Chapter 5

Envelope oscillations and amplitude growth in a compact cyclotron

5.1. Introduction

In this Chapter we discuss the investigation on the amplitude growth and oscillations in the beam envelopes along the accelerated orbit in a compact cyclotron [7]. We have used the coupled beam envelope equations and assumed the beam to be a uniform ellipsoidal bunch.

The study of the beam behavior in the central region of the cyclotron reveals that beam envelope behaves differently due to the coupling of the horizontal and vertical motions arising due to space charge effects [37]. In order to achieve optimum performance, the emittance and orientation of the phase ellipses of the injected beam must be matched to the acceptance of the central region of the cyclotron. It is believed that envelope mismatch is the major cause of emittance growth and halo formation [44]. For mismatch beams, an unbalance between the applied focusing force and the defocusing forces due to space charge and thermal effects, cause whole beam to oscillate in a coherent way. This effect increases the beam size in both the transverse planes and causes severe beam loss. In the case of low beam current the mismatch in one plane affects the beam behavior only in that particular plane. However, in the case of intense beam, where the space charge effect couples the motions of the two transverse planes, a mismatch in one plane affects the beam behavior in both transverse planes.

In a previous work [37] discussed in Chapter 2 we have studied the effect of space charge in a compact cyclotron to get the proper beam matching conditions at the injection.
We have used transverse envelope equations assuming a uniform continuous beam and an analytical hard edge model for the magnetic field. Since hard edge model of the magnetic field overestimates the vertical betatron tunes at lower radii, the results so obtained are limited in accuracy; however, they provide good insight about the behavior of beam envelopes. We like to point out here that studies on space charge effect in a compact cyclotron above 1 mA beam current are still not well understood and lots of research is going on to resolve the physics and technological issues at high current. Commercial compact cyclotrons operating for medical isotope production are limited to beam current <1 mA. Therefore, to avoid the beam loss, a systematic study of the space charge dominated beam behavior in the focusing channel of a compact cyclotron is of practical importance. This is the main objective of the work presented in this Chapter.

In this Chapter, first we have obtained the values of magnetic and electric betatron tunes and then developed the coupled accelerated beam envelope equations. It is assumed that the beam is a uniform ellipsoidal bunch. First the pattern of envelope oscillations and amplitude growth of the beam in both transverse planes at a particular radius (without acceleration) have been analyzed by displacing the initial beam size from the matched beam size at several values of beam current. Then we have discussed the results of our studies on the behavior of beam envelope in the 10 MeV cyclotron at different initial conditions of the beam. Finally we have obtained the proper matching conditions by optimizing the input beam parameters and also estimated the maximum beam current that can be transported through a given aperture of the cyclotron focusing channel.

5.2. Calculation of Betatron tunes

The design of the magnet of 10 MeV cyclotron has been discussed in Chapter 3. We have obtained the radial and axial betatron tunes from the computed magnetic field data.
using the equilibrium orbit (EO) program GENSEO [39]. The variation of the radial \( \nu_r \) and vertical \( \nu_y \) (magnetic) magnetic betatron tunes as a function of the orbit radius \( R \) is shown in Fig. 5.1. During the first few turns in a compact cyclotron, the vertical magnetic focusing is very weak. It is, therefore, necessary to exploit the vertical electric focusing available at the dee gaps by properly adjusting the geometry and phase.

In a cyclotron, the electric field at the acceleration gaps exerts a lens like action in the vertical plane on off-median plane particles during the first few turns. In order to include this effect in the calculations, we have used the first order theory [80] to estimate the electric vertical betatron tune. The electric focusing has negligible effect on the radial motion. We have assumed that particle traverses the electric gaps periodically in a stationary orbit of radius \( R \), without increase of energy (taking the average of initial and final energy at the gaps). As mentioned earlier, the accelerating structure of 10 MeV cyclotron consists of two delta type resonators located in the opposite valleys and each has an angle of \( \pi/4 \). For a particle displaced from the median plane in the vertical direction, the initial position and slope \( y_0 \) and \( y'_0 \) at the first gap and final position and slope \( y \) and \( y' \) at the third gap are related as

\[
\begin{bmatrix}
  y \\
  y'
\end{bmatrix}
= M
\begin{bmatrix}
  y_0 \\
  y'_0
\end{bmatrix}
\tag{5.1}
\]

\[
M = \begin{pmatrix}
1 & 3\pi R/4 \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-1/f_2 & -f_1/f_2 \\
\end{pmatrix} \begin{pmatrix}
1 & \pi R/4 \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-1/f_2 & -f_1/f_2 \\
\end{pmatrix}
\tag{5.2}
\]

where \( M \) is the transfer matrix from the first gap to the third gap. The distance between the first and second gap is \( \pi R/4 \) whereas the distance between the second and third gap is \( 3\pi R/4 \), where \( R \) is the average orbit radius under consideration. The focal length \( f_1 \) and \( f_2 \) are given by the following expressions:
\[
\frac{f_1}{f_2} = \frac{E_c - (qV_g \cos \theta_c)/2}{E_c + (qV_g \cos \theta_c)/2}. \tag{5.3}
\]

\[
\frac{1}{f_2^2} = \frac{\pi f_{rf}}{c} \left(2\pi \frac{V_g}{E_0}\right)^\frac{1}{2} \left(\frac{qV_g}{E_c}\right)^\frac{3}{2} \sin \theta_c + \frac{F}{2\pi H} \left(\frac{qV_g}{E_c}\right)^2 \cos^2 \theta_c \tag{5.4}
\]

where \( f_{rf}, H \) and \( E_0 \) are the rf frequency, half height of the dee and rest mass energy of the particle respectively. The factor \( F \) depends upon the geometry of the dee [80]. A particle with kinetic energy \( E \) will gain energy \( qV_g \cos \theta_c \) in traversing the first gap and the energy of the particle at the middle of the gap will be \( E_c = E + (qV_g \cos \theta_c)/2 \) where \( V_g \) is the gap voltage and \( \theta_c \) is the phase of rf when the particle is at the middle of the gap. The electric vertical betatron tune can be obtained from

\[
\nu_y = \frac{1}{\pi} \cos^{-1} \left[ \frac{1}{2} \text{Tr} \mathbf{M} \right] \tag{5.5}
\]

**Fig 5.1.** Betatron tunes as a function of orbit radius. Dashed curves represent the contribution to the vertical betatron tunes \( \nu_y \) (solid line) from the electric and magnetic focusing. The initial rf phase \( \theta_c \) is equal to \( 5^\circ \).

The parameters used in the calculations are: injection energy \( E = 80 \) keV, dee voltage \( V_g = 125 \) kV, rf frequency \( f_{rf} = 42 \) MHz, dee height from the median plane \( H = 1.5 \) cm
and gap between dee and dummy dee $2W = 2$ cm. The corresponding $F$ factor for this particular dee geometry is 0.82. The electric vertical tune $\nu_y^{(\text{electric})}$ obtained using Eq. (5.5) which is appreciable only at lower radii is shown by dashed line. The effective vertical tune is obtained using the formula, $\nu_y^{(\text{effective})} = [\nu_y^2^{(\text{magnetic})} + \nu_y^2^{(\text{electric})}]^{0.5}$ which is obtained by adding the vertical magnetic and electric focusing forces together and using the fact that focusing force is proportional to the square of the tune value. The result is also shown in Fig. 5.1. We have used values of these betatron tunes $\nu_x$ and $\nu_y^{(\text{effective})}$ in the beam envelope calculations.

### 5.3. Accelerated Beam envelope equations

Our main objective in this work is to study the behavior of beam envelopes along the accelerated orbits in the two transverse planes and to estimate the maximum transportable beam current in a given aperture. In order to obtain the coordinates of the path along the accelerated orbits, we performed the orbit tracing in the median plane of the cyclotron by solving the equations of motion in the combined electric and magnetic fields. The coordinates and velocity of the central ion trajectory obtained at the inflector exit were used as input. The accelerated orbits of the proton from 80 keV to 10 MeV in the median plane of the cyclotron is shown in Fig. 3.10 in Chapter 3.

We consider a bunched beam having uniform density distribution with ellipsoidal symmetry propagating along the accelerated orbit in the focusing channel of a compact cyclotron. The method presented here can also be utilized for any beam distribution using the concept of equivalent beams. According to this concept, the beam must have the same second moment as the actual beam distribution [44, 81]. It is well known that the space charge effect on bunches circulating in a cyclotron is very complex. Bunches rotate in the median plane and the rate of rotation depends upon the charge density. For short bunches
the stationary beam distribution shape is circular (i.e. bunch length = radial width). In the case, where the injected bunches are much longer azimuthally compared to the radial width, the bunch breaks up into small droplets [82, 83]. Since the charge density and hence the perveance reduces in the case of long bunches for the same beam current compared to the short bunches, this break up takes comparatively longer time [82]. Since in our case the average longitudinal bunch size during the first turn (~ 14 mm) is large compared to the matched radial beam width (~ 5 mm) we have simplified our analysis by neglecting the bunch rotation due to the radial and longitudinal coupling. We have also neglected the effect of axial variation of longitudinal field on the transverse motion. Under these assumptions the bunch shape remains ellipsoidal within the cyclotron. Since the longitudinal bunch size in a cyclotron depends on the location of the equilibrium orbit it will change to a new value after the acceleration at each gap.

![Fig. 5.2](image.png)

**Fig. 5.2.** A schematic of coordinate system attached to the beam bunch moving along the accelerated orbit in the median plane.

We now introduce a local coordinate system \( x, y \) and \( z \) as shown in Fig. 5.2 with the centre of the ellipsoid which moves with velocity \( v \) along the accelerated orbit in the median plane. Here \( x \) and \( y \) are the two transverse coordinates measure the distances from the bunch centre \( (x = y = z = 0) \) along the radial and vertical directions respectively.
whereas $z$ measures distance along the longitudinal direction. The $z$ axis is always tangent to the accelerated orbit. If $s$ be the path length along the accelerated orbit in the median plane from the starting point, then a particle with coordinates $(x, y, z)$ in the beam frame has coordinates $(R + x, y, s + z)$ in the laboratory frame where $R$ is the instantaneous orbit radius from the machine centre. The differential equations for beam envelopes $X(s)$ and $Y(s)$ in the two transverse planes [84, 85] can be written as:

$$X^* + \left( \frac{\beta \gamma}{\beta \gamma} \right) X' + \left[ \frac{\nu_x^2}{R^2} - \frac{3Ic}{2I_o \beta^2 \gamma^3 f_{cf}} \frac{1}{X^3} G1\left( \frac{Y}{X}, \frac{Z}{X} \right) \right] X - \frac{\epsilon_{nx}^2}{\beta^2 \gamma^2 X^3} = 0$$ (5.6)

$$Y^* + \left( \frac{\beta \gamma}{\beta \gamma} \right) Y' + \left[ \frac{\nu_y^2}{R^2} - \frac{3Ic}{2I_o \beta^2 \gamma^3 f_{cf}} \frac{1}{Y^3} G2\left( \frac{X}{Y}, \frac{Z}{Y} \right) \right] Y - \frac{\epsilon_{ny}^2}{\beta^2 \gamma^2 Y^3} = 0$$ (5.7)

and the average beam current $I$ as

$$I = \frac{4}{3} \pi XYZn_b q f_{cf} = Q_b f_{cf}$$ (5.8)

Here $X(s), Y(s)$ and $Z(s)$ are the semi axes of the ellipsoid, which are also the envelope sizes in the $x, y$ and $z$ directions respectively. $Q_b$ is the total charge in the bunch assumed to remain constant during the motion, $f_{cf}$ is the rf frequency, $R$ is the average orbit radius, $\nu_x$ and $\nu_y$ are the betatron tunes and $\epsilon_{nx}$ and $\epsilon_{ny}$ are the normalized emittances of the beam in the $x$ and $y$ planes respectively. $I_0 = 4\pi \epsilon_0 mc^3 / q$ is the characteristic current and for proton, $I_0 = 31$ MA. The second term in both the envelope equations (5.6) and (5.7) represent the acceleration effect and prime denotes the differentiation with respect to $s$. Integrals, $G1$ and $G2$ in eqs. (5.6) and (5.7) can be expressed in terms of the elliptic integrals of the first kind $F(\alpha, p)$ and second kind $E(\alpha, p)$. The form of these integrals depends on the relative magnitude of the envelope functions. For the ellipsoid with $X < Y < Z$, the integrals $G1$ and $G2$ have the form:
\[ G_1 \left( \frac{Y}{X'}, \frac{Z}{X'} \right) = -\frac{2X^3E(\alpha, p)}{(Y^2 - X^2)\sqrt{(Z^2 - X^2)}} + \frac{2X^2Y}{Z(Y^2 - X^2)} \]  
(5.9)

\[ G_2 \left( \frac{X}{Y'}, \frac{Z}{Y'} \right) = 2Y^3\sqrt{(Z^2 - X^2)} \left[ \frac{E(\alpha, p)}{(Y^2 - X^2)} - \frac{F(\alpha, p)}{(Z^2 - X^2)} \right] - \frac{2XY^2}{Z(Y^2 - X^2)} \]  
(5.10)

where \( \alpha, p \) and the integrals \( F(\alpha, p) \) and \( E(\alpha, p) \) are given by

\[ \alpha = \sin^{-1}\left(1 - \frac{X^2}{Z^2}\right)^{1/2}, \quad p = \left(\frac{Z^2 - Y^2}{Z^2 - X^2}\right)^{1/2}, \]

\[ F(\alpha, p) = \int_0^\alpha \frac{dx}{\sqrt{(1 - p^2 \sin^2 x)}}, \quad E(\alpha, p) = \int_0^\alpha \sqrt{(1 - p^2 \sin^2 x)} \, dx. \]  
(5.11)

For oblate ellipsoid and spheroid with different relative magnitudes of \( X, Y \) and \( Z \), integrals \( G_1 \) and \( G_2 \) have been obtained using the standard integrals given in ref. [86].

The transverse envelope equations with acceleration and linear focusing forces for a long uniform continuous beam having elliptical symmetry [87] can be expressed as

\[ X'' + \frac{(\beta'\gamma')'}{\beta'\gamma'} X' + \frac{\nu_x^2}{R^2} X = \frac{4I}{I_0 \beta^3 \gamma^3} \frac{2\pi}{\Delta\phi} \frac{1}{(X + Y)} - \frac{\epsilon_{nx}^2}{\beta^2 \gamma^2 X^3} = 0 \]  
(5.12)

\[ Y'' + \frac{(\beta'\gamma')'}{\beta'\gamma'} Y' + \frac{\nu_y^2}{R^2} Y = \frac{4I}{I_0 \beta^3 \gamma^3} \frac{2\pi}{\Delta\phi} \frac{1}{(X + Y)} - \frac{\epsilon_{ny}^2}{\beta^2 \gamma^2 Y^3} = 0 \]  
(5.13)

The term \( (2\pi / \Delta\phi) \) is included with the beam current \( I \) to account for the phase acceptance in the central region. Knowing the functional dependence of the focusing strength and \( (\beta'\gamma')'(\beta'\gamma') \) along the path length \( s \), one can easily obtain the evolution of envelopes around the central ion trajectory in the median plane of the cyclotron. The space charge term couples the envelope equations, and hence plays an important role in the evolution of envelopes \( X(s) \) and \( Y(s) \), particularly when the beam current is sufficiently large. For low beam current and without acceleration there are special solutions of Eqs. (5.6) and (5.7), where \( X = X_m = \text{const.} \) and \( Y = Y_m = \text{const.} \) and are given by
These correspond to the so called matched solutions for which the beam envelope sizes preserve the initial shape throughout its path \( i.e. \) beam envelopes are straight lines. The matched solutions with space charge terms and without acceleration, can be obtained by solving the following coupled equations simultaneously

\[
\frac{\nu_s^2}{R} X - \frac{3I_c}{2I_0 \beta^2 \gamma^3 f_{y'}} \frac{1}{X^2} G_1 \left( \frac{Y}{X}, \frac{Z}{X} \right) - \frac{\varepsilon_{nx}^2}{\beta^2 \gamma^2 X^3} = 0
\]

\[
\frac{\nu_s^2}{R} Y - \frac{3I_c}{2I_0 \beta^2 \gamma^3 f_{y'}} \frac{1}{Y^2} G_2 \left( \frac{X}{Y}, \frac{Z}{Y} \right) - \frac{\varepsilon_{ny}^2}{\beta^2 \gamma^2 Y^3} = 0
\]

In the case of space charge dominated beam together with acceleration it is not possible to obtain the matched sizes. In such situation one needs to solve Eqs. (5.6) and (5.7) for beam envelopes along the path of the accelerated orbit and then to optimize the initial conditions for which the envelopes \( X(s) \) and \( Y(s) \) exhibit minimum amplitude of oscillations.

### 5.4. Numerical results and discussions

In this section we present the results of studies on behavior of the beam envelopes in the focusing channel of the 10 MeV compact cyclotron under various initial conditions of the beam parameters. We performed numerical solutions of Eqs. (5.6) and (5.7) and tried to optimize the input beam conditions in the presence of space charge to get the beam envelopes within the specified acceptance in the 10 MeV cyclotron. We have used same values for emittances equal \( 60 \pi \) mm mrad in both the planes in the present calculation. This is the typical value of the emittance one expects at the injection radius in the cyclotron after transporting and inflecting the beam by a spiral inflector using the microwave ion source. The values of \( (\beta \gamma)',(\beta \gamma) \) have been estimated from the change of energy of the accelerated particle along the path length \( s \) using the orbit integration code.
Figure 5.3(a) shows the variation of \( \frac{(\beta \gamma)'}{(\beta \gamma)} \) as a function \( s \) up to five turns. The bunch size corresponding to 30° of rf phase at the injection energy of 80 keV and rf frequency of 42 MHz, is equal to 8 mm. Therefore, we have chosen the initial value of the longitudinal semi axis of the ellipsoidal bunch equal to 4 mm. As the beam energy increases the bunch size also increases along the longitudinal direction. The variation of bunch size \( 2Z \) as a function of the path length \( s \) for five turns is shown in Fig. 5.3(b). This change of bunch size has been incorporated while solving Eqs. (5.6) and (5.7) along the accelerated orbits of the beam.

![Figure 5.3](image)

**Fig. 5.3.** The variation of (a) \( \frac{(\beta \gamma)'}{(\beta \gamma)} \) and (b) bunch size \( 2Z \) as a function path length \( s \) along the accelerated orbits up to five turns. Long ticks on the middle horizontal line indicate the turn number. There are four kicks at four acceleration gaps in each turn.

### 5.4.1. Beam envelopes in a particular orbit

To understand the beam envelopes behaviour along the accelerated orbits we have first explored the beam envelopes under various conditions at a particular orbit radius of 30 cm (it can be any radius) without including the acceleration effects in the envelope equations.
We obtained the matched beam sizes at this radius for three different values of beam current using Eqs. (5.14) to (5.16). The calculated matched beam sizes in the radial and vertical planes are 1.71 mm and 1.98 mm for 0 mA, 1.92 mm and 2.38 mm for 5 mA and 2.11 mm and 2.84 mm for 10 mA respectively as shown by the dotted lines in Fig. 5.4.

**Fig. 5.4.** Radial and vertical beam envelopes, for different initial conditions at radius 30 cm without acceleration for five revolutions and three different values of beam current. Matched beam sizes are shown by dotted lines. All solid curves represent mismatch by 0.5 mm from the matched beam size whereas dashed curves represent mismatch by 1.0 mm from the matched beam size. The values of the matched beam sizes are (a) $X_m = 1.71$ mm, $Y_m = 1.98$ mm for 0 mA, (b) $X_m = 1.92$ mm, $Y_m = 2.38$ mm for 5 mA and (c) $X_m = 2.11$ mm, $Y_m = 2.84$ mm for 10 mA.

We then studied the envelope evolution for five revolutions by changing the initial beam sizes by 0.5 mm and 1.0 mm from the matched values in all the above mentioned
three cases. Results are shown in Fig. 5.4. In the case of \( I = 0 \) mA, the patterns of the envelope oscillations in each plane are similar for the two values of displaced initial beam sizes. The number of oscillations per turn in the radial plane is equal to 2.12 which is twice the value of betatron tune \( \nu_x = 1.06 \) and follows the linear theory of envelope oscillations described in detail for solenoid and quadrupole focusing channels [44, 88]. Similarly in the vertical plane where \( \nu_y = 0.8 \), there are 1.6 oscillations per turn. These numbers of oscillations are independent of the displacement of the initial beam size from the matched value, however, the amplitudes of oscillations are different.

It is readily seen from Fig. 5.4(b) and 5.4(c) that the envelope oscillations pattern is completely different in both the planes when the space charge effect is included in the calculations. We observed the decrease in the number of oscillations per turn as the beam current is increased and a marginal increase in the number of oscillations as the beam size from the matched radius is increased. This behavior is due to the fact that with increase in the beam current there is a depression in the tune values and hence the decrease in the number of oscillations. However it is difficult to predict the exact behavior of these oscillations due to mixture of two modes of oscillations because of different values of betatron tunes in the two transverse planes.

We have also studied the envelope behaviors by increasing and decreasing the beam sizes from the matched beam size in one plane and keeping the beam size fixed in the other plane equal to the matched size. In the case of \( I = 0 \) mA, as expected, we did not observe any change in the envelope of \( y \) plane due to mismatch in the \( x \) plane and vice versa. However, with beam current, a mismatch in \( x \) plane not only produced oscillations in the \( x \) plane but also produced a small ripples in the \( y \) plane and vice versa. We observed this effect in the case of 5 mA and 10 mA beam current which is due to the fact that space charge term couples the two transverse motions. The behavior of envelopes in
$x$ and $y$ planes around the matched beam sizes (dotted line) under various conditions is shown in Fig. 5.5 for 5 mA beam current.

**Fig. 5.5.** Radial and vertical beam envelopes without acceleration, for different initial conditions at radius 30 cm for five revolutions at $I = 5$ mA beam current. Dotted lines represent the matched envelope sizes $X_m = 1.92$ mm and $Y_m = 2.38$ mm. The beam envelopes are for (a) $X = X_m - 0.5$ mm, $Y = Y_m + 0.5$ mm, (b) $X = X_m - 0.5$ mm, $Y = Y_m - 0.5$ mm, (c) $X = X_m - 0.5$ mm, $Y = Y_m$, (d) $X = X_m + 0.5$ mm, $Y = Y_m$.

In Fig. 5.5(a) the initial beam sizes in $x$ plane is reduced by 0.5 mm whereas in $y$ plane it is increased by 0.5 mm from the corresponding matched beam sizes. In this case the envelope oscillations start initially with so called “out of phase” mode, quickly change to “in phase” mode and then again to “out of phase” mode and so on. This mixed mode of oscillation is due to the fact that betatron tunes in both planes are different and hence the
number of oscillations per turn. Fig. 5.5(b), in which the initial beam sizes in both $x$ and $y$ planes are reduced by 0.5 mm from the matched beam sizes, shows the almost identical behavior where the initial oscillations start first with “in phase” mode and keep on changing between the two modes along the path length. In Fig. 5.5(c) and 5.5(d) we have shown the behavior of envelope oscillations where the initial beam sizes in $y$ plane is kept equal to matched beam sizes and in $x$ plane the beam size is decreased by 0.5 mm in one case and increased by 0.5 mm in other case from the matched sizes. Here we observe that the induced envelope oscillations in $y$ plane are completely different in both cases.

### 5.4.2. Beam envelopes along accelerated orbit

Now we will discuss about the behaviour of the beam envelopes along the accelerated orbits in the 10 MeV cyclotron. Since there is a wide variation of betatron tunes with radius in a cyclotron as shown in Fig. 5.1, the calculated matched beam sizes at different orbit radii are also different. Figure 5.6 shows the variation of matched beam sizes $X_m$ and $Y_m$ in the radial and vertical planes respectively as a function of the orbit radius for various values of the beam current. The increase in the matched beam sizes with beam current in both the cases is due to the depression in tune values with beam current. Since the matched beam sizes at different orbit radii are not same, it is not possible to find out a unique matched beam sizes at the injection radius. The matching of the beam size at one radius automatically becomes an un-matched beam size at other radii and hence produce oscillations as well as growth in the beam envelopes. It is well known that amount of beam current that can be transported through a focusing channel is a maximum when the beam is perfectly matched to the acceptance of the channel. It is not possible to obtain such condition in a cyclotron focusing channel. Therefore, one needs to optimize the initial
beam conditions in both the planes at the injection radius to reduce the amplitude growth and envelope oscillations as minimum as possible throughout the focusing channel.

Fig. 5.6. Variation of matched beam envelope sizes $X_m$ and $Y_m$ as a function of orbit radius for various values of beam current.

The beam envelopes in the two transverse planes as a function of distance $s$ along the accelerated orbit for beam current $I = 0$ mA are shown in Fig. 5.7(a). The initial beam sizes used here are the matched beam sizes $X_m = 1.92$ mm and $Y_m = 3.26$ mm at the injection radius (7.05 cm). We see that due to the acceleration there is a large growth in the amplitude of the envelope oscillations in both the planes together with a distinct slow modulation on the radial beam envelope amplitude. It is clear that these initial conditions are not at all suitable. In order to reduce the amplitude of oscillations we studied the behavior of the envelopes by changing the input beam conditions. Figure 5.7(b) shows the optimized envelopes after adjusting the initial beam sizes as well as orientations of the
phase ellipses in both the planes. There is a substantial reduction in the envelope amplitude with these initial conditions. The pattern of modulation on the amplitude is also altered with the reduction in the frequency of modulation.

**Fig. 5.7.** Beam envelopes along the path of the accelerated orbits up to final radius for \( I = 0 \) mA. (a) The initial beam sizes are the matched beam sizes \( X_m = 1.92 \) mm and \( Y_m = 3.26 \) mm at the injection radius (7.05cm) in both the planes. (b) Envelopes obtained after optimization of initial beam conditions to yield minimum amplitude of oscillations in the beam envelopes. Ticks on the horizontal central line indicate the number of turns.

The behavior of beam envelopes for 5 mA is shown in **Fig. 5.8.** The input conditions of the beam in **Fig. 5.8(a)** are the same as that of optimized input conditions of \( I = 0 \) mA. As we see these input conditions produce more amplitude of oscillations in the beam envelopes. At different values of beam current the pattern of oscillations are different. The optimized beam envelopes are shown in **Fig. 5.8(b)** after further adjusting the initial beam conditions at the injection radius. **Figure 5.9** shows the phase ellipses in the radial and
vertical directions used as initial conditions for beam current $I = 0$ mA and $I = 5$ mA. It is to be pointed out here that orientations of the phase ellipses and beam sizes at the injection are very crucial parameters and need to be adjusted properly each time, if the beam current is changed. For different values of the beam current, the orientation of the phase ellipses are different in both the planes to yield minimum growth in the amplitude of beam envelope oscillations.

![Graph showing beam envelopes](image)

Fig. 5.8. Radial and vertical beam envelopes along the path of the accelerated orbit up to final radius for $I = 5$ mA. (a) The initial beam sizes are the optimized sizes with $I = 0$ mA; $X_0 = 2.4$ mm, $X'_0 = -7$ mrad in the radial plane and $Y_0 = 2.6$ mm, $Y'_0 = -10$ mrad in the vertical plane at the injection radius. (b) Envelopes obtained after optimization of initial beam conditions to yield minimum oscillation in the envelopes ($X_0 = 3.7$ mm, $X'_0 = -27$ mrad, $Y_0 = 4.8$ mm, $Y'_0 = -32$ mrad).

A comparison of the behavior of envelopes of uniform ellipsoidal bunch with that of a uniform continuous beam obtained by solving Eqs. (5.12) and (5.13) shows the patterns of
oscillations almost similar. However, the growth in the amplitude is less in the case of a uniform cylindrical beam. This kind of result is expected because for the same value of beam current in a given phase width, the charge density and hence the space charge defocusing force is more in the case of ellipsoidal bunch. Since ellipsoidal bunch is more close to the laboratory beam, we can say that the uniform cylindrical beam underestimates the space charge effects.

Fig. 5.9. Input phase ellipses in $x$ and $y$ planes for (a) matched envelope sizes at injection radius for $I = 0$ mA (Fig. 5.7(a)), (b) the optimized envelopes with acceleration and $I = 0$ mA (Fig. 5.7(b)), (c) the optimized envelopes with acceleration and $I = 5$ mA (Fig. 5.8(b)).

5.4.3. Estimation of limiting current

We have also estimated the maximum transverse limiting current that can be transported through the focussing channel of the 10 MeV cyclotron within 6 mm half
aperture. Figure 5.10(a) shows the radial and vertical beam envelopes along the accelerated orbits up to 21 turns with optimized initial conditions for ellipsoidal beam. In the case of ellipsoidal bunch the limiting current is approximately 7 mA. The limiting current in the case of uniform continuous beam is slightly higher \(i.e.\) 8.2 mA. These limiting currents can be increased if we remove the restriction on the aperture sizes from 6 mm. We have also carried out optimization of the beam envelope by varying the normalized beam emittances from 0.5 to 1.5 \(\pi\) mm mrad and found that the limiting current reduces slightly with the increase in the beam emittance. At normalized emittances of 0.5, 0.7, 1.0, 1.2, 1.5 \(\pi\) mm mrad the limiting currents, whose beam envelopes remain within 6 mm in both planes, are 7.2 mA, 7 mA, 6.6 mA, 6.3 mA and 6.1 mA respectively.

Fig. 5.10. Radial (\(X\)) and vertical (\(Y\)) beam envelopes along the accelerated orbits up to 10 MeV. (a) uniform ellipsoidal bunched beam with initial conditions \(X_0 = 4.9\) mm, \(X'_0 = -30\) mrad, \(Y_0 = 5.3\) mm, \(Y'_0 = -35\) mrad and (b) envelopes when the betatron tunes are scaled of by 1.15 times with same initial conditions as in (a).
The vertical blow up of the beam near the injection caused by the small tune value is the main factor responsible for the limiting current. In our design, the chosen maximum height of the dee from the median plane is equal to 15 mm. Therefore, a 5 mA beam current can be comfortably injected and accelerated in the present design of the cyclotron.

We believe that slow modulation on the radial beam envelope ($X$) is due to the betatron tune $\nu_x$, which is very close to one at the lower radii. A scaling of $\nu_x$ either up or down from the present value reduces the amplitude as well as these oscillations considerably. Fig 5.10(b) shows the envelope patterns where the $\nu_x$ and $\nu_y$ are scaled up by a factor of 1.15. In a compact cyclotron, one can easily manipulate the values of vertical tune $\nu_y$ by changing the flutter and the shape of sectors. It is not possible to change the profile of the radial tune $\nu_x$ as desired in an isochronous cyclotron because it follows the profile of relativistic term $\gamma$ as the energy of the beam increases. This value remains very close to unity at lower radii where the beam energy is not sufficiently relativistic. The best way to control the beam envelope oscillations and amplitude growth is then to optimize the initial beam conditions properly.

Till now we have discussed the beam optimization in the focusing channel of the cyclotron and results indicate that for maximum beam transmission one needs converging initial phase ellipses in both the transverse planes. However, in reality it is difficult to get such initial conditions because the spiral inflector which is used to inject the beam in the central region puts sever restrictions on the beam emittance due to inter-plane coupling effect. Generally the optimum phase ellipse in the vertical direction at the inflector exit is diverging. The optimized phase ellipses at the exit of the inflector are shown in Fig. 5.11(a) and 5.11(b) for equal input emittances of 45 $\pi$ mm mrad in both the planes. It can be readily seen that there is a substantial growth of emittances in both the planes and the
orientations of phase ellipses are also different. The behavior of beam envelopes in $x$ and $y$ planes with these input conditions of the beam are shown in Fig. 5.11(c). Here we see that there is a slight reduction in the beam current ($I_{lin} = 5.3 \text{ mA}$) within the specified 6 mm half aperture sizes in both the planes.

Fig. 5.11. Optimized phase ellipses at the exit of spiral inflector in the central region of 10 MeV cyclotron and beam envelopes in $x$ and $y$ planes along the accelerated orbits with these input beam conditions.

5.5. Summary

The behavior of transverse beam oscillations has been studied in a compact cyclotron along the accelerated orbits for space charge dominated beam. The emphasis has been on the determination of the input beam conditions at the injection to reduce the oscillations and amplitude growth in both the transverse planes. The most important conclusion that
can be extracted from this study is the critical dependence of the input beam conditions on the injected beam current. Our results suggest that for different values of beam current, the beam sizes and orientations of the phase ellipses are different in both the transverse planes that lead to the minimum amplitude growth in envelope oscillations.

The evolution of beam bunch in the combined electric and magnetic fields of a cyclotron is very complex due to the coupling between the radial and longitudinal motions which leads to the rotation of the bunch. In this work we have presented a simplified model and results so obtained present a good insight about the behavior of the transverse motion of the space charge dominated beam with uniform density distribution in a compact cyclotron.