CHAPTER IV

APPROXIMATION OF A CLASS OF FUNCTIONS BY EULER

MEANS OF FOURIER SERIES

4.1. Definitions. Let $f$ be a Lebesgue integrable function with period $2\pi$ and let

\[
4.1.1 \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

be its Fourier series.

An infinite series $\sum_{n=0}^{\infty} a_n$ with the sequence of partial sums $\{s_n\}$ is said to be summable by Euler means or more precisely, summable $(E, q)$, $q > 0$ to $s$ if

\[
P_n = (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k
\]

tends to a finite limit $s$ as $n \to \infty$.

Also, let $f(x,y)$ be a Lebesgue-integrable function defined on the square $K(-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$. Let the double Fourier series of $f(x,y)$ be

\[
4.1.2 \quad f(x,y) \sim \sum_{m,n=0}^{\infty} a_{mn} \cos mx \cos ny + b_{mn} \sin mx \cos ny + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny
\]

\[\text{(*) Pure and Applied Mathematica Sciences. 22 (1985), 59 - 62.}\]
where

\[ \lambda_{mn} = \begin{cases} 
\frac{1}{4} & \text{for } m = n = 0 \\
\frac{1}{2} & \text{for } m > 0, n = 0 \text{ or } m = 0, n > 0 \\
1 & \text{for } m > 0, n > 0 ,
\end{cases} \]

and the coefficients are calculated by the formulas

\[ a_{mn} = \frac{1}{x^2} \int f(x,y) \cos mx \cos ny \, dx \, dy \]
\[ b_{mn} = \frac{1}{x^2} \int f(x,y) \sin mx \cos ny \, dx \, dy \]
\[ c_{mn} = \frac{1}{x^2} \int f(x,y) \cos mx \sin ny \, dx \, dy \]
\[ d_{mn} = \frac{1}{x^2} \int f(x,y) \sin mx \sin ny \, dx \, dy . \]

A double series \( \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{mn} \) with the sequence of partial sums \( \langle s_{m,n} \rangle \) is said to be summable by Euler means or more precisely, summable \( (E,p,q) \) \((p,q > 0) \) to \( s \) provided that

\[ P_{m,n} = \frac{1}{(p+1)^m(q+1)^n} \sum_{k=0}^{p-1} \sum_{l=0}^{q-1} \binom{m}{k} \binom{n}{l} (m+l) (n+k) \frac{2^{-k} p^{-k} s_{l,k}}{s_{m,n}} \]

tends to a finite limit \( s \) as \( n \to \infty \).

A function \( f \) is said to belong to the class \( \text{Lip } \alpha \) \((0 < \alpha \leq 1) \) if
Let $J(t)$ be a positive and non-decreasing function. Izumi [12] replaced $|h|^a$ in (4.1.2) by the function $J(t)$ and obtained a more general class Lip $J(t)$. Hasegawa [10] gave the following definition:

A function $f(x,y)$ is said to belong to the class Lip $(x,\beta)$ if

$$f(x+u, y+v) - f(x,y) = O(|u|^\alpha + |v|^\beta)$$

uniformly at the point $(x,y)$ as $u$ and $v$ tend to zero, independent of each other, where $0 < \alpha \leq 1$ and $0 < \beta \leq 1$.

4.2 In this chapter, we extend Theorem B of Chapter I by obtaining the degree of approximation of periodic functions belonging to classes Lip $\alpha$ and Lip $J(t)$ by Riesz means of Fourier series. The results obtained have also been generalized for double Fourier series. We prove the following theorem:

**Theorem 1.** Let $f$ be a periodic function with period $2\pi$ and belonging to the class Lip $\alpha$. Then

$$\max_{0 \leq x \leq 2\pi} |p_n(x) - f(x)| = O(n^{-\alpha}), \quad 0 < \alpha \leq 1.$$

**Theorem 2.** Let $f$ be a periodic function with period $2\pi$ and belonging to the class Lip $J(t)$. If $J(t)$ satisfies the condition
(4.2.1) \[ \int_{x/n}^{t} \frac{J(t)}{t} \, dt = O[J(1/n)] \quad 0 < t \leq x \] Then
\[ \max_{0 \leq x \leq 2n} |P_n(x) - f(x)| = O[J(1/n)] \]

**Theorem 3.** The degree of approximation of a periodic function \( f \) of period \( 2\pi \) and belonging to the class \( \text{Lip} (\alpha, \beta) \) by Euler means of its double Fourier series is given by
\[ \max_{0 \leq x \leq 2n} |P_{n,a}(x,y) - f(x,y)| = O \left[ n^{-\alpha} \log n + n^{-\beta} \log n \right] \]

Putting \( J(t) = t^\alpha \), we observe that Theorem 2 is a generalization of Theorem 1 for \( 0 < \alpha < 1 \).

### 4.3. Proof of Theorem 1.
Let \( S_k(x) \) denotes the partial sum of the series (4.1.1). Then
\[ S_k(x) - f(x) = \frac{1}{\pi} \int_{0}^{\pi} \phi(t) \frac{\sin(k+\frac{1}{2})t}{2\sin\frac{1}{2}} \, dt \]
where
\[ \phi(t) = f(x+t) + f(x-t) - 2f(x) \]
Next
\[ P_n(x) - f(x) = (q+1)^{-n} \sum_{k=0}^{n} \frac{\binom{n}{k} q^{-k}}{x^{(q+1)n}} \int_{0}^{x} \phi(t) P_k(t) \, dt \]
where \( R_k(t) \) denotes the Dirichlet's kernel.

Thus

\[
|P_n(x) - f(x)| \leq \frac{1}{n(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \int_0^{x/n} t^{s} |R_k(t)| \, dt
\]

\[
= \frac{1}{n(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left[ \int_0^{x/n} t^{s} |\hat{\phi}(t)| \, dt \right]
\]

\[
= I_1 + I_2, \text{ say.}
\]

Now

\[
I_1 = 0 \left[ \frac{1}{n(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \int_0^{x/n} t^{s} |R_k(t)| \, dt \right]
\]

\[
= 0\left[ \frac{1}{n(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \int_0^{x/n} t^{a} \, dt \right]
\]

\[
= 0\left[ \frac{n}{n(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left( \frac{1}{a} \right)^{a+1} \right]
\]

\[
= O(n^{-a}),
\]

\[
I_2 = 0 \left[ \frac{1}{n(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \int_0^{x/n} t^{s} \, dt \right]
\]

\[
= 0\left[ \frac{1}{n(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \int_0^{x/n} \frac{t^{a}}{t} \, dt \right]
\]

\[
= O(n^{-a}).
\]

This proves the theorem.
4.4. Proof of Theorem 2. As in the proof of Theorem 1, we have

\[ |P_n(x) - f(x)| = I_1 + I_2, \text{ say.} \]

Now

\[ I_1 = O \left[ \frac{1}{x(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{x^n}{x^n} \int_0^{J(t)} |D_k(t)| \, dt \right] \]

\[ = O \left[ \frac{1}{x(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \int_0^{J(t)} \, dt \right] \]

\[ = O \left[ \frac{1}{x(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \int_0^{J(t)} \frac{J(t)}{t} \, dt \right] \]

\[ = O \left[ J(1/n) \right]. \]

The use of condition (4.2.1), yields

\[ I_2 = O \left[ \frac{1}{x(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{x^n}{x^n} \int_0^{J(t)} |D_k(t)| \, dt \right] \]

\[ = O \left[ \frac{1}{x(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \int_0^{J(t)} \frac{J(t)}{t} \, dt \right] \]

\[ = O \left[ J(1/n) \right], \]

and this completes the proof of the theorem.

4.5. Proof of Theorem 3. Let \( \langle S_{m,n}(x,y) \rangle \) denote the
partial sum of the double Fourier series (4.1.2). Then

\[
S_{m,n}(x,y) = f(x,y) - \frac{1}{x^2} \int_0^x \int_0^y \phi(u,v) \frac{\sin(m+1/2)u \sin(n+1/2)v}{\sin \frac{u}{2} \sin \frac{v}{2}} du dv
\]

where

\[
\phi(u,v) = f(x+u, y+v) + f(x-u, y+v) + f(x+u, y-v) + f(x-u, y-v) - 4f(x,y).
\]

Now

\[
P_{m,n}(x,y) - f(x,y) = \frac{1}{x^2(p+1)q(q+1)^n} \sum_{l=0}^{m} \sum_{k=0}^{n} \frac{m \frac{n}{l} \frac{m}{l} \frac{n}{k} \frac{m-1}{p} \frac{n-k}{q} x^2}{(p+1)(q+1)}
\]

\[
\left[ S_{1,k}(x,y) - f(x,y) \right]
\]

\[
= \frac{4}{x^2(p+1)q(q+1)^n} \sum_{l=0}^{m} \sum_{k=0}^{n} \frac{m \frac{n}{l} \frac{m}{l} \frac{n}{k} \frac{m-1}{p} \frac{n-k}{q} x^2}{(p+1)(q+1)}
\]

\[
\int_0^x \int_0^y \phi(u,v) \frac{\sin(l+1/2)u \sin(k+1/2)v}{\sin \frac{u}{2} \sin \frac{v}{2}} du dv
\]

where \(D_1(u)\) and \(D_k(v)\) denote Dirichlet's kernels.

Thus
\[ |p_{n,m}(x,y) - z(x,y)| \leq \frac{4}{x^2(p+1)^m(q+1)^n} \sum_{l=0}^{n} \sum_{k=0}^{n} \binom{n}{l} \binom{n}{k} \int_0^1 \int_0^1 |\phi(u,v)| x^{l/n} y^{k/n} du dv \]

\[ \int_0^1 \int_0^1 |\phi(u,v)| du dv \]

\[ = I_1 + I_2 + I_3 + I_4, \text{ say.} \]

**Now**

\[ I_1 = O\left[ \frac{1}{(p+1)^m(q+1)^n} \sum_{l=0}^{n} \binom{n}{l} \binom{n}{k} x^{l/n} y^{k/n} \int_0^1 \int_0^1 (u^\alpha + v^\beta) l \text{ du dv} \right] \]

\[ = O\left[ \frac{n}{(p+1)^m(q+1)^n} \sum_{l=0}^{n} \binom{n}{l} \binom{n}{k} x^{l/n} y^{k/n} \left( \frac{x^\alpha}{n} + \frac{y^\beta}{n} \right) \right] \]

\[ = O\left[ \frac{n^\alpha + n^\beta}{n} \right] \]

\[ I_2 = O\left[ \frac{1}{(p+1)^m(q+1)^n} \sum_{l=0}^{n} \binom{n}{l} \binom{n}{k} x^{l/n} y^{k/n} \int_0^1 \int_0^1 l \frac{u^\alpha + v^\beta}{v} \text{ du dv} \right] \]

\[ = O\left[ \frac{n}{(p+1)^m(q+1)^n} \sum_{l=0}^{n} \binom{n}{l} \binom{n}{k} x^{l/n} y^{k/n} \left( \frac{\log n}{x^\alpha} + \frac{x^\beta}{n} \right) \right] \]
\[-O\left[ n \left( \frac{\log n}{n^{\alpha + 1}} + \frac{1}{n^\beta} \right) \right] \]

\[-O\left[ n^{-\alpha} \log n + n^{-\beta} \right]. \]

\[ I_3 = O\left[ \frac{1}{(p+1)^m(q+1)^n} \sum_{l=0}^n \left( \frac{m}{1} \right) p^{-1} \sum_{k=0}^{n-k} \left( \frac{n}{k} \right) q^{-k} \int_{x/m}^{x/n} \int_{x/n} \right. \]

\[ x \frac{u^\alpha + v^\beta}{uv} \, du \, dv \]

\[-O\left[ \frac{n^\alpha}{(p+1)^m(q+1)^n} \sum_{l=0}^n \left( \frac{m}{1} \right) p^{-1} \sum_{k=0}^{n-k} \left( \frac{n}{k} \right) q^{-k} \left( \frac{1}{n^{\alpha + 1}} + \frac{\log n}{n^{\beta + 1}} \right) \right] \]

\[-O\left[ n^{-\alpha} \log n + n^{-\beta} \log n \right]. \]

\[ I_4 = O\left[ \frac{1}{(p+1)^m(q+1)^n} \sum_{l=0}^n \left( \frac{m}{1} \right) p^{-1} \sum_{k=0}^{n-k} \left( \frac{n}{k} \right) q^{-k} \int_{x/m}^{x/n} \int_{x/n} \right. \]

\[ \frac{n^\alpha}{uv} \, du \, dv \]

\[-O\left[ \frac{1}{(p+1)^m(q+1)^n} \sum_{l=0}^n \left( \frac{m}{1} \right) p^{-1} \sum_{k=0}^{n-k} \left( \frac{n}{k} \right) q^{-k} \left( n^{-\alpha} \log n + n^{-\beta} \log n \right) \right] \]

\[-O\left[ n^{-\alpha} \log n + n^{-\beta} \log n \right]. \]

Hence

\[ | P_{m,n}(x,y) - f(x,y) | = O\left[ n^{-\alpha} \log n + n^{-\beta} \log n \right]. \]

This completes the proof of the theorem.