Chapter 2

Basics on Tannakian Categories

In this chapter, we recall some basic definitions and results from the theory of Tannakian categories. It is a well-known theorem due to Saavedra that any neutral Tannakian category over a field $k$ is equivalent to the category of finite dimensional representations of some affine group scheme $G$ defined over $k$.

In Section 2.1, we recall some basic definitions and results about affine group scheme defined over a field $k$. In Section 2.2, we recall some basic definitions from the theory of tensor categories. In Section 2.3, we recall the definition of neutral Tannakian category and state the main theorem of the theory of neutral Tannakian categories (Theorem 2.3.3).

2.1 Affine Group Schemes and Hopf Algebras

In this section, we explain the correspondence between affine group schemes and Hopf algebras. In Subsection 2.1.1, we show that a representation of an affine group scheme corresponds to a comodule over a Hopf algebra. Results and proofs in this section are mainly from [Wa79].

Let $k$ be a field. Let $\text{Alg}_k$ denote the category of $k$–algebras, and let $\text{Grp}$ denote the category of groups.

**Definition 2.1.1** We say that a functor $G$ from the category $\text{Alg}_k$ to the category $\text{Grp}$ is an affine group scheme over $k$ if it is representable by some $k$–algebra $A$. We call $A$ the coordinate ring of $G$, and denote it by $k[G]$.

**Remark 2.1.2** In general a group scheme over $k$ can be defined as a group object in the category of $k$–schemes, i.e. a $k$–scheme $G$ together with $k$–morphisms
$m : G \times G \rightarrow G$ (multiplication), $e : \text{Spec}(k) \rightarrow G$ (unit) and $i : G \rightarrow G$ (inverse) subject to the usual group axioms. These morphisms induce a group structure on the set $G(S) := \text{Hom}_k(S, G)$ of $k$–morphisms into $G$ for each $k$–scheme $S$. Therefore, the contravariant functor $S \mapsto \text{Hom}_k(S, G)$ on the category of $k$–schemes represented by $G$ is in fact group-valued. Restricting it to the full subcategory of affine $k$–schemes we obtain a contravariant functor $\text{Spec}(R) \mapsto \text{Hom}_k(\text{Spec}(R), G)$. Since the contravariant functor $R \mapsto \text{Spec}(R)$ induces an isomorphism of the category of affine schemes with the opposite category of commutative rings with unit, it follows that when $G = \text{Spec}(A)$ is itself affine, then this is none but the above functor.

Let $G$ be an affine group scheme over $k$. The coordinate ring $k[G]$ of an affine group scheme $G$ carries additional structure coming from the group operations. To see this, note first that the functor $G \times G$ given by

$$R \mapsto G(R) \times G(R)$$

is representable by the tensor product $A \otimes_k A$ in view of the functorial isomorphism

$$\text{Hom}_{\text{Alg}_k}(k[G], R) \times \text{Hom}_{\text{Alg}_k}(k[G], R) \xrightarrow{\sim} \text{Hom}_{\text{Alg}_k}(k[G] \otimes_k k[G], R)$$

induced by $(\varphi, \psi) \mapsto \varphi \otimes \psi$ (the inverse map is given by $\lambda \mapsto (a \mapsto \lambda(a \otimes 1), a \mapsto \lambda(1 \otimes a))$, and the functor

$$R \mapsto \{1\}$$

is representable by $k$. Therefore, by the Yoneda Lemma, the morphism of functors

$$m : G \times G \rightarrow G,$$

given by multiplication is induced by a $k$–algebra homomorphism

$$\Delta : k[G] \rightarrow k[G] \otimes_k k[G].$$

Similarly, morphism of functors

$$e : \{1\} \rightarrow G$$
given by the identity element is induced by a $k$–algebra homomorphism

$$\varepsilon : k[G] \to k,$$

and the morphism of functors

$$i : G \to G$$

is induced by a $k$–algebra homomorphism

$$S : k[G] \to k[G].$$

The homomorphisms $\Delta$, $\varepsilon$ and $S$ are called the comultiplication, counit, and coinverse (or antipode), of $k[G]$, respectively. The following diagrams commute:

In the last two diagrams $c$ is the constant map $G \to \{1\}$, $\gamma$ the composite $k[G] \to k \to k[G]$ and $m' : k[G] \otimes_k k[G] \to k[G]$ the algebra multiplication.

The above diagrams indicate the translation of the associativity, unit and inverse axioms for groups on the left-hand side to the corresponding compatibility conditions on $k[G]$ on the right-hand side. They are called the coassociativity,
counit and antipode (or coinverse) axioms, respectively.

**Definition 2.1.3** A $k$–algebra (not necessarily commutative) equipped with the $k$–algebra maps $\Delta$, $\varepsilon$ and $S$ above and satisfying the coassociativity, counit and coinverse axioms is called *Hopf algebra*.

Hopf algebras associated with affine group schemes are always commutative.

**Remark 2.1.4** In calculations, it is often useful to write down the Hopf algebra axioms explicitly for concrete elements. For instance, if we write $\Delta(a) = \sum a_i \otimes b_i$ for the comultiplication map, then the counit axiom says that

$$a = \sum \varepsilon(a_i)b_i = \sum a_i\varepsilon(b_i)$$

and the antipode axiom says that

$$\varepsilon(a) = \sum S(a_i)b_i = \sum a_iS(b_i).$$

**Proposition 2.1.5** The functor $A \mapsto \text{Spec}(A)$ defines an anti-equivalence between the category of commutative Hopf algebras over $k$ and the category of affine group scheme over $k$.

**Example 2.1.1** The functor $R \mapsto \mathbb{G}_a(R)$ mapping a $k$–algebra $R$ to its underlying additive group $R^+$ is an affine group scheme with coordinate ring $k[x]$, in view of the functorial isomorphism $R^+ \cong \text{Hom}_k(k[x], R)$. The comultiplication map on $k[x]$ is given by $\Delta(x) = 1 \otimes x + x \otimes 1$, the counit is the zero map and the antipode is induced by $x \mapsto -x$.

**Example 2.1.2** The functor $R \mapsto \mathbb{G}_m(R)$ mapping a $k$–algebra $R$ to the group $R^\times$ of invertible elements is an affine group scheme with coordinate ring $k[x, x^{-1}]$, because an invertible element in $R$ corresponds to a $k$–algebra homomorphism $k[x, x^{-1}] \rightarrow R$. The comultiplication map on $k[x, x^{-1}]$ is given by $\Delta(x) = x \otimes x$, the counit sends $x$ to $1$ and the antipode is induced by $x \mapsto x^{-1}$.

**Example 2.1.3** More generally, the functor that maps a $k$–algebra $R$ to the group $\text{GL}_n(R)$ of invertible matrices with entries in $R$ is an affine group scheme.
To find its co-ordinate ring $A$, notice that an $n \times n$ matrix $M$ over $R$ is invertible if and only if $\det M$ is invertible in $R$. This allows us to recover $A$ as the quotient of the polynomial ring in $n^2 + 1$ variables $k[x_{11}, \ldots, x_{nn}, x]$ by the ideal generated by $\det(x_{ij})x - 1$. The isomorphism $GL_n(R) \cong \text{Hom}_{\text{Alg}}(A, R)$ is induced by sending a matrix $M = (m_{ij})$ to the homomorphism given by $x_{ij} \mapsto m_{ij}$, $x \mapsto \det(m_{ij})^{-1}$. The comultiplication is induced by $x_{ij} \mapsto \sum_k x_{ik} \otimes x_{kj}$, the counit sends $x_{ij}$ to $\delta_{ij}$ (Kronecker delta), and the antipode comes from the formula for the inverse matrix.

**Definition 2.1.6** Let $A$ be a finite dimensional $k$–algebra. We say that $A$ is **separable** if $A \otimes \bar{k}$ is reduced.

Recall that if $k$ is perfect field, then a finite $k$–algebra $A$ is separable if and only if $A$ is reduced.

**Definition 2.1.7** Let $G$ be an affine group scheme defined over a field $k$. We say that $G$ is finite if $k[G]$ is a finitely generated $k$–module.

**Definition 2.1.8** A finite group scheme $G$ over $k$ is called **étale** if $k[G]$ is separable $k$–algebra.

The proof of the following result can be found in [Wa79, p. 86-87].

**Theorem 2.1.9** All finite group schemes in characteristic zero are étale.

**Example 2.1.4** Let $\Gamma$ be a finite group. Let $A$ denote the $k$–algebra $k^\Gamma$ of functions from $\Gamma$ to $k$. Then $A$ is a product of copies of $k$ indexed by the elements of $\Gamma$. More precisely, let $e_\sigma$ be the function that is 1 on $\sigma$ and 0 on the remaining elements of $\Gamma$. The $e_\sigma$‘s are a complete system of orthogonal idempotents for $A$:

$$e_\sigma^2 = e_\sigma, \quad e_\sigma e_\tau = 0 \quad \text{for} \quad \sigma \neq \tau, \quad \sum e_\sigma = 1.$$ 

The maps

$$\Delta(e_\rho) = \sum_{\rho = \sigma \tau} (e_\sigma \otimes e_\tau), \quad \varepsilon(e_\sigma) = \begin{cases} 1 & \text{if } \sigma = 1 \\ 0 & \text{otherwise} \end{cases}, \quad S(e_\sigma) = e_{\sigma^{-1}}.$$

define a Hopf-algebra structure on $A$. Let $(\Gamma)_k$ be the associated affine group scheme, so that $(\Gamma)_k(R) = \text{Hom}_{\text{Alg}}(A, R)$. 


If $R$ has no idempotents other than 0 or 1, then a $k$–algebra homomorphism $A \rightarrow R$ must send one $e_\sigma$ to 1 and the remainder to 0; therefore, $(\Gamma)_k(R) \simeq \Gamma$, and it is easy to check that the group structure provided by the maps $\Delta, \varepsilon, S$ is the given one. For this reason, $(\Gamma)_k$ is called the constant group scheme defined by $\Gamma$.

Remark 2.1.10 Let $G$ be a finite étale group scheme over $k$. Let $k_s$ denote the separable closure of $k$. By [Wa79, Theorem 6.4, p. 49], the category of finite étale $k$–group schemes is equivalent to the category of finite groups $\Gamma$ carrying a continuous $\Gal(k_s/k)$–action. In this equivalence, the constant group scheme $(\Gamma)_k$ corresponds to $\Gamma$ with trivial Galois action. A finite group scheme $G$ is étale if and only if $G \times_k k_s$ is constant group scheme.

By ignoring the $k$–algebra structure on a Hopf algebra, we obtain the following more general notion.

Definition 2.1.11 A coalgebra $C$ over $k$ is a $k$–vector space equipped with a comultiplication $\Delta : C \rightarrow C \otimes_k C$ and a counit $\varepsilon : C \rightarrow k$ subject to the coassociativity and counit axioms.

In this definition, the map $\Delta$ and $\varepsilon$ are assumed to be only $k$–linear maps. Coalgebra over $k$ forms a category: morphisms are defined to be $k$–linear maps compatible with the $k$–coalgebra structure.

We now define right comodules over a coalgebra by dualizing the notion of left modules over a $k$–algebra $B$. Note that to give a unitary left $B$–module is to give a $k$–vector space $V$ together with a $k$–linear multiplication $l : B \otimes_k V \rightarrow V$ so that the following diagrams commute:

\[
\begin{array}{ccc}
B \otimes_k B \otimes_k V & \xrightarrow{1 \otimes l} & B \otimes_k V \\
\downarrow m' \otimes 1 & & \downarrow l \\
B \otimes_k V & \xrightarrow{l} & V
\end{array}
\]

\[
\begin{array}{ccc}
k \otimes_k V & \xrightarrow{\otimes 1} & B \otimes_k V \\
\downarrow \simeq & & \downarrow l \\
V & \xrightarrow{1} & V
\end{array}
\]

where $i : k \rightarrow B$ is the natural map sending 1 to the unit element of $B$.

Definition 2.1.12 Let $C$ be a coalgebra over a field $k$. A right $C$–comodule is a $k$–vector space $V$ together with a $k$–linear map $\rho : V \rightarrow V \otimes_k C$ such that
Remark 2.1.13 We can write out the comodule axioms explicitly on elements as follows. Assume \( \rho \) is given by
\[
\rho(v) = \sum v_i \otimes a_i, \quad \rho(v_i) = \sum v_{ij} \otimes c_j
\]
and furthermore \( \Delta(a_i) = \sum a_{ik} \otimes b_k \). Here \( v, v_i, v_{ij} \) are in \( V \) and the other elements lie in \( C \). Then the commutativity of the first diagram is described by the equality
\[
\sum_{i,k} v_i \otimes a_{ik} \otimes b_k = \sum_{i,j} v_{ij} \otimes c_j \otimes a_i.
\]
(2.2)

The second diagram reads
\[
\sum_i \varepsilon(a_i)v_i = v.
\]
(2.3)

Definition 2.1.14 A subcoalgebra of a coalgebra \( C \) is defined as a \( k \)-subspace \( B \subset C \) with the property that \( \Delta(B) \subset B \otimes_k B \). The restrictions of \( \Delta \) and \( \varepsilon \) then turn \( B \) into a coalgebra over \( k \). One defines a subcomodule of an \( C \)-comodule \( V \) as a \( k \)-subspace \( W \subset V \) with the property that \( \rho(W) \subset W \otimes_k C \).

A subcoalgebra \( B \subset C \) is also naturally a subcomodule of \( C \) considered as a right comodule over itself. Subcomodules and subcoalgebras enjoy the following basic finiteness property.

Proposition 2.1.15 Let \( C \) be a \( k \)-coalgebra and \((V, \rho)\) a comodule over \( C \). Any finite subset of \( V \) is contained in a subcomodule of \( V \) having finite dimension over \( k \). In particular, any finite subset of \( C \) is contained in a subcoalgebra of \( C \) having finite dimension over \( k \).

Proof. Let \( \{a_i\} \) be a basis for \( C \) over \( k \). If \( v \) is in the finite subset, write
\[
\rho(v) = \sum v_i \otimes a_i \text{ (finite sum).}
\]
(2.4)

The \( k \)-space generated by the \( v \) and \( v_i \) is a subcomodule of \( V \). For, from the
\[ (\rho \otimes \text{id}_A)(\rho(v)) = \sum_i \rho(v_i) \otimes a_i. \]

On the other hand, by the first comodule axiom, we have
\[ (\rho \otimes \text{id}_A)(\rho(v)) = \sum_i v_i \otimes \Delta(a_i). \]

Writing \( \Delta(a_i) = \sum_{j,k} \lambda_{ijk} (a_j \otimes a_k) \), we obtain
\[ \sum_k \rho(v_k) \otimes a_k = \sum_i v_i \otimes \sum_{j,k} \lambda_{ijk} (a_j \otimes a_k). \]

Since \( \{a_i\} \) is a basis for \( C \) over \( k \), it follows that
\[ \rho(v_k) = \sum_i v_i \otimes \sum_{j,k} \lambda_{ijk} a_j \]
for all \( k \). This proves that \( k \)–space generated by the \( v \) and \( v_i \) is a subcomodule of \( V \). Note that by the \( k \)–linearity of \( \rho : V \to V \otimes_k C \), \( k \)–linear span of finitely many subcomodules of \( V \) is again a subcomodule. \( \square \)

**Corollary 2.1.16** Any comodule \((V, \rho)\) over \( C \) is direct limit of its finite dimensional subcomodules.

### 2.1.1 Representations and Comodules

Let \( G \) be an affine group scheme over \( k \), and let \( V \) be a vector space over \( k \) (not necessarily finite dimensional).

**Definition 2.1.17** A linear representation of \( G \) on \( V \) is a natural homomorphism
\[ \Phi : G(R) \to \text{Aut}_R(V \otimes_k R). \]

In other words, for each \( k \)–algebra \( R \), we have an action
\[ G(R) \times (V \otimes_k R) \to (V \otimes_k R) \]
of $G(R)$ on $(V \otimes_k R)$ in which each $g \in G(R)$ acts $R$–linearly, and for each homomorphism of $k$–algebra $R \rightarrow S$, the following diagram

$$
\begin{array}{ccc}
G(R) \times (V \otimes_k R) & \rightarrow & (V \otimes_k R) \\
\downarrow & & \downarrow \\
G(S) \times (V \otimes_k S) & \rightarrow & (V \otimes_k S)
\end{array}
$$

commutes.

Let $\Phi$ be a linear representation of $G$ on $V$. Given a homomorphism $\alpha : R \rightarrow S$ and an element $g \in G(R)$ mapping to $h$ in $G(S)$, we get a diagram:

$$
\begin{array}{ccc}
V \otimes_k R & \overset{\Phi(g)}{\rightarrow} & V \otimes_k R \\
\downarrow^{1 \otimes \alpha} & & \downarrow^{1 \otimes \alpha} \\
V \otimes_k S & \overset{\Phi(h)}{\rightarrow} & V \otimes_k S
\end{array}
$$

Now let $g \in G(R) = \text{Hom}_k(A, R)$. Then $g : A \rightarrow R$ sends the “universal” element $1_A \in G(A) = \text{Hom}_k(A, A)$ to $g$, and so the picture becomes the bottom part of the following diagram:

$$
\begin{array}{ccc}
V = V \otimes_k k & \overset{\Phi(1_A)|_V}{\rightarrow} & V \otimes_k k \\
\downarrow^{1 \otimes g} & & \downarrow^{1 \otimes g} \\
V \otimes_k A & \overset{\Phi(1_A)}{\rightarrow} & V \otimes_k A \\
\downarrow^{A-\text{linear}} & & \downarrow^{A-\text{linear}} \\
V \otimes_k R & \overset{\Phi(g)}{\rightarrow} & V \otimes_k R
\end{array}
$$

In particular, we see that $\Phi$ defines a $k$–linear map $\rho := \Phi(1_A)|_V : V \rightarrow V \otimes_k A$.

Moreover, it is clear from the diagram that $\rho$ determines $\Phi$, because $\Phi(1_A)$ is the unique $A$–linear extension of $\rho$ to $V \otimes_k A$, and $\Phi(g)$ is the unique $R$–linear extension of $\Phi(1_A)$ to $V \otimes_k R$. 
Conversely, suppose we have a $k$–linear map $\rho : V \to V \otimes_k A$. Then the diagram shows that we get a natural map

$$\Phi : G(R) \to \text{Aut}_R(V \otimes_k R),$$

namely, given $g : A \to R$, $\Phi(g)$ is the unique $R$–linear map making the following diagram commute.

These maps will be homomorphisms if and only if the following diagram commutes:

$$V \xrightarrow{\rho} V \otimes_k A \xrightarrow{1 \otimes g} V \otimes_k R \xrightarrow{\Phi(g)} V \otimes_k R$$

For, we must have $\Phi(1_{G(R)}) = 1_{V \otimes_k R}$. By definition, $1_{G(R)} = (A \xrightarrow{\epsilon} k \xrightarrow{} R)$ as an element of $\text{Hom}_k(A, R)$, and so the following diagram must commute:

$$V \xrightarrow{\rho} V \otimes_k A \xrightarrow{1 \otimes \epsilon} V \otimes_k A \xrightarrow{\rho \otimes 1} V \otimes_k A \otimes_k A$$

This means that the upper part of the diagram must commute with the map $V \otimes_k k \to V \otimes_k k$ being the identity map, which is the second of the diagrams (2.5).

Similarly, the first diagram in (2.5) commutes if and only if the formula $\Phi(gh) = \Phi(g) \circ \Phi(h)$ holds.
For, by definition, \( gh \) is the composition

\[
A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{(g,h)} R
\]

and so \( \Phi(gh) \) is the extension of

\[
V \xrightarrow{\rho} V \otimes_k A \xrightarrow{1 \otimes \Delta} V \otimes_k A \otimes_k A \xrightarrow{1 \otimes (g,h)} V \otimes_k R
\] (2.6)

which is equal to

\[
V \xrightarrow{\rho} V \otimes_k A \xrightarrow{\rho \otimes 1_A} V \otimes_k A \otimes_k A \xrightarrow{1 \otimes (g,1)} V \otimes_k R
\] (2.8)

Now (2.6) and (2.8) agree for all \( g, h \) if and only if the first diagram in (2.5) commutes. Therefore, we have

**Proposition 2.1.18** Let \( G \) be an affine group scheme over \( k \) with corresponding Hopf algebra \( A \), and let \( V \) be a \( k \)-vector space. To give a linear representation of \( G \) on \( V \) is canonically equivalent to giving an \( A \)-comodule structure on \( V \).  

**Example 2.1.5** For any Hopf algebra \( A \) over \( k \), the map \( \Delta : A \rightarrow A \otimes_k A \) is a comodule structure on \( A \). The corresponding representation of \( A \) is called the regular representation.

**Remark 2.1.19**  
1. An element \( g \) of \( G(R) = \text{Hom}_k(A, R) \) acts on \( v \in V \otimes_k R \) according to the rule:  
\[
g \cdot v = ((1_V, g) \circ \rho)(v)
\]

2. Recall that a \( k \)-subspace \( W \) of an \( A \)-comodule \( V \) is a subcomodule if \( \rho(W) \subset W \otimes_k A \). Then, \( W \) itself is an \( A \)-comodule, and the linear representation of \( G \) on \( W \) defined by this comodule structure is the restriction of that on \( V \).
2.2 Tensor Categories

We begin this section from the definition of tensor category. In Subsections 2.2.2, 2.2.4, 2.2.5 we recall definitions of invertible objects, tensor functors and morphisms of tensor functors. The main reference for this section is [DM82] (see also [Sa72], [Sz09]).

2.2.1 Tensor Categories and Tensor Functors

Let $\mathcal{C}$ be a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a functor. An *associativity constraint* for $(\mathcal{C}, \otimes)$ is an isomorphism $\Phi$ of functors from $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ given on a triple $(X, Y, Z)$ of objects by

$$\Phi_{X,Y,Z} : (X \otimes Y) \otimes Z \sim X \otimes (Y \otimes Z)$$

such that the diagram

$$
\begin{array}{c}
(X \otimes (Y \otimes Z)) \otimes W \\
\downarrow \Phi \\
X \otimes ((Y \otimes Z) \otimes W)
\end{array}
\quad
\begin{array}{c}
\Phi \otimes 1 \\
\downarrow \Phi
\end{array}
\quad
\begin{array}{c}
((X \otimes Y) \otimes Z) \otimes W \\
\downarrow \Phi
\end{array}
\quad
\begin{array}{c}
(X \otimes Y) \otimes (Z \otimes W) \\
\Phi
\end{array}
\quad
\begin{array}{c}
X \otimes (Y \otimes (Z \otimes W)) \\
1 \otimes \Phi
\end{array}
$$

commutes for each four-tuple $(X, Y, Z, W)$ of objects in $\mathcal{C}$. The commutativity of the above diagram is usually referred to as the *pentagon axiom*.

A *commutativity constraint* for $(\mathcal{C}, \otimes)$ is an isomorphism $\Psi$ of functors from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ given on a pair $(X, Y)$ of objects by

$$\Psi_{X,Y} : X \otimes Y \sim Y \otimes X$$

such that $\Psi_{X,Y} \circ \Psi_{Y,X} = 1_{X \otimes Y}$ for all objects $X, Y$.

An associativity constraint $\Phi$ and a commutativity constraint $\Psi$ are compat-
ible if, for all objects \(X, Y, Z\), the diagram

\[
\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{1 \otimes \Psi} & (X \otimes Y) \otimes Z \\
\downarrow^{\Phi} & & \downarrow^{\Psi} \\
X \otimes (Z \otimes Y) & \xrightarrow{\phi} & (X \otimes Z) \otimes Y \\
\downarrow^{\Phi} & & \downarrow^{\Psi \otimes 1} \\
(X \otimes Z) \otimes Y & \xrightarrow{\psi \otimes 1} & (Z \otimes X) \otimes Y
\end{array}
\]

commutes. This compatibility is called the *hexagon axiom*.

A pair \((1, \iota)\) consisting of an object \(1\) of \(C\) and an isomorphism \(\iota : 1 \rightarrow 1 \otimes 1\) is an *identity object* of \((C, \otimes)\) if the functor \(C \rightarrow C\) given by \(X \mapsto 1 \otimes X\) is an equivalence of categories.

**Proposition 2.2.1** Let \((1, \iota)\) be an identity object for \((C, \otimes)\). Then

1. there exists a unique functorial isomorphism
   \[
l_X : X \rightarrow 1 \otimes X
   \]
   such that \(l_1\) is \(\iota\) and the diagrams
   
   \[
   \begin{array}{ccc}
   X \otimes Y & \xrightarrow{\iota} & 1 \otimes (X \otimes Y) \\
   \downarrow^{l_1} & & \downarrow^{\Phi} \\
   X \otimes (1 \otimes Y) & \xrightarrow{1 \otimes l_1} & (1 \otimes X) \otimes Y \\
   \end{array}
   \]
   \[
   \begin{array}{ccc}
   X \otimes Y & \xrightarrow{l_1} & (1 \otimes X) \otimes Y \\
   \downarrow^{\Phi} & & \downarrow^{\Psi \otimes 1} \\
   X \otimes (1 \otimes Y) & \xrightarrow{\Phi} & (X \otimes 1) \otimes Y
   \end{array}
   \]
   are commutative.

2. if \((1', \iota')\) is another identity object of \((C, \otimes)\) then there is a unique iso-
morphism $a : 1 \rightarrow 1'$ making the following diagram

\[
\begin{array}{ccc}
1 & \xrightarrow{\iota} & 1 \otimes 1 \\
\downarrow a & & \downarrow a \otimes a \\
1' & \xrightarrow{\iota'} & 1' \otimes 1'
\end{array}
\]

commute.

Proof.

(1) Since $X \mapsto 1 \otimes X$ is an equivalence of categories, to define $l_X$, it suffices to define $1 \otimes l_X : 1 \otimes X \rightarrow 1 \otimes (1 \otimes X)$; this we take to be

\[
1 \otimes X \xrightarrow{\iota \otimes 1} (1 \otimes 1) \otimes X \xrightarrow{\Phi} 1 \otimes (1 \otimes X).
\]

(2) The map

\[
1 \xrightarrow{l_1} 1' \otimes 1 \xrightarrow{\Psi} 1 \otimes 1' \xrightarrow{l_{1'}} 1'
\]

has the required properties.

\[\square\]

Remark 2.2.2 The functorial isomorphism $r_X = \Psi_{1,X} \circ l_X : X \rightarrow X \otimes 1$ has analogous properties to $l_X$.

Definition 2.2.3 A tensor category is a system $(\mathcal{C}, \otimes, \Phi, \Psi)$, where $\mathcal{C}$ is a category, $\Phi$ is an associativity constraint satisfying the pentagon axiom, and $\Psi$ is a commutativity constraint compatible with $\Phi$, together with an identity element $(1, \iota)$.

2.2.2 Invertible objects

Let $(\mathcal{C}, \otimes)$ be a tensor category. An object $L$ of $\mathcal{C}$ is invertible if the functor $X \rightarrow L \otimes X : \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence of categories.

Thus, if $L$ is invertible, there exists an $L'$ such that $L \otimes L' \simeq 1$; the converse assertion is also true. An inverse of $L$ is a pair $(L^{-1}, \delta)$, where $\delta : L \otimes L^{-1} \xrightarrow{\simeq} 1$. Note that this definition is symmetric: $(L, \delta)$ is an inverse of $L^{-1}$. If $(L_1, \delta_1)$ and
(L_2, \delta_2) are both inverse of L, then there exists a unique isomorphism \alpha : L_1 \to L_2 such that
\[ \delta_1 = \delta_2 \circ (1 \otimes \alpha) : L \otimes L_1 \to L \otimes L_2 \to \mathbb{1}. \]

### 2.2.3 Internal Hom

Let \((C, \otimes)\) be a tensor category. If the functor \(T \mapsto \text{Hom}(T \otimes X, Y) : C^0 \to \text{Set}\) is representable, then we denote by \(\text{Hom}(X, Y)\) the representing object and by \(\text{ev}_{X,Y} : \text{Hom}(X, Y) \otimes X \to Y\) the morphism corresponding to \(1_{\text{Hom}(X,Y)}\).

Thus, to any element \(g \in \text{Hom}(T \otimes X, Y)\) corresponds a unique \(f \in \text{Hom}(T, \text{Hom}(X, Y))\) such that \(\text{ev} \circ (f \otimes \text{id}) = g;\)

\[
\begin{array}{ccc}
T & \xrightarrow{T \otimes X} & T \otimes X \\
\downarrow & & \downarrow \text{id} \\
\text{Hom}(X, Y) & \xrightarrow{f \otimes \text{id}} & \text{Hom}(X, Y) \otimes X \\
& & \downarrow \text{id} \\
& & Y
\end{array}
\]

(2.9)

Note that \(\text{Hom}(1, Y) = Y\) and
\[
\text{Hom}(1, \text{Hom}(X, Y)) = \text{Hom}(1 \otimes X, Y) = \text{Hom}(X, Y)
\]

(2.10)

The dual \(X^\vee\) of an object \(X\) is defined to be \(\text{Hom}(X, 1)\). Therefore, there is a map \(\text{ev}_X : X^\vee \otimes X \to 1\) inducing a functorial isomorphism
\[
\text{Hom}(T, X^\vee) \xrightarrow{\cong} \text{Hom}(T \otimes X, 1)
\]

(2.11)

**Remark 2.2.4** The dual of an object \(X\) is an object \(X^\vee\) together with the morphisms \(\varepsilon : X^\vee \otimes X \to 1\) and \(\delta : 1 \to X \otimes X^\vee\) such that the following diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{\cong} & X \otimes 1 \\
\downarrow \delta & & \downarrow 1 \otimes \varepsilon \\
1 \otimes X & \xrightarrow{\delta \otimes 1} & X \otimes X^\vee \otimes X
\end{array}
\]

and

\[
\begin{array}{ccc}
X^\vee & \xrightarrow{\cong} & 1 \otimes X^\vee \\
\downarrow \varepsilon \otimes 1 & & \downarrow 1 \otimes \delta \\
X^\vee \otimes 1 & \xrightarrow{1 \otimes \delta} & X^\vee \otimes X \otimes X^\vee
\end{array}
\]

(2.12)

commutes.

A dual of an object \(X\) is uniquely determined up to isomorphism. If we fix one of the maps \(\varepsilon\) or \(\delta\), then this isomorphism is unique.
To verify this, fix $X$ and $\varepsilon$. It then suffices to show that $X^\vee$ represents the contravariant functor $Z \mapsto \text{Hom}(X \otimes Z, 1)$, that is, it suffices to give a functorial bijection

$$\text{Hom}(X \otimes Z, 1) \longrightarrow \text{Hom}(Z, X^\vee).$$

Let $\varphi \in \text{Hom}(X \otimes Z, 1)$. Define $\nu_\varphi : Z \longrightarrow X^\vee$ to be the composite

$$Z \xrightarrow{\cong} 1 \otimes Z \xrightarrow{\delta \otimes 1} (X^\vee \otimes X) \otimes Z \xrightarrow{1_{X^\vee} \otimes \varphi} X^\vee \otimes 1 \xrightarrow{\cong} X^\vee.$$

Then the map $\varphi \mapsto \nu_\varphi$ is the required bijection. Its inverse is given by $\nu \mapsto \nu_\varphi \nu$, where $\varphi_\nu$ is the composite

$$X \otimes Z \xrightarrow{1 \otimes \varphi} X \otimes X^\vee \xrightarrow{\varepsilon} 1.$$

Let $i_X : X \longrightarrow X^{\vee\vee}$ be the map corresponding in (2.11) to $\text{ev}_X \circ \Psi : X \otimes X^\vee \longrightarrow 1$. If $i_X$ is an isomorphism, then $X$ is said to be reflexive. If $X$ has an inverse $(X^{-1}, \delta : X^{-1} \otimes X \xrightarrow{\cong} 1)$, then $X$ is reflexive and $\delta$ determines an isomorphism $X^{-1} \xrightarrow{\cong} X^\vee$ as in (2.9).

For any pair $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ of finite families of objects, there is a morphism

$$\otimes_{i \in I} \text{Hom}(X_i, Y_i) \longrightarrow \text{Hom}(\otimes_{i \in I} X_i, \otimes_{i \in I} Y_i)$$

(2.13)

corresponding to

$$\left( \otimes_{i \in I} \text{Hom}(X_i, Y_i) \right) \otimes \left( \otimes_{i \in I} X_i \right) \xrightarrow{\cong} \otimes_{i \in I} \left( \text{Hom}(X_i, Y_i) \otimes X_i \right) \xrightarrow{\text{ev}} \otimes_{i \in I} Y_i$$

in (2.9).

In particular, there are morphisms

$$\otimes_{i \in I} X_i^\vee \longrightarrow \left( \otimes_{i \in I} X_i \right)^\vee$$

(2.14)

and

$$X^\vee \otimes Y \longrightarrow \text{Hom}(X, Y)$$

(2.15)

obtained from (2.13).

**Definition 2.2.5** A tensor category $(\mathcal{C}, \otimes)$ is rigid if $\text{Hom}(X, Y)$ exists for all objects $X$ and $Y$, the maps (2.13) are isomorphism for all finite families of objects,
and all objects of $\mathcal{C}$ are reflexive.

If $(\mathcal{C}, \otimes)$ is a rigid tensor category, then the map $X \mapsto X^\vee$ can be made into a contravariant functor: to $f : X \to Y$, we associate the unique map $t_f : Y^\vee \to X^\vee$ defined as the composite

$$Y^\vee \cong Y^\vee \otimes 1 \xrightarrow{1 \otimes Y} Y^\vee \otimes X \otimes X^\vee \cong Y^\vee \otimes X \otimes Y^\vee \xrightarrow{\varepsilon \otimes 1} 1 \otimes X^\vee \cong X^\vee$$

such that the diagram

$$\begin{array}{ccc}
Y^\vee \otimes X & \xrightarrow{t_f \otimes 1} & X^\vee \otimes X \\
1 \otimes f \downarrow & & \downarrow \text{ev}_X \\
Y^\vee \otimes Y & \xrightarrow{\text{ev}_Y} & 1
\end{array}$$

commutes.

The map $f \mapsto t_f$ induces a bijection

$$\text{Hom}(X, Y) \to \text{Hom}(Y^\vee, X^\vee).$$

**Remark 2.2.6** Recall that there is an isomorphism

$$\text{Hom}(X, Y) \cong \text{Hom}(Y^\vee, X^\vee) ; f \mapsto t_f.$$
2.2.4 Tensor Functors

Let \((\mathcal{C}, \otimes)\) and \((\mathcal{C}', \otimes')\) be tensor categories.

**Definition 2.2.7** A tensor functor \((\mathcal{C}, \otimes) \rightarrow (\mathcal{C}', \otimes')\) is a functor \(F : \mathcal{C} \rightarrow \mathcal{C}'\) together with an isomorphism \(c\) of functors from \(\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}'\) given on a pair \((X, Y)\) of objects of \(\mathcal{C}\) by a morphism

\[ c_{X,Y} : F(X) \otimes' F(Y) \xrightarrow{\simeq} F(X \otimes Y) \]

with the following properties:

(a) for a triple \((X,Y,Z)\) of objects of \(\mathcal{C}\), the diagram

\[
\begin{array}{ccc}
F(X) \otimes' (F(Y) \otimes' F(Z)) & \xrightarrow{1 \otimes' c} & F(X) \otimes' F(Y \otimes Z) \\
\Phi \downarrow & & \downarrow F(\Phi) \\
(F(X) \otimes F(Y)) \otimes' F(Z) & \xrightarrow{c \otimes' 1} & F(X \otimes Y) \otimes' F(Z) \\
\end{array}
\]

is commutative;

(b) for a pair \((X,Y)\) of objects of \(\mathcal{C}\), the diagram

\[
\begin{array}{ccc}
F(X) \otimes' F(Y) & \xrightarrow{c} & F(X \otimes Y) \\
\Psi \downarrow & & \downarrow F(\Psi) \\
F(Y) \otimes' F(X) & \xrightarrow{c} & F(Y \otimes X) \\
\end{array}
\]

is commutative;

(c) if \((U,u)\) is an identity object of \(\mathcal{C}\), then \((F(U), F(u))\) is an identity object of \(\mathcal{C}'\).

The conditions (a), (b), (c) imply that for any finite family \((X_i)_{i \in I}\) of objects of \(\mathcal{C}\), \(c\) gives rise to a well-defined isomorphism

\[ c : \otimes'_{i \in I} F(X_i) \xrightarrow{\simeq} F\left( \otimes_{i \in I} X_i \right). \]
In particular, \((F,c)\) maps invertible objects to invertible objects. Also, the morphism
\[
F(ev) : F(\text{Hom}(X,Y) \otimes F(X)) \rightarrow F(Y)
\]
gives rise to morphisms
\[
F_{X,Y} : F(\text{Hom}(X,Y)) \rightarrow \text{Hom}(F(X), F(Y))
\]
and
\[
F_X : F(X^\vee) \rightarrow F(X^\vee).
\]

**Proposition 2.2.8** Let \((F,c) : C \rightarrow C'\) be a tensor functor. If \(C\) and \(C'\) are rigid, then \(F_{X,Y} : F(\text{Hom}(X,Y)) \rightarrow \text{Hom}(F(X), F(Y))\) is an isomorphism for all pairs \((X,Y)\) of objects of \(C\).

**Proof.** It suffices to show that \(F\) preserves duality, but this is obvious from the following characterization of the dual of an object \(X\) in \(C\):

The dual of an object \(X\) in \(C\) is a pair \((X,Y \otimes X \xrightarrow{ev} 1)\), for which there exists a morphism \(\delta : 1 \rightarrow X \otimes Y\) such that
\[
X = 1 \otimes X \xrightarrow{\delta \otimes 1} (X \otimes Y) \otimes X = X \otimes (Y \otimes X) \xrightarrow{1 \otimes ev} X,
\]
and the same map with \(X\) and \(Y\) interchanged, are identity maps.

\[\square\]

### 2.2.5 Morphisms of Tensor Functors

**Definition 2.2.9** Let \((F,c)\) and \((G,d)\) be tensor functor from \(C\) to \(C'\); a *morphism of tensor functors* \((F,c) \rightarrow (G,d)\) is a morphism of functors \(\lambda : F \rightarrow G\) such that, for all finite families \((X_i)_{i \in I}\) of objects in \(C\), the diagram

\[
\begin{array}{ccc}
\otimes_{i \in I} F(X_i) & \xrightarrow{c} & F\left( \otimes_{i \in I} X_i \right) \\
\downarrow_{\otimes \lambda_{X_i}} & & \downarrow_{\lambda} \\
\otimes_{i \in I} G(X_i) & \xrightarrow{d} & G\left( \otimes_{i \in I} X_i \right)
\end{array}
\]

(2.16)
Note that it is enough to require that the above diagram commutes for \( I = \{1, 2\} \).

For \( I \) the empty set, (2.16) becomes,

\[
\begin{array}{ccc}
I' & \xrightarrow{\approx} & F(1) \\
\parallel & & \downarrow \lambda_1 \\
I' & \xrightarrow{\approx} & G(1)
\end{array}
\]  

(2.17)

in which the horizontal maps are the unique isomorphisms compatible with the structure of \( I', F(1) \) and \( G(1) \) as identity objects of \( C' \). In particular, when diagram (2.17) commutes, the morphism \( \lambda_1 \) is an isomorphism.

We write \( \text{Hom}^\otimes(F,F') \) for the set of morphisms of tensor functors from \((F,c)\) to \((F',c')\).

**Definition 2.2.10** A tensor functor \((F,c) : C \to C'\) is a tensor equivalence (or an equivalence of tensor categories) if \( F : C \to C' \) is an equivalence of categories.

The above definition is justified by the following proposition.

**Proposition 2.2.11** Let \((F,c) : C \to C'\) be a tensor equivalence; then there is a tensor functor \((F',c') : C' \to C\) and an isomorphism of functors \( F' \circ F \xrightarrow{\approx} 1_C \) and \( F \circ F' \xrightarrow{\approx} 1_{C'} \), commuting with tensor product (i.e., isomorphism of tensor functors).

**Proof.** See [Sa72, I4.4]. \(\square\)

**Proposition 2.2.12** Let \((F,c)\) and \((G,d)\) be tensor functors from \( C \) to \( C' \). If \( C \) and \( C' \) are rigid, then any morphism of tensor functors \( \lambda : F \to G \) is an isomorphism.

**Proof.** The morphism \( \mu : G \to F \), making the diagrams

\[
\begin{array}{ccc}
F(X^\vee) & \xrightarrow{\lambda_X^\vee} & G(X^\vee) \\
\approx & & \approx \\
F(X)^\vee & \xrightarrow{\mu_X} & G(X)^\vee
\end{array}
\]

commutative for all \( X \in \text{Obj}(C) \), is an inverse for \( \lambda \). \(\square\)
Remark 2.2.13 For any field $k$ and $k$–algebra $R$, there is a canonical tensor functor

$$\Phi_R : \text{Vect}(k) \rightarrow \text{Modfg}_R,$$

defined by $V \mapsto V \otimes_k R$, where $\text{Modfg}_R$ is the category of finitely generated $R$–modules. If $(F,c)$ is a tensor functor $C \rightarrow \text{Vect}(k)$, then we can define set-valued functors $\text{End}^\otimes(F)$ on the category of $k$–algebras by

$$\text{End}^\otimes(F)(R) = \text{Hom}^\otimes(\Phi_R \circ F, \Phi_R \circ F) \quad (2.18)$$

Thus, we get a functor $\text{End}^\otimes(F) : \text{Alg}_k \rightarrow \text{Set}$.

Tensor Subcategories

Definition 2.2.14 Let $C'$ be a strictly full subcategory of a tensor category $C$. We say $C'$ is a tensor subcategory of $C$ if it is closed under finite tensor products (equivalently, if it contains an identity object of $C$ and if $X_1 \otimes X_2 \in \text{Ob}(C')$ whenever $X_1, X_2 \in \text{Ob}(C')$).

A tensor subcategory of a rigid tensor category is said to be a rigid tensor subcategory if it contains $X^\vee$ whenever it contains $X$.

A tensor subcategory of a tensor category becomes a tensor category under the induced tensor structure. Similarly, a rigid tensor subcategory of a rigid tensor category becomes a rigid tensor category.

Abelian Tensor Categories

Definition 2.2.15 An additive (resp. abelian) tensor category is a tensor category $(C, \otimes)$ such that $C$ is an additive (resp. abelian) category and $\otimes$ is a bi-additive functor.

When $(C, \otimes)$ is an abelian, then we say that a family $X = (X_i)_{i \in I}$ of objects of $C$ is a tensor generating family for $C$ if every object of $C$ is isomorphic to a sub-quotient of $P((X_i)_{i \in I})$ for some $P \in \mathbb{Z}[T_i]_{i \in I}$ with non-negative coefficients. Note that for every polynomial $P \in \mathbb{Z}[T_i]_{i \in I}$, there exists a finite subset $\{i_1, \ldots, i_n\}$ of $I$ such that

$$P = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} T_{i_1}^{\alpha_1} \cdots T_{i_n}^{\alpha_n},$$
where $a_\alpha \in \mathbb{Z}$ and $a_\alpha = 0$ for all but finitely many $\alpha$. If $P \in \mathbb{Z}[T_{i}]_{i \in I}$ is a polynomial with non-negative integral coefficients, then

$$P((X_i)_{i \in I}) = \bigoplus_{\alpha \in \mathbb{N}^n} (X_{i_1}^{\otimes \alpha_1} \otimes \cdots \otimes X_{i_n}^{\otimes \alpha_n})^{\otimes a_\alpha}.$$

### 2.3 Neutral Tannakian Categories

In this section, we recall the definition of neutral Tannakian category and state the main theorem due to Saavedra. Let $k$ be a field, and let $\text{Vect}(k)$ be the category of finite dimensional vector spaces over $k$.

**Definition 2.3.1** A category $\mathcal{C}$ is said to be $k$–linear if the set of morphisms between two arbitrary objects of $\mathcal{C}$ is a $k$–vector space and for objects $X, Y$ and $Z$ in $\mathcal{C}$, the composite map

$$\text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \to \text{Hom}_{\mathcal{C}}(X, Z)$$

is $k$–bilinear with respect to the $k$–vector space structures on the Hom-sets involved.

**Definition 2.3.2** A neutral Tannakian category over $k$ is a rigid $k$–linear abelian tensor category $\mathcal{C}$ equipped with an exact faithful $k$–linear tensor functor $\omega : \mathcal{C} \to \text{Vect}(k)$ into the category of finite dimensional $k$–vector spaces. The functor $\omega$ is called a neutral fibre functor.

Let $\text{Aut}^\otimes(\omega)$ be the group-valued functor defined on the category of $k$–algebras by sending $R$ to the set of $R$–linear tensor functor isomorphisms. By Proposition 2.2.12, the natural morphism $\text{Aut}^\otimes(\omega) \to \text{End}^\otimes(\omega)$ of functors is an isomorphism (see, Remark 2.2.13).

Given an affine group scheme $G$, the category $\text{Rep}_k(G)$ of finite dimensional representations of $G$ with its usual tensor structure and the forgetful functor $\omega_G : \text{Rep}_k(G) \to \text{Vect}(k)$ as fibre functor is a neutral Tannakian category. Conversely, we have:

**Theorem 2.3.3** Let $(\mathcal{C}, \otimes, 1, \omega)$ be a neutral Tannakian category over $k$. Then

(a) the functor $G = \text{Aut}^\otimes(\omega)$ of $k$–algebras is an affine group scheme;
(b) there is an equivalence \( \omega' : \mathcal{C} \rightarrow \text{Rep}_k(G) \) of tensor categories such that the following diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\omega'} & \text{Rep}_k(G) \\
\downarrow{\omega} & & \downarrow{\omega_G} \\
\text{Vect}(k) & & \\
\end{array}
\]

commutes.

The affine group scheme determined by the Theorem 2.3.3 is called the Tannakian fundamental group scheme of \((\mathcal{C}, \otimes, I, \omega)\).

**Proposition 2.3.4** Let \( f : G \rightarrow G' \) be a homomorphism of affine group schemes and \( \omega^f : \text{Rep}_k(G') \rightarrow \text{Rep}_k(G) \) the corresponding tensor functor.

1. The homomorphism \( f \) is faithfully flat if and only if \( \omega^f \) is fully faithful and every subobject of \( \omega^f(X') \) \((X' \in \text{Rep}_k(G'))\) is isomorphic to the image of a subobject of \( X' \).

2. The homomorphism \( f \) is closed immersion if and only if every object of \( \text{Rep}_k(G) \) is isomorphic to a subquotient of \( \omega^f(X') \) for some \( X' \in \text{Rep}_k(G') \).