have $\sigma_r(a_{r+1}) = 0$.

Thus there exists $a_{r+1} \in R$ such that $\sigma_r(a_{r+1}) < \sigma_{r+1}(a_{r+1})$.

Hence we get

$$\sigma_1 \subsetneq \sigma_2 \subsetneq \sigma_3 \subsetneq \dots$$

a sequence of fuzzy left ideals of R, which contradicts the fact that R is fuzzy left Noetherian.

So $\sup_n k(n)$ is finite.

Let $k_0 = \sup_n k(n)$.

Suppose if possible $\delta^{-1}(\lambda_k) \neq \phi$.

Let $f(x)$ be a polynomial of least degree in $\delta^{-1}(\lambda_k)$ and let a be its leading coefficient.

Let r_0 be the least degree of $f(x)$. Then obviously $r_0 \geq 1$.

It follows that $\mu_{r_0}(a) \geq \lambda_{k_0} > \lambda_k = \delta(f(x))$.

By definition of $\mu_{r_0}(a)$, there exists a polynomial $g(x)$ of degree $r \leq r_0$ with leading coefficient a such that $\delta(g(x)) > \mu_{r_0} - \epsilon$, where $\epsilon > 0$ is chosen so that $\mu_{r_0} - \epsilon > \lambda_k$.

We can assume $g(x)$ is of degree r_0 for otherwise we can replace $g(x)$ by $x^{r_0 - r} g(x)$.

Then

$$\delta(f(x)) \geq \min(\delta(f(x) - g(x)), \delta(g(x)) \geq \delta(f(x) - g(x))$$

and

$$\delta(f(x) - g(x)) \geq \min(\delta(f(x)), \delta(g(x)) \geq \delta(f(x))$$

. Thus $\delta(f(x)) = \delta(f(x) - g(x)) = \lambda_k$. But $deg(f(x) - g(x)) < r_0$, which contradicts that $f(x)$ is a polynomial of minimal degree in $\delta^{-1}(\lambda_k)$.

Hence $Im\delta \subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_{k_0}\} \cup \{0\}$. Consequently from lemma 3.3.6, it follows that $R[x]$ is fuzzy left Noetherian. ∎

3.4 Module Satisfying DCC on Essential Fuzzy Submodules

In this section we study some characteristics of modules with descending chain condition on fuzzy submodules.

Definition 3.4.1. *A fuzzy submodule μ of M is called a fuzzy min-E module if every descending chain on essential fuzzy submodules of μ is finite.*

Definition 3.4.2. [24] *A fuzzy submodule μ of M is called a essential fuzzy submodule of M, denoted by $\mu \subseteq_e M$ if for every nonzero fuzzy submodule θ of M,*

$\mu \cap \theta \neq \chi_\theta$.

Definition 3.4.3. [24] *Let μ and σ be two nonzero fuzzy submodules of M such that $\mu \subseteq \sigma$. Then μ is called fuzzy essential in σ, denoted by $\mu \subseteq_e \sigma$ if for every nonzero fuzzy submodule ν of M satisfying $\nu \subseteq \sigma, \mu \cap \nu \neq \chi_\theta$.*

Definition 3.4.4. [57] *A fuzzy submodule δ of M is said to be fuzzy simple submodule if $\mu \subseteq \delta$ where $\mu \in F(M)$ implies either $\mu = \chi_\theta$ or $\mu = \delta$.*

Definition 3.4.5. [57] *If μ is fuzzy submodule of M then the socle of μ, denoted by $soc\mu$ is defined as the sum of all fuzzy simple submodules of μ. If μ has no fuzzy simple submodules then $soc\mu = \chi_\theta$.*

Lemma 3.4.1. [57] *Let μ and σ be two nonzero fuzzy submodules of M such that μ is fuzzy essential in σ. Then for any fuzzy submodule δ of M, $\mu \cap \delta \subseteq_e \sigma \cap \delta$.*

Lemma 3.4.2. [48] *Let $\nu \in F(M)$ and A be a submodule of M. Define $\xi \in [0,1]^{\frac{M}{A}}$ as follows:*

$$\xi([x]) = \bigvee\{\nu(u) \mid u \in [x]\}, \textit{for all } x \in M$$

where $\frac{M}{A}$ denotes the quotient module of M with respect to A and $[x]$ represents the coset $x + A$. Then $\xi \in F(\frac{M}{A})$.

Let $\mu, \nu \in F(M)$ be such that $\mu \subseteq \nu$. Then both μ^* and ν^* are submodules of M. Clearly, $\mu^* \subseteq \nu^*$. Thus μ^* is a submodule of ν^*. Moreover, it is also clear that $\nu\mid_{\nu^*}\in F(\nu^*)$. Therefore, it follows from the above lemma that if we define $\xi \in F(\frac{\nu^*}{\mu^*})$ as follows:

$$\xi([x]) = \bigvee\{\nu(z) \mid z \in [x]\}$$

$\forall x \in \nu^*$,where $[x]$ denotes the coset $x + \nu^*$, then $\xi \in F(\frac{\nu^*}{\mu^*})$. The fuzzy submodule ξ is called the quotient of ν with respect to μ and written as $\frac{\nu}{\mu}$.

Lemma 3.4.3. [57] *If $\mu \in F(M)$ and ξ is the intersection of all essential fuzzy submodules of μ then $soc\mu = \xi$.*

Lemma 3.4.4. [24] *Let μ,ν and σ be nonzero fuzzy submodules of M such that $\mu \subseteq \nu \subseteq \sigma$. Then $\mu \subseteq_e \sigma$ if and only if $\mu \subseteq_e \nu \subseteq_e \sigma$.*

Note: For the last two results we assume that equality of support implies the equality of the fuzzy sets.

Theorem 3.4.5. *A fuzzy R -module μ is a fuzzy min-E module if and only if $\frac{\mu}{soc\mu}$ is fuzzy Artinian.*

Proof. First suppose, μ is a fuzzy min-E module.

Then because finite intersection of essential fuzzy submodules is fuzzy essential,$soc\mu$ is fuzzy essential in μ.

Also for any submodule μ_i of μ containing $soc\mu$, we have $soc\mu \subseteq \mu_i \subseteq \mu$.

Thus any submodule of μ containing $soc\mu$ is essential.

Let $\frac{\mu_1}{soc\mu} \supseteq \frac{\mu_2}{soc\mu} \supseteq \frac{\mu_3}{soc\mu} \supseteq \ldots$ be a descending chain.

Then as $soc\mu \subseteq \mu_i; \forall i$ we have each μ_i is fuzzy essential.

Thus,$\mu_1 \supseteq \mu_2 \supseteq \mu_3 \supseteq \ldots$ is a descending chain of essential fuzzy submodules of μ,which is a fuzzy min-E module and this gives $\mu_k = \mu_{k+1} = \ldots$, for some $k \in I^+$.

This gives $\frac{\mu_k}{soc\mu} = \frac{\mu_{k+1}}{soc\mu} = \ldots$.

Hence $\frac{\mu}{soc\mu}$ is fuzzy Artinian.

Conversely,suppose,$\frac{\mu}{soc\mu}$ is fuzzy Artinian.

Let $\mu_1 \supset \mu_2 \supset \mu_3 \ldots$ be any descending chain of essential fuzzy submodules of μ. As $soc\mu$ is the intersection of all essential fuzzy submodules of μ, we get $soc\mu \subseteq \mu_i \subseteq \mu$.

Thus we get,

$\frac{\mu_1}{soc\mu} \supseteq \frac{\mu_2}{soc\mu} \supseteq \frac{\mu_3}{soc\mu} \ldots$ a descending chain of fuzzy submodules of $\frac{\mu}{soc\mu}$, which is fuzzy Artinian.

So we have,$\frac{\mu_n}{soc\mu} = \frac{\mu_{n+j}}{soc\mu}$, for some $n \in I^+, j \geq 1$.

This implies $\frac{\mu_n^*}{(soc\mu)^*} = \frac{\mu_{n+j}^*}{(soc\mu)^*}$ and this gives $\mu_n^* = \mu_{n+j}^*; n \in I^+, j \geq 1$. So we have, $\mu_n = \mu_{n+j}$. ∎

Theorem 3.4.6. *Let μ and σ be fuzzy submodules of M. If σ is a fuzzy min-E module and $\frac{\mu}{\sigma}$ is fuzzy Artinian, then μ is also a fuzzy min-E module.*

Proof. We consider the descending sequence,

$$\mu_1 \supset \mu_2 \supset \mu_3 \supset \ldots$$

of essential fuzzy submodules of M contained in μ,

Consider now the sequence,

$$\frac{\mu_1 + \sigma}{\sigma} \supset \frac{\mu_2 + \sigma}{\sigma} \supset \frac{\mu_3 + \sigma}{\sigma} \supset \ldots$$

of fuzzy submodules of $\frac{\mu}{\sigma}$ as well as a sequence,

$$(\mu_1 \cap \sigma) \supset (\mu_2 \cap \sigma) \supset (\mu_3 \cap \sigma) \supset \dots$$

of essential fuzzy submodules of σ. But both these sequences are stationary,say after n-steps. So we have,

$$(\mu_n \cap \sigma) = (\mu_{n+1} \cap \sigma) = \dots$$

and

$$\frac{\mu_n + \sigma}{\sigma} = \frac{\mu_{n+1} + \sigma}{\sigma} = \dots$$

. From the first relation we have,

$(\mu_n \cap \sigma)^* = (\mu_{n+1} \cap \sigma)^* = \dots$

$\Rightarrow (\mu_n^* \cap \sigma^*) = (\mu_{n+1}^* \cap \sigma^*) = \dots$

and from the second relation we get,

$(\frac{\mu_n+\sigma}{\sigma})^* = (\frac{\mu_{n+1}+\sigma}{\sigma})^* = \dots$

$\Rightarrow \frac{(\mu_n+\sigma)^*}{\sigma^*} = \frac{(\mu_{n+1}+\sigma)^*}{\sigma^*} = \dots$

$\Rightarrow \mu_n^* + \sigma^* = \mu_{n+1}^* + \sigma^* = \dots.$

Now, $\mu_n^* = \mu_n^* \cap (\mu_n^* + \sigma^*) = \mu_n^* \cap (\mu_{n+1}^* + \sigma^*) = \mu_{n+1}^* + (\mu_n^* \cap \sigma^*) = \mu_{n+1}^* + (\mu_{n+1}^* \cap \sigma^*) = \mu_{n+1}^*$. Thus $\mu_n = \mu_{n+1}$. Hence the result follows. ∎

3.5 Fuzzy Aspects of Rings with Chain Condition on Annihilators

In this section we introduce the concept of fuzzy Goldie ring using the notion of fuzzy annihilators. Here we also try to fuzzify the results of ring theory which are stated in section 1 of this chapter.

Definition 3.5.1. [28] *Let Ω denote a non empty set. Let $x, x_\alpha \in M$ where $\alpha \in \Omega$. By the summation $\sum_{\alpha\in\Omega} x_\alpha$ we mean all but a finite number of x_α are non zero. If $\{A_\alpha\}$ is the collection of submodules of M, then we let $\sum_{\alpha\in\Omega} A_\alpha$ denote the set $\{\sum_{\alpha\in\Omega} x_\alpha | x_\alpha \in A_\alpha, \alpha \in \Omega\}$ and we let $\oplus_{\alpha\in\Omega} A_\alpha$ denote the direct sum of A_α. For $\beta \in \Omega$, we let Ω_β denote $\Omega - \{\beta\}$.*

Definition 3.5.2. [28] *Let $\{\mu_\alpha | \alpha \in \Omega\} \cup \{\mu\} \subseteq F(M)$. Then μ is said to be the direct sum of the μ_α's if $\mu = \sum_{\alpha\in\Omega} \mu_\alpha$ and $\forall x \in M, x \neq \theta$, $(\mu_\beta \cap \sum_{\alpha\in\Omega_\beta} \mu_\alpha)(x) = 0$. If μ is the direct sum of the μ_α's, then we write $\mu = \oplus_{\alpha\in\Omega}\mu_\alpha$.*

Definition 3.5.3. *Let M be a left R-module. Then a fuzzy submodule μ of M is said to be* uniform *if any two non-zero fuzzy submodules of M contained in μ has non-zero intersection. Thus each non-zero fuzzy submodule of M is an essential fuzzy submodule of μ. If μ does not contain a direct sum of infinite number of non-zero fuzzy submodules of M, then we say that μ has* finite fuzzy Goldie dimension.

Definition 3.5.4. *Let R be a ring with unity. Then R is called a* fuzzy left Goldie ring *if any chain of fuzzy left annihilators satisfies the ascending chain condition and R does not contain an infinite direct sum of fuzzy left ideals.*

Definition 3.5.5. *If a fuzzy annihilator μ is such that it satisfies*

$$\mu = r(l(\mu)) = l(r(\mu)),$$

then μ is called a fuzzy annihilator ideal.

Definition 3.5.6. *A fuzzy prime ideal μ of a ring R is called* minimal fuzzy prime *if μ does not properly contain any fuzzy prime ideal of R.*

Definition 3.5.7. *A fuzzy ideal μ is called* fuzzy nilpotent *if $\mu^n \subseteq \chi_0$, for some positive integer n.*

Definition 3.5.8. *A fuzzy point x_t is called* left fuzzy regular *if $l(x_t) = \chi_0$ and* right fuzzy regular *if $r(x_t) = \chi_0$. If $l(x_t) = \chi_0 = r(x_t)$, then we call x_t is* fuzzy regular.

Definition 3.5.9. *Let $\mu, \eta, \nu \in F(M)$. Then μ is said to be the* direct sum *of η and ν if $\mu = \eta + \nu$ and $\eta \cap \nu = \chi_0$ In this case, we write $\mu = \eta \oplus \nu$. R is called* fuzzy semi prime *ring if R has no nonzero fuzzy nilpotent ideal.*

Theorem 3.5.1. *If R is a fuzzy semi-prime commutative ring then, $Z_f(R) = 0$*

Proof.

$Z_f(R) = \{r \in R : \mu r = \chi_0, \text{for some essential fuzzy left ideal } \mu \text{ of } R \text{ satisfying } \mu(0) = 1\}$.

First we assume R is fuzzy semi-prime.

Let $z \in Z_f(R)$. Now we set

$$\mu = \nu \cap l(\chi_{(z)}) \tag{3.5.1}$$

where ν is defined as follows

$$\nu(x) = \begin{cases} t \text{ if } x \in Rz - \{0\} \\ 1 \text{ if } x = 0 \\ 0 \text{ if } x \notin Rz \end{cases}$$

From definition of ν it follows that $\nu_t = Rz$, where $t \in (0, 1]$. We have

$$l(\chi_{(z)}) = \chi_{l(z)}(x) = \begin{cases} 1\ if\ x \in l(z) \\ 0\ if\ x \notin l(z) \end{cases}$$

Now for $x(\neq 0) = yy'$, We must have, $y \notin Rz$ and $y' \notin l(z)$.

For if , $y \in Rz$ and $y' \in l(z) \;\Rightarrow\; y = rz$ and $y'z = 0$

$$\Rightarrow\; x = yy' = rzy' = ry'z = r.0 = 0, \text{ a contradiction.}$$

$$\begin{aligned} \text{Thus } (\nu l(\chi_{(z)}))(x) &= \bigvee_{x=yy'} \{\nu(y) \wedge l(\chi_{(z)}))(y') | y, y' \in R\} \\ &= \bigvee_{x=yy'} \{\nu(y) \wedge \chi_{l(z)}(y') | y, y' \in R\} \\ &= 0 \end{aligned}$$

Also for $x = 0$, $(\nu l(\chi_{(z)}))(x) = 1$.

Hence $\nu l(\chi_{(z)}) = \chi_0$.

From equation [3.5.1], we get $\mu \subseteq \nu$ and $\mu \subseteq l(\chi_{(z)})$.

$\Rightarrow \mu^2 \subseteq \nu l(\chi_{(z)}) = \chi_0$.

$\Rightarrow \mu^2 = \chi_0$.

Since R is fuzzy semi-prime we have $\mu = \chi_0$ i.e. $\nu \cap l(\chi_{(z)}) = \chi_0$.

Since $z \in Z_f(R)$, by theorem 2.4.8 we have $l(\chi_{(z)}) \subseteq_e R \Rightarrow \nu = \chi_0$ i.e.

$$\nu(x) = \begin{cases} 1 \; if \; x = 0 \\ 0 \; if \; x \neq 0 \end{cases}$$

Thus from the definition of $\nu(x)$ it follows that $Rz = 0$, this implies $z = 0$. ∎

Theorem 3.5.2. *Let R be a commutative ring with unity. If $Z_f(R) = 0$, then R is a semi-prime ring*

Proof. Suppose $Z_f(R) = 0$.

Let a be any element of R such that $a^2 = 0$ and let x be a non-zero element of R.

Then either $xa = 0 \Rightarrow x \in l(a)$,

or $xa \neq 0 \Rightarrow xa$ is non zero element of $l(a)$.

Thus in either case we have , $Rx \cap l(a) \neq 0$ i.e. $l(a)$ is essential.

Now $l(a)$ is an essential left ideal of R

$\Rightarrow \chi_{l(a)}$ is fuzzy essential in R [by lemma 2.3.2]

$\Rightarrow l(\chi_{(a)})$ is fuzzy essential in R [by lemma 2.2.2]

$\Rightarrow a \in Z_f(R) = 0$ [by theorem 2.4.8]

$\Rightarrow a = 0$.

i.e. R has no non-zero nilpotent elements. Hence R is a semi-prime ring. ∎

Theorem 3.5.3. *Let R be a ring with the a.c.c. for left annihilators, then the fuzzy left singular ideal of R is nilpotent.*

Proof. We write $Z_f(R) = Z_f$.

We have,

$Z_f \supseteq Z_f^2 \supseteq Z_f^3 \supseteq \ldots$

$\Rightarrow \; l(Z_f) \subseteq l(Z_f^2) \subseteq l(Z_f^3) \subseteq \ldots$

Since, R is a ring with the a.c.c. for left annihilators, there exists a positive integer n such that $l(Z_f^n) = l(Z_f^{n+1})$.

We now show that $Z_f^{n+1} = 0$.

Suppose if possible $Z_f^{n+1} \neq 0$.

$\Rightarrow$ there exists an element a of Z_f such that $aZ_f^n \neq 0$.

Choose an element a with $l(a)$ as large as possible.

Let $a, b \in Z_f$.

$\Rightarrow l(\chi_{(b)})$ is an essential fuzzy left ideal of R [by theorem 2.4.8]

$\Rightarrow \chi_{l(b)}$ is an essential fuzzy left ideal of R [by lemma 2.2.2]

$\Rightarrow l(b)$ is an essential left ideal of R [by lemma 2.3.2]

$\Rightarrow$ there exists an element $r \in R$ such that $ra \neq 0$ and $ra \in l(b)$.

Now we claim that $ab \in Z_f$.

Let θ be any non-zero fuzzy left ideal of R.

$$
\begin{aligned}
\text{Now, } a \in Z_f \;&\Rightarrow\; l(\chi_{(a)}) = \chi_{l(a)} \subseteq_e R \text{ [by theorem 2.4.8]}\\
&\Rightarrow\; \text{there exists } x(\neq 0) \in R \text{ such that } x_t \in \theta \text{ and } x_t \in \chi_{l(a)}\\
&\Rightarrow\; x \in \theta_t \text{ and } x \in \left(\chi_{l(a)}\right)_t = l(a)
\end{aligned}
$$

Again $x \in l(a) \Rightarrow xa = 0$. Therefore $x(ab) = (xa)b = 0b = 0$.

$\Rightarrow x \in l(ab)$

$\Rightarrow x \in \chi_{l(ab)}$.

Thus for any θ, there exists $x(\neq 0) \in R$ such that $x_t \in \theta$ and $x_t \in \chi_{l(ab)} = l(\chi_{(ab)})$.

Therefore $l(\chi_{(ab)}) \subseteq_e R \Rightarrow ab \in Z_f$ [by theorem 2.4.8].

Also we have $l(a) \subseteq l(ab)$.

But $ra \neq 0$ and $rab = 0$

$\Rightarrow r \notin l(a)$ and $r \in l(ab)$.

Therefore $l(a)$ is properly contained in $l(ab)$.

Then from choice of a it follows that $abZ_f^n = 0$. But b is arbitrary element of Z_f.

Therefore, $aZ_f^{n+1} = 0$.

i.e. $a \in l(Z_f^{n+1}) = l(Z_f^n)$

$\Rightarrow aZ_f^n = 0$, which is a contradiction.

Hence $Z_f^{n+1} = 0$. i.e. $Z_f = Z_f(R)$ is nilpotent. ∎

Theorem 3.5.4. *Let R be a semi-prime left fuzzy Goldie ring and μ be a essential fuzzy left ideal of R, then μ contains a left fuzzy regular point.*

Proof. Let μ be a non-nil fuzzy left ideal .

Let a_{1_α} be non-nilpotent fuzzy point of μ with $l(a_{1_\alpha})$ as large as possible.

We have, $l(a_{1_\alpha}) \subseteq l(a_{1_\alpha}^2)$

Also, $a_{1_\alpha}^2$ is fuzzy non-nil point of μ . By our choice $l(a_{1_\alpha}) = l(a_{1_\alpha}^2)$

If $l(a_{1_\alpha}) = \chi_0$, we stop.

If not, $l(a_{1_\alpha}) \cap \mu \neq \chi_0$.

Let a_{2_α} be a fuzzy non-nil point of $l(a_{1_\alpha}) \cap \mu$.

Now we claim that $l(a_{1_\alpha} + a_{2_\alpha}) = l(a_{1_\alpha}) \cap l(a_{2_\alpha})$

We know ,

$a_{1_\alpha} \subseteq a_{1_\alpha} + a_{2_\alpha}$ and $a_{2_\alpha} \subseteq a_{1_\alpha} + a_{2_\alpha}$

$\Rightarrow l(a_{1_\alpha} + a_{2_\alpha}) \subseteq l(a_{1_\alpha})$ and $l(a_{1_\alpha} + a_{2_\alpha}) \subseteq l(a_{2_\alpha})$

$\Rightarrow l(a_{1_\alpha} + a_{2_\alpha}) \subseteq l(a_{1_\alpha}) \cap l(a_{2_\alpha})$

Let $(l(a_{1_\alpha}) \cap l(a_{2_\alpha}))(x) = t$

$\Rightarrow l(a_{1_\alpha})(x) \geq t$ and $l(a_{2_\alpha})(x) \geq t$

Let $l(a_{1_\alpha})(x) = t_1 \Rightarrow x \in l(a_{1_\alpha})_{t_1}$ and

$l(a_{2_\alpha})(x) = t_2 \Rightarrow x \in l(a_{2_\alpha})_{t_2}$

Also, let $s = max^m\{t_1, t_2\}$

$s \geq t_1 \Rightarrow (a_{1_\alpha})_s \subseteq (a_{1_\alpha})_{t_1}$

$\Rightarrow l[(a_{1_\alpha})_{t_1}] \subseteq l[(a_{1_\alpha})_s]$

$\Rightarrow x \in l(a_{1_\alpha})_s$

Similarly , $x \in l[(a_{2_\alpha})_s]$

$\Rightarrow x \in l[(a_{1_\alpha})_s] \cap l[(a_{2_\alpha})_s] \subseteq l((a_{1_\alpha})_s + (a_{2_\alpha})_s) = l((a_{1_\alpha} + a_{2_\alpha})_s) \subseteq (l(a_{1_\alpha} + a_{2_\alpha}))_s$

$\Rightarrow l(a_{1_\alpha} + a_{2_\alpha})(x) \geq s \geq t$

$\Rightarrow l(a_{1_\alpha} + a_{2_\alpha})(x) \geq (l(a_{1_\alpha}) \cap l(a_{2_\alpha}))(x)$

Therefore, $l(a_{1_\alpha} + a_{2_\alpha}) \supseteq l(a_{1_\alpha}) \cap l(a_{2_\alpha})$

Thus we have ,

$l(a_{1_\alpha} + a_{2_\alpha}) = l(a_{1_\alpha}) \cap l(a_{2_\alpha})$

If $l(a_{1_\alpha} + a_{2_\alpha}) = \chi_0$, then we stop.

If not we continue the process.

Now, $a_{2_\alpha} \subseteq l(a_{1_\alpha}) \cap \mu \Rightarrow a_{2_\alpha} a_{1_\alpha} = \chi_0$ [by lemma 3.2.1]

Now, $a_{1_\alpha} \cap a_{2_\alpha} \subseteq a_{1_\alpha}$ and $a_{1_\alpha} \cap a_{2_\alpha} \subseteq a_{2_\alpha}$

$\Rightarrow (a_{1_\alpha} \cap a_{2_\alpha})^2 \subseteq a_{2_\alpha} a_{1_\alpha} = \chi_0$

$\Rightarrow a_{2_\alpha} \cap a_{1_\alpha} = \chi_0$

Hence the sum $a_{1_\alpha} + a_{2_\alpha}$ is direct.

If $l(a_{1_\alpha} + a_{2_\alpha}) = \chi_0$,we stop.

If not , let a_{3_α} be a non-nilpotent element of $l(a_{1_\alpha} + a_{2_\alpha}) \cap \mu$ with $l(a_{3_\alpha})$ as large as possible. Then $l(a_{3_\alpha}) = l(a_{3_\alpha}^2)$ and the sum $(a_{1_\alpha} + a_{2_\alpha} + a_{3_\alpha})$ is direct.

Thus $l(a_{1_\alpha} + a_{2_\alpha} + a_{3_\alpha}) = l(a_{1_\alpha} + a_{2_\alpha}) \cap l(a_{3_\alpha}) = l(a_{1_\alpha}) \cap l(a_{2_\alpha}) \cap l(_{3_\alpha})$.

But R has finite fuzzy Goldie dimension,the process stops after finite number of steps and when it does there fuzzy points $a_{1_\alpha}, a_{2_\alpha}, \ldots, a_{n_\alpha}$ of μ such that

$l(a_{1_\alpha} + a_{2_\alpha} + \ldots + a_{n_\alpha}) = \chi_0 \Rightarrow a_{1_\alpha} + a_{2_\alpha} + \ldots + a_{n_\alpha}$ is a fuzzy left regular point of μ . ∎

Theorem 3.5.5. *Let R be a ring with left finite fuzzy left Goldie dimension and $Z_f(R) = 0$, then R satisfies a.c.c. for left fuzzy annihilators.*

Proof. Let μ_1 and μ_2 be left fuzzy annihilators in R with $\mu_1 \subseteq \mu_2$ and $\mu_1 \subseteq_e \mu_2$.

Suppose $a_t \in \mu_2,\ t \in (0,1]$.

Then by theorem 2.4.1, there exists a essential fuzzy left ideal σ of R such that $a_t\sigma \subseteq \mu_1 \Rightarrow (a_t\sigma)_t \subseteq (\mu_1)_t$.

Therefore $(a_t)_t\sigma_t \subseteq (a_t\sigma)_t \subseteq (\mu_1)_t$

$\Rightarrow a\sigma_t \subseteq (\mu_1)_t$

$\Rightarrow (l(\mu_1)_t).a\sigma_t = 0.$

Let $r \in Z(R)$. Then $Kr = 0$, for some essential left ideal K of R.

We define a fuzzy subset μ of R as:

$$\mu(x) = \begin{cases} t \; if \; x \in K - \{0\} \\ 1 \; if \; x = 0 \\ 0 \; if \; x \notin K \end{cases}$$

Then μ is a fuzzy left ideal of R . Also $\mu_t = K, t \in (0, 1)$, which gives $\mu_t \subseteq_e R$.

Hence by lemma 2.3.1, $\mu \subseteq_e R$.

Clearly $(\mu r)(0) = \bigvee\{\mu(y) : y \in R, yr = x\} = 1$.

For $x(\neq 0)$,

$(\mu r)(x) = \bigvee\{\mu(y) : y \in R, yr = x\} = 0$, if $yr = x(\neq 0)$ then $y \notin K$, because if $y \in K$, then $x = yr \in Kr = 0$, a contradiction. Thus $\mu r = \chi_0$ implies $r \in Z_f(R) = 0 \Rightarrow r = 0$ a contradiction.

Hence $Z(R) = 0$. This gives R is non-singular.

So $l\big((\mu_1)_t\big)a = 0$

$\Rightarrow a \in r\big(l(\mu_1)_t\big) = a \in (\mu_1)_t$

$\Rightarrow a_t \in \mu_1$.

Thus $\mu_1 = \mu_2$. Therefore if μ_1 and μ_2 are left fuzzy annihilators in R and μ_1 is strictly contained in μ_2 then there is a non-zero fuzzy left ideal ν of R such that $\nu \subseteq \mu_2$ and $\mu_1 \cap \nu = \chi_0$.

If $\mu_1 \subset \mu_2 \subset \mu_3 \subset \ldots$, then

$\exists\, \nu_1 \subseteq \mu_2$, $\nu_2 \subseteq \mu_3$, $\nu_3 \subseteq \mu_4 \ldots$such that

$\mu_1 \cap \nu_1 = \chi_0$, $\mu_2 \cap \nu_2 = \chi_0$, $\mu_3 \cap \nu_3 = \chi_0 \ldots$

Thus we get a chain of direct sum of non-zero fuzzy left ideals as:

$$\mu_1 \oplus \nu_1 \subseteq \mu_1 \oplus \nu_1 \oplus \nu_2 \subseteq \mu_1 \oplus \nu_1 \oplus \nu_2 \oplus \nu_3 \subseteq \dots$$

The above chain cannot be infinite, as R has left fuzzy Goldie dimension. Hence R has the a.c.c. for fuzzy left annihilators. ∎

Theorem 3.5.6. *Let R be a fuzzy semi-prime ring with the a.c.c. for fuzzy annihilator ideals, then R has only finite number of minimal fuzzy prime ideals. If $\sigma_1, \sigma_2 \dots \sigma_n$ are the minimal fuzzy prime ideals of R then*

$$\sigma_1 \cap \sigma_2 \cap \dots \cap \sigma_n = \chi_0.$$

Also a fuzzy prime ideal of R is minimal if and only if it is an annihilator ideal.

Proof. We first show that every fuzzy annihilator ideal of R contains a product of fuzzy annihilator primes.

Suppose not, then there is fuzzy annihilator ideal μ which is maximal with respect to not containing a product of fuzzy annihilators primes. Because μ cannot itself be prime, there are fuzzy ideals σ and θ of R which strictly contain μ such that $\sigma\theta \subseteq \mu$ and therefore by lemma 3.2.2, $l(\mu)\sigma\theta = \chi_0$, so that we may take $\theta = r(l(\mu)\sigma)$.

Similarly we can take, $\sigma = l(\theta r(\mu))$. Thus σ and θ are fuzzy annihilator ideals which strictly contain μ and $\sigma\theta \subseteq \mu$. Therefore σ and θ each contain a product of fuzzy annihilators primes and hence so does μ, a contradiction.

Now, $r(\chi_0)(x) = \bigvee\{t : x \in r((\chi_0)_t))\}$, $t \in (0, 1]$ and

$l(\chi_0)(x) = \bigvee\{t : x \in l((\chi_0)_t))\}$, $t \in (0, 1]$.

Since the zero ideal of R is an annihilator,

$l(0) = r(0) \Rightarrow l((\chi_0)_t)) = r((\chi_0)_t))$.

Thus $l(\chi_0)(x) = r(\chi_0)(x), \forall x$.

$\Rightarrow l(\chi_0) = r(\chi_0)$

$\Rightarrow \chi_0$ is a fuzzy annihilator.

So, there are fuzzy primes $\sigma_1, \sigma_2, \ldots, \sigma_n$ of R such that $\sigma_1\sigma_2\ldots\sigma_n = \chi_0$.

Since $(\sigma_1 \cap \sigma_2 \cap \ldots \cap \sigma_n)^n \subseteq \sigma_1\sigma_2\ldots\sigma_n$

$\Rightarrow \sigma_1 \cap \sigma_2 \cap \ldots \cap \sigma_n = \chi_0$.

If ν is a fuzzy prime ideal of R, then $\sigma_1\sigma_2\ldots\sigma_n \subseteq \nu$ so that $\sigma_i \subseteq \nu$, for some i.

Hence $\nu = \sigma_i$.

Conversely, let ν be a fuzzy annihilator prime ideal.

To show ν is minimal fuzzy prime.

Let ν' be a fuzzy prime ideal of R with $\nu \subseteq \nu'$.

Suppose $l(\nu) \subseteq \nu'$,then $l(\nu) \subseteq \nu \Rightarrow (l(\nu))^2 = \chi_0 \Rightarrow l(\nu) = \chi_0$.

Also we have ,

$$\begin{aligned} l(\chi_R)(x) = \chi_{l(R)} &= 1, if \quad x \in l(R) = \{0\} \\ &= 0, if \quad x \notin l(R) = \{0\} \end{aligned}$$

Thus, $l(\nu) = \chi_0 \Rightarrow \nu = \chi_R$, which is not allowed.

Therefore $l(\nu)\nu \subseteq \nu'$ and ν' does not contain $l(\nu) \Rightarrow \nu \subseteq \nu'$.

Therefore , $\nu = \nu'$ i.e. ν is minimal fuzzy prime. ▪