Chapter 6

Elliptical solenoid and matching of intense beam to the spiral inflector

6.1. Introduction

Since the overall goal of this thesis is to maximize the amount of beam that can be injected into the cyclotron, a transverse beam matching at the inflector entrance is thus necessary. Results of beam dynamics in a spiral inflector carried out in Chapter 4, indicate that convergent phase ellipses with different orientations in $x$ and $y$ planes and a comparatively smaller width in the $y$ plane give better beam transmission. The transformation of an axisymmetric beam from the ion source to a non-axisymmetric beam at the entrance of the spiral inflector can’t be achieved by using cylindrical symmetric magnets such as Glaser and solenoid magnets as used in our transport line. In this case one needs either an elliptical solenoid [89-91] or a quadrupole doublet. In our case an elliptical solenoid is more suitable than the quadrupole doublet due to space constraint.

In this Chapter, first we discuss the beam optical properties of an elliptical solenoid magnet in the presence of linear space charge effects, as the beam passes through it. Then we present the feasibility of using an elliptical solenoid in the solenoid based low energy beam transport line of our 10 MeV cyclotron to match the beam at the input of the spiral inflector [92]. Generally the Kapchinskij-Vladimirskij (K-V) beam envelope equations [93] are used to understand the high intensity beam dynamics and evolution of beam envelope through a transport system. However, these equations can be used only in an uncoupled lattice i.e. where the two transverse motions are uncoupled. In the case of an
elliptical solenoid it is not straightforward to decouple the two transverse motions and hence the applications of K-V beam envelope equations are difficult [102].

A simple way to study the dynamics of intense beam in the elliptical solenoid is to find out the transfer matrix of it in the presence of space charge effect. Since both equations of motion and beam envelope quantities are coupled with each other, it is not easy to obtain the transfer matrix analytically. In such situations a convenient way is to follow the infinitesimal transfer matrix approach [81, 94]. We have obtained the paraxial ray equations of motion in the combined fields of elliptical solenoid and space charge. From these paraxial equations we have obtained the infinitesimal transfer matrix of an elliptical solenoid for a non-axisymmetric beam and at the same time we have also calculated the beam envelope through the magnet by employing the recursive sigma matrix method [81].

6.2. Theoretical analysis

Consider a space charge dominated continuous beam propagating through the magnetic field $\vec{B}$ of an elliptical solenoid with average axial velocity $v = \beta c$ where $\beta$ is the relativistic parameter and $c$ is the speed of light in vacuum. In the laboratory frame, we use a right handed Cartesian coordinate system $x$, $y$ and $z$ with unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$ respectively. As it is customary in accelerator physics we use $s = z$, the distance along the axial direction aligned with the beam axis and $x$, $y$ represent the transverse coordinates from the beam axis. In the present analysis we assume that the particle trajectory will remain very close to the axis and transverse beam sizes $X = 2\sqrt{\langle x^2 \rangle}$ and $Y = 2\sqrt{\langle y^2 \rangle}$ in $x$ and $y$ directions are very small compared to the radii of beam pipes, coils etc. where $\langle \cdots \rangle$ denotes a transverse statistical average over the beam distribution function. We also assume that the transverse velocities are much smaller compared to the average axial velocity of the particles (i.e. $\dot{x}, \dot{y} \ll v$).
6.2.1. Potential for elliptical solenoid

The magnetic field and its components in an elliptical solenoid can be obtained by solving the Laplace equation for magnetic potential \( \Phi \) in a cylindrical coordinate system

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial s^2} = 0
\]  

(6.1)

For two fold geometrical symmetry with respect to \( \theta = 0 \) and \( \theta = \pi / 2 \) such as the case of elliptical solenoid, the general solution of Eq. (1) can be written as

\[
\Phi(r, \theta, s) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \Phi_{2l,2m}(s) r^{2l} \cos(2m\theta)
\]  

(6.2)

By substituting Eq. (6.2) in (6.1) and comparing the coefficients of \( r \) and \( \theta \) for all values of \( l \) and \( m \), which satisfy the recursion relations of both \( r \) and \( \theta \) simultaneously, we find that all \( \Phi_{2,4} = \Phi_{2,6} = \Phi_{2,8} = \ldots = 0 \) as well as all \( \Phi_{0,2} = \Phi_{0,4} = \Phi_{0,6} = \ldots = 0 \). The other coefficients can be obtained using the recursion relations. The general solution (6.2) can now be expressed as

\[
\Phi(r, \theta, s) = \Phi(0, s) - \Phi^{(2)}(0, s) \left( \frac{r}{2} \right)^2 + \Phi^{(4)}(0, s) \left( \frac{r}{2} \right)^4 - \ldots
\]

\[
+ \left[ 4\Phi_{2,2}(0, s) - \frac{4\Phi_{2,4}(0, s)}{3} \left( \frac{r}{2} \right)^2 + \frac{4\Phi_{2,6}(0, s)}{24} \left( \frac{r}{2} \right)^4 - \ldots \right] \times \left( \frac{r}{2} \right)^2 \cos(2\theta)
\]

\[
+ \left[ 16\Phi_{4,4}(0, s) - \frac{16\Phi_{4,4}(0, s)}{5} \left( \frac{r}{2} \right)^2 + \frac{16\Phi_{4,6}(0, s)}{60} \left( \frac{r}{2} \right)^4 - \ldots \right] \times \left( \frac{r}{2} \right)^4 \cos(4\theta)
\]

\[+ \ldots \]

(6.3)

where \( \Phi^{(n)} \) denotes the \( n \)-th derivative of \( \Phi \) with respect to the axial distance \( s \). Since in Eq. (6.3) the potential on the axis has only \( s \) dependence, therefore, only off axis particles will experience radial and azimuthal fields. Under the paraxial approximation of beam transport as stated above, only the second order terms in the expansions of the potential...
will be important. Therefore, in our further analysis, we retain the terms only up to \( r^2 \) in Eq. (6.3) and ignore all other higher order terms in the potential. The potential then takes the form of

\[
\Phi(r, \theta, s) = A(s) - A^{(2)}(s) \left( \frac{r}{2} \right)^2 + D(s) \left( \frac{r}{2} \right)^2 \cos(2\theta)
\]  

(6.4)

where \( A(s) = \Phi(0, s) \) and \( D(s) = 4\Phi_{2,2}(0, s) \). The first two terms in Eq. (6.4) are the usual terms used in the conventional solenoid. The third term is the contribution due to the asymmetric pole face of the elliptical solenoid. This term is similar to a quadrupolar term and produces focusing and defocusing in the two transverse planes in addition to the usual solenoidal focusing given by the first two terms. In terms of coordinates \( x, y \) and \( s \), the potential \( \Phi \) as given in Eq. (6.4), can be expressed as

\[
\Phi(x, y, s) = A(s) - \frac{1}{4} \left( A^{(2)}(s) - D(s) \right) x^2 - \frac{1}{4} \left( A^{(2)}(s) + D(s) \right) y^2
\]

(6.5)

The components of the magnetic field at any point \( (x, y, s) \) near the axis can be obtained easily from Eq. (6.5) as

\[
B_x(x, y, s) = -\frac{1}{2} \left( B^{(1)}(s) - D(s) \right) x
\]

(6.6a)

\[
B_y(x, y, s) = -\frac{1}{2} \left( B^{(1)}(s) + D(s) \right) y
\]

(6.6b)

\[
B_z(x, y, s) = B(s) - \frac{1}{4} \left( B^{(2)}(s) - D^{(1)}(s) \right) x^2 - \frac{1}{4} \left( B^{(2)}(s) + D^{(1)}(s) \right) y^2 \approx B(s)
\]

(6.6c)

where \( B(s) = A^{(1)}(s) \) is the field on the axis of the solenoid. In the last equation (6.6c) we have neglected the second order terms in \( x \) and \( y \). The function \( D(s) \) is related with the field gradient along \( x \) and \( y \) directions and depends upon the shape of the elliptic cross-section of the solenoid. The value of \( B^{(1)}(s) \) and elliptical parameter \( D(s) \) can be easily obtained using Eqs. (6.6) as
\[ B^{(1)}(s) = \left( \frac{B_x(R_x)}{R_x} + \frac{B_y(R_y)}{R_y} \right) \]  

\[ D(s) = \left( \frac{B_x(R_x)}{R_x} - \frac{B_y(R_y)}{R_y} \right) \] 

where \( R_x \) and \( R_y \) are the semi major and semi minor axes of the elliptic pole face of the solenoid and \( B_x(R_x) \) and \( B_y(R_y) \) are the magnetic fields at the tip of the pole face along \( x \) and \( y \) directions respectively. It is easy to see that for \( R_x = R_y \), the value of \( D(s) \) becomes zero and \( B^{(1)}(s) \) reduces to the same value as that of a conventional solenoid.

### 6.2.2. Equations of motion

The transverse equations of motion of a particle of rest mass \( m \) and charge \( q \) in the external magnetic field of the elliptical solenoid and beam self fields can be written as

\[
\gamma m \ddot{x} = q [E_x^S + \dot{y}(B_y + B_y^S) - \dot{s}(B_y + B_y^S)] \\
\gamma m \ddot{y} = q [E_y^S + \dot{x}(B_x + B_x^S) - \dot{s}(B_x + B_x^S)]
\]

Here \( \gamma \) is the usual relativistic parameter and \( E^S \) and \( B^S \) are the space charge electric and magnetic fields respectively. For an intense continuous beam with small transverse dimensions and slow axial variations \(((\partial / \partial s) \approx 0)\), the \( s \) variation of the self scalar and vector potentials can be treated as negligibly small. Under these conditions, the self-electric and self-magnetic fields associated with an intense continuous beam of charged particles can be obtained from

\[
E^S(x, y, s) = -\left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} \right) \phi^S(x, y, s),
\]

\[
B^S(x, y, s) = \left( \frac{\partial}{\partial y} \hat{x} - \frac{\partial}{\partial x} \hat{y} \right) A^S_x(x, y, s)
\]
where the scalar potential \( \phi^S(x,y,s) \) for the self electric field obeys the Poisson equation,

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi^S(x,y,s) = -\frac{qn(x,y,s)}{\varepsilon_0}
\]  

(6.10)

and the vector potential for the self magnetic field is

\[
\vec{A}^S(x,y,s) = \frac{\beta}{c} \vec{\phi}^S(x,y,s) \hat{z}
\]  

(6.11)

Here \( n(x,y,s) \) is the number density of particles.

In an uncoupled system, a beam having constant density and pulsating in transverse dimensions \( X \) and \( Y \), the elliptical cross-section of the beam always remains upright. In this case one can obtain the space charge potential inside the beam by solving Eq. (6.10) and hence self fields as discussed in ref. [93] as well as in many text books [44, 84].

Fig. 6.1. A schematic of rotated coordinate system \((\tilde{x}, \tilde{y})\) with respect to the laboratory coordinate system \((x, y)\) by an angle \( \theta \), in which the semi axes of elliptical cross-section of the beam are aligned with the coordinate axes.

In the cases where the two transverse motions are coupled, the elliptical cross section of the beam not only pulsates in two transverse planes but also rotates about the propagation axis. Both the dimensions \( X \) and \( Y \) as well as the tilt angle \( \theta \) are functions of
s and thus it is not straightforward to obtain the space charge potential within the beam. In order to obtain the space charge potential in a coupled system at any axial location \( s \) we use a coordinate system which is rotated by an angle \( \theta \) with respect to the laboratory coordinate system where the elliptical cross-section is upright. Here one can calculate the space charge potential assuming a uniform beam distribution. This potential can then be transformed to the laboratory coordinate system to calculate the required self-electric and self-magnetic fields as desired in the equations of motion.

We have introduced a coordinate system \( (\tilde{x}, \tilde{y}) \) which is rotated with respect to the laboratory coordinate system \( (x, y) \) by an angle \( \theta \). From here and onwards we use tilde \( \sim \) on a variable to represent the quantity in rotated coordinate system. In the rotated coordinate system the semi axes of elliptical cross-section of the beam are aligned with the coordinate axes as shown in Fig. 6.1. The coordinates are related by

\[
\tilde{x} = x \cos \theta + y \sin \theta, \quad \tilde{y} = -x \sin \theta + y \cos \theta
\]  

To determine the self-electric and self-magnetic fields of the beam self consistently we assume that the equilibrium particle density is uniform within the upright elliptical boundary in the rotated system and zero elsewhere. Therefore the uniform particle density profile of the beam in this system can be expressed as

\[
n(\tilde{x}, \tilde{y}, s) = \frac{n_0}{\pi \tilde{X} \tilde{Y}} \Theta \left( 1 - \frac{\tilde{x}^2}{\tilde{X}^2} - \frac{\tilde{y}^2}{\tilde{Y}^2} \right)
\]

where \( \Theta(x) = 1 \) if \( x > 0 \) and \( \Theta(x) = 0 \) if \( x < 0 \) and \( \tilde{X}(s) = 2\sqrt{\langle \tilde{x}^2 \rangle} \) and \( \tilde{Y}(s) = 2\sqrt{\langle \tilde{y}^2 \rangle} \) are the beam sizes along the transverse \( \tilde{x} \) and \( \tilde{y} \) coordinate axes respectively.

Here \( n_0 = \int_0^\infty d\tilde{x}d\tilde{y}n(\tilde{x}, \tilde{y}, s) = \text{const} \) is the number of particles per unit axial length. The density profile \( n(x, y, s) \) in Eq. (6.10) will also remain constant inside the rotated ellipse whose transverse dimensions are \( \tilde{X} \) and \( \tilde{Y} \) and the tilt angle is \( \theta \) [95-97]. The self
electrostatic potential in the rotated coordinate system where elliptical cross-section is upright with constant density distribution specified in Eq. (6.13) can be solved using Poisson’s equation [98]. The result is

\[ \phi^S(\tilde{x}, \tilde{y}, s) = -\frac{I}{2\pi\varepsilon_0\beta c} \left( \frac{\tilde{x}}{X} + \frac{\tilde{y}}{Y} \right)^2 \]  

(6.14)

where \( I = qn_0\pi\bar{X}\bar{Y}\beta c \) is the beam current and \( \varepsilon_0 \) is the permittivity of free space. By substituting the expression of \( \tilde{x} \) and \( \tilde{y} \) from Eq. (6.12) into Eq. (6.14), we have

\[ \phi^S(x, y, s) = -\frac{I}{4\pi\varepsilon_0\beta c} \left( \phi_{xx} x^2 - 2\phi_{xy} xy + \phi_{yy} y^2 \right) \]  

(6.15)

where

\[ \phi_{xx} = \frac{\bar{X} + \bar{Y} - (\bar{X} - \bar{Y})\cos2\theta}{\bar{X}\bar{Y}(\bar{X} + \bar{Y})}, \quad \phi_{xy} = \frac{(\bar{X} - \bar{Y})\sin2\theta}{\bar{X}\bar{Y}(\bar{X} + \bar{Y})}, \]

\[ \phi_{yy} = \frac{\bar{X} + \bar{Y} + (\bar{X} - \bar{Y})\cos2\theta}{\bar{X}\bar{Y}(\bar{X} + \bar{Y})} \]  

(6.16)

Eq. (6.15) gives the self potential in terms of laboratory coordinates \( x \) and \( y \). The expressions for the self field of a rotated beam in the case of a linearly coupled system are also mentioned in ref. [99, 100].

We are now in a position to express the equations of motion given in Eq. (6.8) in terms of field components of elliptical solenoid obtained in Eq. (6.6) and self field of the beam. Replacing \( t \) with \( s \) (\( s = \beta ct \)) and using the fact that \( E^S_x - s\dot{B}^S_y = -(1 - \beta^2)\partial\phi^S / \partial x \), \( E^S_y + s\dot{B}^S_x = -(1 - \beta^2)\partial\phi^S / \partial y \) and \( B^S_x = 0 \), the equations of motion can be written as

\[ x'' = \frac{qB'(s)}{2m\gamma\beta c} y + J(s)y + 2K(s)y' + Q(x\phi_{xx} - y\phi_{xy}) \]

\[ y'' = -\frac{qB'(s)}{2m\gamma\beta c} x + J(s)x - 2K(s)x' + Q(y\phi_{yy} - x\phi_{xy}) \]  

(6.17)
where $K(s) = qB(s)/2m\gamma\beta c$, $J(s) = qD(s)/2m\gamma\beta c$ and $Q = qI / 2\pi\varepsilon_0 mc^2 \beta^3 \gamma^3$ is the beam perveance and the prime denotes the derivative with respect to $s$. The first term in Eqs (6.17) depends upon the gradient of the magnetic field. It is effective only at the entrance and exit of the elliptical solenoid where there is a sharp rise and fall in the magnetic field. Its effect is to impart an impulse to the particle which causes a sudden change in the direction of the trajectory [91, 101] and can be expressed in matrix form for the entrance as

$$
\begin{bmatrix}
x \\
x' \\
y \\
y'
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & K(s) & 0 \\
0 & 0 & 1 & 0 \\
-K(s) & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x'_0 \\
y_0 \\
y'_0
\end{bmatrix} = R_{\text{end}}(K(s))
\begin{bmatrix}
x_0 \\
x'_0 \\
y_0 \\
y'_0
\end{bmatrix}
$$

(6.18)

where $x_0, y_0$ are the initial positions and $x'_0, y'_0$ are the initial divergences of the particle in $x$ and $y$ planes respectively at the entrance of the magnet. For the exit of the solenoid the matrix will be $R_{\text{end}}(-K(s))$. Inside the elliptical solenoid the particle trajectory is obtained by solving Eqs (6.17) without the first term. Thus the equations of motion inside the elliptical solenoid are

$$
x'' = J(s)y + 2K(s)y' + Q(x\phi_{xx} - y\phi_{xy})
$$

$$
y'' = J(s)x - 2K(s)x' + Q(y\phi_{yy} - x\phi_{yx})
$$

(6.19)

These are coupled differential equations and their analytical solutions are known in the absence of space charge ($Q = 0$) and can be represented in terms of transfer matrix [91].

The transfer matrix then can be utilized to obtain the beam properties using the standard sigma matrix method. However, in the case of space charge effect simple matrix multiplication cannot be used because the space charge forces are dependent on the properties of the beam itself. A convenient way for dealing with the space charge is to employ the infinitesimal transfer matrix approach.
6.2.3. Infinitesimal transfer matrix and beam envelopes

Since Eqs. (6.19) are linear, therefore any solution of these equations can be represented by a linear combinations of four linearly independent solutions. Since we are primarily interested in the transverse beam dynamics we need to construct $4 \times 4$ infinitesimal transfer matrix. Let $x(s) = (x, x', y, y')^T$ represents the coordinates of a paraxial ray in the laboratory coordinate system at a location $s_0$ inside the elliptical solenoid magnet. At some other location $s$, the coordinates will be transformed according to the matrix equation, $x(s) = R(s,s_0)x(s_0)$ where $R(s,s_0)$ is a $4 \times 4$ transfer matrix inside the magnet whose elements are functions of $s$ and $s_0$. In order to generate the matrix $R(s,s_0)$, we need to solve Eqs. (6.19) for four different initial conditions with sufficiently small interval $ds = s - s_0$, with $ds << L$, where $L$ is the length of the elliptical solenoid. A simple way is to choose the initial conditions in which one coordinate say $x$ is equal to 1 and all other coordinates are equal to 0. The solution of differential equations will yield the first column of transfer matrix $R(s,s_0)$. By repeating the same procedure with other coordinates one can easily get the all other columns of matrix $R(s,s_0)$. The entry and exit matrix of magnet at $s_0$ and $s$ has to be multiplied with $R(s,s_0)$ to get the coordinate transformation through the small portion $ds$ of elliptical solenoid magnet. This can be expressed as

$$M(s,s_0) = R_{end}(-K(s))R(s,s_0)R_{end}(K(s_0)) \quad (6.20)$$

This is the infinitesimal transfer matrix.

It is to be noted here that the space charge term in equations of motion (6.19) depends on the properties of the beam itself i.e. on the beam sizes $(\tilde{X}(s)$ and $\tilde{Y}(s))$ and the rotation angle $\theta$ which are function of the axial distance $s$. These quantities must be evaluated
after each interval $ds$ to obtain the self field. The beam envelope quantities are basically related to the beam matrix $\sigma$. For a continuous beam the sigma matrix defines the shape of a 4D hyper-ellipsoid of the beam i.e.

$$ x^T \sigma^{-1} x = 1 $$

(6.21)

where the superscript “$T$” denotes the transpose of the matrix. The 4D hyper-ellipsoid when projected into 2D subspace (say $x, y$) in the laboratory coordinate system, the equation of the projected ellipse becomes

$$ x_{13}^T \sigma_{13}^{-1} x_{13} = 1 $$

(6.22)

where

$$ x_{13} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \sigma_{13} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} $$

(6.23)

Here the elements of sub-matrix $\sigma_{13}$ are chosen from elements of the matrix $\sigma$ defined in Eq. (6.21). Expanding Eq. (6.22) we have

$$ \sigma_{xx} \sigma_{yy} - \sigma_{xy}^2 = \sigma_{xy} x^2 + \sigma_{xx} y^2 - 2 \sigma_{xy} xy $$

(6.24)

In the rotated coordinate system, the beam ellipse with semi axes $\tilde{X}(s)$ and $\tilde{Y}(s)$ as shown in Fig. 6.1 is an upright ellipse and its equation can be written as

$$ \frac{\tilde{x}^2}{\tilde{X}^2} + \frac{\tilde{y}^2}{\tilde{Y}^2} = 1 $$

(6.25)

Now using the values of $\tilde{x}$ and $\tilde{y}$ from Eq. (6.12) into Eq. (6.25) and then comparing the coefficients of $x^2$, $y^2$ and $xy$ with Eq. (6.24) we have

$$ \sigma_{xx} = \tilde{X}^2 \cos^2 \theta + \tilde{Y}^2 \sin^2 \theta $$

(6.26a)

$$ \sigma_{yy} = \tilde{Y}^2 \cos^2 \theta + \tilde{X}^2 \sin^2 \theta $$

(6.26b)

$$ \sigma_{xy} = (\tilde{X}^2 - \tilde{Y}^2) \sin \theta \cos \theta $$

(6.26c)
The semi axes of upright ellipse $\tilde{X}(s)$ and $\tilde{Y}(s)$ in the rotated coordinate system can be easily obtained using Eqs. (6.26) as

$$\tilde{X}(s) = \frac{1}{\sqrt{2}} \left[ \sigma_{xx}(s) + \sigma_{yx}(s) + \sqrt{[\sigma_{xx}(s) - \sigma_{yx}(s)]^2 + 4\sigma_{xy}^2(s)} \right]$$

$$\tilde{Y}(s) = \frac{1}{\sqrt{2}} \left[ \sigma_{xx}(s) + \sigma_{yx}(s) - \sqrt{[\sigma_{xx}(s) - \sigma_{yx}(s)]^2 + 4\sigma_{xy}^2(s)} \right]$$

$$\theta(s) = \frac{1}{2} \tan^{-1} \left( \frac{2\sigma_{xy}(s)}{\sigma_{xx}(s) - \sigma_{yx}(s)} \right)$$  \hspace{1cm} (6.27)

The beam sizes $\tilde{X}(s)$ and $\tilde{Y}(s)$ and the rotation angle $\theta(s)$ have been used to determine the self field potential in the laboratory frame as given in Eq. (6.15) to solve Eqs. (6.19).

In order to determine the beam sigma matrix along the axial distance $s$ in the laboratory coordinate system, we use the recursive sigma matrix method. In this method we divided the elliptical solenoid magnet into large number of small intervals $ds = s - s_0$ and beam matrix $\sigma(s)$ at location $s$ is obtained by using the relation

$$\sigma(s) = M(s, s_0)\sigma(s_0)M(s, s_0)^T$$  \hspace{1cm} (6.28)

where $\sigma(s_0)$ is the beam matrix at location $s_0$ and $M(s, s_0)$ is the infinitesimal transfer matrix defined in Eq. (6.20). The beam sizes and the emittances in the $x$ and $y$ planes at an axial location $s$ can be obtained from the following relations

$$X(s) = \sqrt{\sigma_{xx}(s)}, \quad Y(s) = \sqrt{\sigma_{yy}(s)}$$

$$\varepsilon_x(s) = \sqrt{[\sigma_{xx}(s)\sigma_{xx}(s) - \sigma_{xx}^2(s)]}, \quad \varepsilon_y(s) = \sqrt{[\sigma_{yy}(s)\sigma_{yy}(s) - \sigma_{yy}^2(s)]}$$  \hspace{1cm} (6.29)

### 6.3. Beam optical properties of an elliptical solenoid

In this section we first discuss the beam optical properties of the elliptical solenoid magnet in the presence of space charge using the parameters of our low energy beam
transport line. We have chosen injection energy equal to 80 keV and parameters $K = 0.04$ cm$^{-1}$ and $J = 0.0003$ cm$^2$. These parameters are taken constant along the length $s$ of the magnet. Throughout the analysis we have assumed that initial emittances in both the planes are uncoupled. The beam line parameters are: length of the elliptic solenoid $L = 30$ cm and drift lengths before and after the magnet are 40 cm and 100 cm respectively.

6.3.1. Focusing characteristics

We have written a computer code which solves the paraxial trajectory given by Eqs. (6.19) for specified initial beam conditions from point $s_0$ to $s = s_0 + ds$, where $ds$ is the small interval. We have used the step size $ds = 1$ mm. Using the solution of paraxial ion trajectories for four different initial conditions, the infinitesimal transfer matrix $M(s, s_0)$ is obtained for the small interval $ds$. It is then used to find out the beam matrix $\sigma(s)$ at point $s$ from Eq. (6.28) utilizing the initial value of $\sigma(s_0)$. To determine the self field potential $\phi^S(x, y, s)$ in the optical coordinate system we have used the elements of $\sigma(s)$ in Eq. (6.27) to get the beam sizes $\tilde{X}(s)$, $\tilde{Y}(s)$ and the rotation angle $\theta(s)$. The beam envelope quantities are obtained from the elements of $\sigma(s)$ using Eqs. (6.29).

In Fig. 6.2 the behavior of beam envelope and focusing properties of elliptical solenoid magnet are compared with the conventional solenoid ($J = 0$) at two values of beam current $I = 0$ mA and $I = 10$ mA. The input beam conditions for both cases are $X(0) = Y(0) = 0.25$ cm, $X'(0) = Y'(0) = 0$ mrad and equal total emittances in both the planes i.e. $\epsilon_x(0) = \epsilon_y(0) = 60 \pi$ mm mrad at 40 cm before the magnet. The variation of the beam envelopes through the solenoid magnet for beam current $I = 0$ mA and $I = 10$ mA are shown in Fig. 6.2(a). As expected, the beam envelopes in both the planes are similar for this circular symmetric input beam because solenoid magnet exerts equal focusing forces in both the planes. The effect of space charge is clearly evident in terms of
location and size of the waist. For 0 mA beam current, the waist with size ~0.2 cm is formed at a distance ~30 cm from the exit of the magnet. In the case of 10 mA, not only the size of the waist ~0.33 cm of the beam is larger but the location of the waist ~50 cm is also at a farther distance from the exit of the magnet. However, in both the cases the circular symmetry of the beam is maintained.

Fig. 6.2. Beam envelopes for (a) solenoid magnet and (b) elliptical solenoid magnet, for same initial axisymmetric beam with $X(0) = Y(0) = 0.25$ cm, $X'(0) = Y'(0) = 0$ mrad and $\varepsilon_x(0) = \varepsilon_y(0) = 60 \pi$ mm mrad for two different values of beam current $I = 0$ mA (solid curve) and $I = 10$ mA (dashed curve).

In Fig 6.2(b) we have shown the evolution of beam envelopes through the elliptical solenoid with parameters $K = 0.04$ cm$^{-1}$ and $J = 0.0003$ cm$^{-2}$ for two values of beam current with the same input conditions as in the previous case. The effect of asymmetric focusing and inter-plane coupling effect is clearly visible from the envelope behavior. The beam envelopes in both the planes for $I = 0$ mA are different because of the parameter $J$ which causes an extra gain in focusing force in $y$ plane and a reduction in the $x$ plane. As a result the beam waist in $y$ plane is formed at a shorter distance compared to that of the
The effect of space charge, as shown in Fig 6.2(b) by dashed curve for 10 mA beam current, not only increases the waist sizes but also the location of the waists in both planes.

6.3.2. Study of inter-plane coupling effect

In order to understand the coupling effect in the beam caused by an elliptical solenoid magnet we now explore the magnitude of projected emittances under various input beam conditions. Fig. 6.3 shows the behavior of the projected $x$ and $y$ emittances as a function of axial distance $s$, for an axisymmetric input beams with $X(0) = Y(0) = 0.25$ cm, $X'(0) = Y'(0) = 0$ mrad and with equal emittances $\varepsilon_x(0) = \varepsilon_y(0) = 60 \pi$ mm mrad in both the planes. In the case of a conventional solenoid ($J = 0$) the projected emittances in the $x$ and $y$ planes are same at all the points downstream, they are equal to the initial emittances and are not affected by the beam current. This is due to the fact that both the external as well as space charge forces are symmetric with respect to the two semi-axes of the ellipse in the local $x$-$y$ plane.

![Graph showing projected $x$ and $y$ emittances through the solenoid (dashed line) and elliptical solenoid (solid line) for axisymmetric input beam. The input conditions of the beam are same as in the case of Fig. 6.2.](image)

**Fig. 6.3.** Transverse projected $x$ and $y$ emittances through the solenoid (dashed line) and elliptical solenoid (solid line) for axisymmetric input beam. The input conditions of the beam are same as in the case of Fig. 6.2.
The situation is completely different in the case of elliptical solenoid \((J = 0.0003 \text{ cm}^{-2})\) where we see a growth in the projected \(x\) and \(y\) emittances even for \(I = 0 \text{ mA}\). This happens because of the fact that the initial axisymmetric beam becomes non-axisymmetric in the local \(x-y\) plane even in the absence of space charge. The magnitude of the projected emittances in both the planes grows rapidly as the beam passes through the magnet, reaches to a maximum value inside the magnet and then decreases for both values of beam current. It is interesting to note that \(x\) and \(y\) projected emittances are always equal to each other at any point downstream and independent of the magnitude of the \(x-y\) coupling. There is a substantial growth in the magnitude of the projected emittances at the exit in both the planes due to the coupling effects. The estimated values of the emittances for \(I = 0 \text{ mA}\) and \(I = 10 \text{ mA}\) at the exit of the elliptical solenoid magnet are \(66.4 \pi \text{ mm mrad}\) and \(96.4 \pi \text{ mm mrad}\) respectively.

It can be readily seen from Fig. 6.3 that the behavior of projected emittances in the drift after the exit of the elliptical solenoid is different for the case of beam with space charge where the emittances do not remain constant as in the case of \(I = 0 \text{ mA}\). This is happening due to the coupled motions in \(x-y\) planes at the exit of elliptical solenoid i.e. the tilted transverse cross-section of the beam. The space charge effect further introduces coupling due to which there will be either a growth or reduction in the projected emittances as the beam travels in the drift space after the exit.

In Fig. 6.4 we have plotted the behavior of transverse projected emittances through the elliptical solenoid and compared the results with that of conventional solenoid. We have chosen the initial emittances in \(x\) and \(y\) planes as \(\varepsilon_x(0) = 70 \pi \text{ mm mrad}, \varepsilon_y(0) = 50 \pi \text{ mm mrad}\) respectively and unequal input beam sizes in the two transverse directions i.e. \(X(0) = 0.5 \text{ cm}, Y(0) = 0.25 \text{ cm}\). The behavior of projected emittances for a conventional solenoid \((J = 0)\) is shown in Fig. 6.4(a). It is interesting to note here that there is an
exchange of emittance from one plane to the other plane. The projected emittance reduces in the plane where the initial emittance is high and it grows in the other plane where the initial emittance is low. The projected $x$ and $y$ emittances at any point downstream are always unequal except at a point as shown by the dotted vertical line where these are equal in both the planes. There is very little growth in the projected emittances due to the space charge effect for the present initial condition of the beam.

![Projected $x$ and $y$ emittances](image)

**Fig. 6.4.** Projected $x$ and $y$ emittances for $I = 0$ mA (solid line) and $I = 10$ mA (dashed line) through the (a) solenoid and (b) elliptical solenoid magnet for non-axisymmetric input beam. The input beam conditions in both cases are $X'(0) = 0.5$ cm, $Y'(0) = 0.25$ cm, $X'(0) = Y'(0) = 0$ mrad, $\varepsilon'_x(0) = 70 \ \pi$ mm mrad and $\varepsilon'_y(0) = 50 \ \pi$ mm mrad.

The behavior of projected emittances through the elliptical solenoid is presented in **Fig 6.4(b)** with the same initial conditions. Apart from an exchange of emittances from one plane to the other plane, there is a substantial growth in the projected emittances inside as well as outside the magnet due to the space charge effect. Two points are noteworthy from **Fig. 6.4(b)**. In the case of 10 mA beam current, the projected emittance in both the planes
rises very fast inside the magnet, goes to a peak value and then drops to a smaller value at the exit. This effect is very small in the case of low beam current as well as in the case of the conventional solenoid. The second interesting point is that the projected emittances in both the planes are also equal at only one point and the location of that point is unaffected by the space charge effect.

**Fig. 6.5.** Phase ellipses in $x$ and $y$ planes at the exit of elliptical solenoid along with contributions from the inter-plane coupling for initial uncoupled phase ellipses. Parameters of elliptical solenoid are $K = 0.04 \text{ cm}^{-1}$, $J = 0.0003 \text{ cm}^{-2}$ and $L = 30 \text{ cm}$.

In an elliptical solenoid, there is a strong coupling between the two transverse planes which results in growth of projected emittances at the exit. This growth in projected emittances depends on the parameter $J$ and is very sensitive to the orientation of the
phase ellipses of the beam at the entrance of the elliptical solenoid. In order to demonstrate
the coupling effect of the two transverse planes on the projected emittances, tracing of
paraxial rays of 40 representative particles that belong to the boundary of the contours in
\((x-x')\) and \((y-y')\) planes of the phase ellipse at the entrance in both the planes have been
performed through the magnet. The initial conditions for the coordinates and the beam
divergences are chosen using tilted phase ellipses with cross-section \(X(0) = 1\, \text{cm},\)
\(Y(0) = 0.5\, \text{cm}\) and emittances \(\varepsilon_x(0) = \varepsilon_y(0) = 60\, \pi\, \text{mm mrad}\) in both planes.

Fig. 6.5 shows the orientations of phase ellipses, at the entrance and exit of the
elliptical solenoid for 0 mA and 10 mA along with the contributions from the inter-plane
coupling. Figures 6.5(a) and 6.5(b) represent the initial phase ellipses at the entrance of the
elliptical solenoid whereas Fig. 6.5(c) and 6.5(d) represent the phase ellipses at the exit for
\(I = 0\, \text{mA}\). Solid lines indicate the area pertaining to the particular phase plane whereas
dashed lines indicate the area contributed from the other plane. The total area of these
subspaces gives the effective emittance at the exit of the magnet as shown by the dotted
curve. The inter-plane coupling effects in the case of 10 mA beam current are shown in
Fig. 6.5(e) and 6.5(f). It is to be noted here that the increase in emittances in both the
planes at \(I = 0\, \text{mA}\) is purely due to the inter-plane coupling effects. The estimated
effective emittances at the exit in \(x\) and \(y\) planes are same and equal to \(75\, \pi\, \text{mm mrad}\)
for \(I = 0\, \text{mA}\) and \(81\, \pi\, \text{mm mrad}\) for \(I = 10\, \text{mA}\). However, the orientations of projected
emittances in both the planes at the exit are different.

6.3.3. Parametric dependence of emittance growth

We have also studied the behavior of projected emittances in both the planes at the exit
with parameter \(J\) for three different values of the beam current. We have considered two
cases. In the first case (Fig. 6.6(a)) we have chosen an axisymmetric beam with
$X(0) = Y(0) = 1.5 \text{ cm}$ whereas in the second case (Fig. 6.6(b)) it is a non-axisymmetric beam with $X(0) = 1.5 \text{ cm}$, $Y(0) = 1.0 \text{ cm}$. In both the cases initial emittances are uncoupled and equal i.e. $\varepsilon_x(0) = \varepsilon_y(0) = 60 \ \pi \text{ mm mrad}$. It can be readily seen from Fig. 6.6 that, although the projected emittances at the exit increase with parameter $J$ and beam current, their values are always equal in both the planes. For $J = 0$, the projected emittances at the exit are equal to the initial emittances in the case of axisymmetric beam. However, there is a substantial growth in the emittances in the case of non-axisymmetric beam even at $J = 0$ and increases with beam current. It is observed from the numerical calculations that there is a comparatively more growth in the projected emittances for non-axisymmetric beam as we increase the value of parameter $J$ and the beam current.

**Fig. 6.6.** Transverse projected emittances at the exit of the elliptical solenoid as a function of the parameter $J$ for three different values of beam current. (a) axisymmetric beam with $X(0) = Y(0) = 1.5 \text{ cm}$. (b) non-axisymmetric beam with $X(0) = 1.5 \text{ cm}$, $Y(0) = 1.0 \text{ cm}$. Other input conditions are, $\varepsilon_x(0) = \varepsilon_y(0) = 60 \ \pi \text{ mm mrad}$ and $X'(0) = Y'(0) = 10 \text{ mrad}$. Parameters of the elliptical solenoid are $k = 0.04 \text{ cm}^{-1}$ and $L = 30 \text{ cm}$.
In Fig. 6.7 we have plotted the transverse projected emittances at the exit of the elliptical solenoid as a function of the parameter \( J \) for three different values of the beam current considering a more general case where the initial emittances are unequal in both the planes i.e. \( \varepsilon_x(0) = 70 \ \pi \text{ mm mrad} \), \( \varepsilon_y(0) = 50 \ \pi \text{ mm mrad} \) and the beam is non-axisymmetric. As usual we observe an increase in the projected emittances with beam current as well as with parameter \( J \). In this case also there is a distinct exchange of emittances from one plane to the other plane at the exit of the magnet.

![Graph showing projected emittances](image)

**Fig 6.7.** Projected emittances at the exit of the elliptical solenoid as a function of the parameter \( J \) for beam currents \( I = 0 \) mA (solid line), \( I = 10 \) mA (dashed line) and \( I = 20 \) mA (dotted line) for an initial non-axisymmetric beam with \( X(0) = 1.5 \text{ cm} \), \( Y(0) = 1 \text{ cm} \), \( X'(0) = Y'(0) = 10 \text{ mrad} \), \( \varepsilon_x(0) = 70 \ \pi \text{ mm mrad} \) and \( \varepsilon_y(0) = 50 \ \pi \text{ mm mrad} \). Parameters of the elliptical solenoid are \( K = 0.04 \text{ cm}^{-1} \) and \( L = 30 \text{ cm} \).

### 6.3.4. Conversion of an elliptical beam to a circular beam

It is not always possible to transfer a beam of elliptical cross-section to a circular cross-section using a conventional solenoid. Such transformation can happen only in a particular situation where the beam rotation through the solenoid is \( (2n + 1)\pi/4 \) [102, 103]. But a beam of circular cross-section can never be transformed to an elliptical cross-
section using an axisymmetric solenoid magnet. However an elliptical solenoid can transform a beam of elliptical cross-section to a circular cross-section and vice-versa.

A single elliptical solenoid, in general can never transform a beam of circular cross-section to a circular cross-section like an axisymmetric solenoid. For this we need a combination of two elliptical solenoids in succession with parameters \( J \) and \(-J\). Figure 6.8 shows the beam envelopes in which the beam waists with equal sizes are formed at the axial location \(\sim 105 \) cm in both the planes for \(I = 10\) mA. The initial beam is axisymmetric with \(X(0) = Y(0) = 0.25\) cm, \(X'(0) = Y'(0) = 0\) mrad and \(\varepsilon_x(0) = \varepsilon_y(0) = 60\pi\) mm mrad.

The optimum locations and lengths of the elliptical solenoids are indicated by boxes in Fig. 6.8. The length of the first magnet is 20 cm with \(K = 0.046\) cm\(^{-1}\) and \(J = 0.00025\) cm\(^{-2}\) and that of the second is 19 cm with \(K = 0.046\) cm\(^{-1}\), \(J = -0.00025\) cm\(^{-2}\).

![Fig. 6.8. Beam envelopes in x and y planes for 10 mA beam current through elliptical solenoid doublet to produce beam waists at the axial location ~105 cm. The initial beam is axisymmetric with \(X(0) = Y(0) = 0.25\) cm and \(\varepsilon_x(0) = \varepsilon_y(0) = 60\pi\) mm mrad. Here “W” indicates the location of equal beam waist in both the planes.](image)

**6.4. Transverse beam matching to the spiral inflector**

As it is mentioned earlier, the injection system of 10 MeV proton cyclotron consists of two solenoid magnets to transport and match the beam at the entrance of a spiral inflector.
A detailed study presented in Chapter 4 indicates that convergent phase ellipses with different orientations in x and y planes and a comparatively smaller width in y plane give better beam transmission. Figure 6.9 shows the results of transverse matching of an axisymmetric beam at the entrance of the spiral inflector. The resulting beam envelopes producing a beam of unequal sizes in x and y planes are shown in Fig. 6.9(a) and the phase ellipses are shown by solid lines in Fig. 6.9(b) and 6.9(c). The required phase ellipses at the entrance of the spiral inflector for better transmission and minimum emittance growth are shown by dotted curves in Fig. 6.9(b) and 6.9(c). The initial beam parameters at the waist position (s = 0) of second solenoid are X(0) = Y(0) = 0.25 cm and equal values of emittances εx(0) = εy(0) = 60 π mm mrad in both the planes. The optimized parameters
of the elliptical solenoid magnet are $K = 0.064 \text{ cm}^{-1}$ and $J = 0.00014 \text{ cm}^{-2}$ and length $L = 25 \text{ cm}$. The location of elliptical solenoid and matching point from the beam waist of the second solenoid are 20 cm and 65 cm respectively. The estimated projected emittances at the matching point are $\epsilon_x(M) = \epsilon_y(M) = 61.4 \pi \text{ mm mrad}$ which indicates very small emittance growth caused due to elliptical solenoid.

6.5. Summary

In this Chapter, we have discussed the focusing properties of an elliptical solenoid and studied the transport and matching of space charge dominated beam using the infinitesimal transfer matrix technique. We have studied the emittance growth that results from the coupling between the two transverse planes as a function of beam current and the coupling parameter of the elliptical solenoid under various input beam conditions. The change of transverse projected emittances as a function of distance does not really indicate any degradation in the beam quality. We like to mention here that in a linearly coupled 4D phase space although the $x$ and $y$ projected emittances vary as a function of distance, there are always two independent generalized invariants analogous to the $x$ and $y$ emittances for the uncoupled system [104, 105]. Any increase in the generalized invariants provides information about the intrinsic degradation of the full 4D phase space of the beam. Thus in order to know the presence of any nonlinearity in the system one needs to calculate these invariants. We have also shown that the elliptical solenoid has an advantage and flexibility over the conventional solenoid for transverse matching of the beam to the spiral inflector which requires unequal beam sizes as well as different orientations of phase ellipses in the two transverse planes.