CHAPTER 5

GRADED SINGULARITY AND GOLDIE DIMENSION

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Chapter 5

Graded Singularity and Goldie Dimension

In this chapter we study various aspects of graded singularity and Goldie dimension. This chapter has three sections. The first section contains the preliminary concepts. The second section deals with the basic results. The third section is devoted to the main results.

5.1 Preliminaries:

Throughout our discussion R is a graded ring where G is an ordered abelian group and M a graded R-module. We start with the following definitions.

Definition 5.1.1: A graded R module M is said to have finite Goldie dimension if it does not contain a direct sum of an infinite number of non-zero graded submodules. Throughout this chapter dim M will mean Goldie dimension of M.

Definition 5.1.2: A graded uniform module means a graded module that does not contain a direct sum of two non-zero graded submodules.

It is to be noted that if M has finite Goldie dimension then every graded submodule of M contains a graded uniform submodule.

Definition 5.1.3: The graded singular submodule \( Z_g(M) \) can also be defined as follows

\[
Z_g(M) = \{ x \in M |Ix=0, \text{ where } I \text{ is an essential left graded ideal in } R \}.
\]

It is clear that \( Z_g(M) \subseteq Z_R(M) \), where \( Z_R(M) \) denotes the singular submodule of M. On the other hand if G is an ordered group then \( Z_g(M) = Z_R(M) \).
**Definition 5.1.4**: Let $M$ be a graded $R$-module. Then graded rank of $M$, denoted by $\rho_M$, is defined as the Goldie dimension of $M$ over $R$.

**Definition 5.1.5**: Let $M$ be a graded $R$-module. We define the graded singularity rank or $s$-rank of $M$, denoted by $s-\rho_R(M)$, as the Goldie dimension of $M/Z_R(M)$ over $R$.

**Definition 5.1.6**: An element $c \in h(R)$ is called right regular homogeneous if $r(c) = 0$ and left regular homogeneous if $l(c) = 0$. $c$ is called regular homogeneous if $r(c) = l(c) = 0$. We write $C_G(0)$ for the set of all regular homogeneous elements of $R$.

**Definition 5.1.7**: A homogeneous element $x$ of $M$ is called a torsion element if $xc = 0$ for some regular homogeneous element $c$ of positive degree.

**Definition 5.1.8**: The graded torsion submodule of $M$, denoted by $T_g(M)$, is defined as follows

$$T_g(M) = \{x \in M/cx = 0, \text{for some regular homogeneous element } c, \deg c > 0\}$$

**Definition 5.1.9** [44]: A graded ring having finite Goldie dimension and satisfying the ascending chain condition on graded left annihilators is called a graded Goldie ring. If $R$ is trivially graded then it refers to the usual notion of Goldie ring.

**Definition 5.1.10**: A homogeneous element $a$ of $R$ is said to be graded uniform if the ideal $aR$ is graded uniform.

**Definition 5.1.11** [44]: A ring $R$ is called an almost strongly graded ring if there exist a decomposition sum (not necessarily direct sum):

$$R = \sum_{\sigma \in G} R_{\sigma}$$

(as additive groups) into additive subgroups $R_{\sigma}, \sigma \in G$ such that
If the decomposition sum is direct \( R \) is called the strongly graded ring.

Note: Let \( M \) be an \( R \)-module and \( N \) be an \( R_e \)-submodule of \( M \). We denote by 
\[
N^* = nR_gN \quad \text{where} \quad g \in G.
\]
Then \( N^* \) is the largest an \( R \)-submodule of \( M \) contained in \( N \).

Next we discuss some preliminary results needed for the sequel.

**Lemma 5.1.12 [46]**: Let \( R \) be a graded ring, \( S \) a multiplicatively closed subset of \( R \) consisting of homogeneous elements. Then \( S \) satisfies Ore's conditions on the left if and only if:

(i) If \( rs = 0 \) with \( r \in h(R) \), \( s \in S \) then there is an \( s' \in S \) such that \( s'r = 0 \).

(ii) For any \( r \in h(R) \), \( s \in S \) there exist \( r' \in h(R) \), \( s' \in S \) such that \( s'r = r's \).

**Proof**: Clearly, the Ore conditions for \( S \) imply (i) and (ii).

Conversely, let \( s \in S \), \( r \in R \) where \( r = r_{\sigma_1} + \ldots + r_{\sigma_n} \) with \( r_{\sigma_i} \in h(R) \), \( \sigma_i \in G \). If \( n = 1 \), then left Ore conditions for \( r \), \( s \) clearly hold because (i) and (ii) hold. Now, we proceed by induction on \( n \), supposing Ore conditions for all \( r \in R \) having the homogeneous decomposition of length less than \( n \). By assumption there is \( r' \in R \), \( s' \in S \) such that \( s'(r_{\sigma_1} + \ldots + r_{\sigma_{n-1}}) = r's \) and \( s^2 \in S \), \( r^2 \in R \) such that \( s^2 r_{\sigma_n} = r^2 s \).

By (i) and (ii) there exist \( u \in S \), \( v \in h(R) \) such that \( us^1 = vs^2 = t \) and \( t \in S \) is non-zero. Then \( tr = (ur^1 + vr^2)s \) and hence (ii) also holds if \( r \) has a decomposition of length \( n \).
Furthermore, if $a \sigma = 0$ with $a = a_{\sigma_1} + \ldots + a_{\sigma_n}$, with $a_{\sigma_1} \in \h(R)$, then $a_{\sigma_n} s = 0$ and 
$(a_{\sigma_1} + \ldots + a_{\sigma_n}) s = 0$. By induction hypothesis we may pick $t_i \in S$ such that 
$t_i (a_{\sigma_1} + \ldots + a_{\sigma_{n-1}}) s = 0$. Now from $t_i a_{\sigma_n} s = 0$ and (i) it follows that there is a $t_2 \in S$ 
such that $t_2 t_i a_{\sigma_n} s = 0$. Putting $s' = t_2 t_i \in S$ we see that $s' a = 0$.

**Lemma 5.1.13 [44]:** Let $R$ be a graded ring satisfying ascending chain condition for graded left annihilators, then left singular radical of $R$ is nilpotent.

**Proof:** Let $J$ be the left singular radical of $R$. It is clear that $J$ is a graded ideal. Since we have an ascending chain as follows

$$
\text{ann}(J) \subset \text{ann}(J^2) \subset \ldots \subset \text{ann}(J^n) \subset \ldots
$$

and hence by the hypothesis we must have $\text{ann}(J^n) = \text{ann}(J^{n+1})$, for some $n \in \mathbb{N}$.

Let $J^{n+1} \neq 0$. Then for some $a \in \h(R)$, $a J^n \neq 0$ and $a$ can be chosen in such a way that 
\text{ann}(a) is maximal with respect to this property. If $b \in J \cap \h(R)$

then we have $\text{ann}(b) \cap R a \neq 0$ as $\text{ann}(b)$ is essential left ideal of $R$.

Thus there exists $c \in \h(R)$ such that $ca \neq 0$ and $cab = 0$. So $\text{ann}(a) \subset \text{ann}(ab)$.

thus we have $ab J^n = 0$. Consequently since $J$ is graded, a $J^{n+1} = 0$ which is a contradiction. So we must have $J^{n+1} = 0$.

**Lemma 5.1.14 [44]:** Let $R$ be a semisimple graded ring satisfying ascending chain conditions on graded left annihilators. If $I$ is a graded ideal such that every
homogeneous element of I is nilpotent, then \( I = 0 \).

**Proof:** Let I be a nonzero graded ideal of R and we consider a homogeneous element \( a \neq 0 \) such that \( \text{ann}_R(a) \) is maximal amongst left annihilators of nonzero homogeneous elements of I.

Let \( b \in h(R) \) such that \( ab \neq 0 \). By our assumption, there exists a \( t > 0 \) such that \( (ab)^t = 0 \) and \( (ab)^{t+1} \neq 0 \). From the inclusion \( \text{ann}_R(a) \subset \text{ann}_R(ab)^{t+1} \), with \( ab \in h(I) \). So we have \( \text{ann}_R(a) = \text{ann}_R(ab)^{t+1} \).

Thus \( aba = 0 \) and hence \( aRa = 0 \) which contradicts that R is a semiprime ring. So we must have \( I = 0 \).

*From lemma 5.1.15 to theorem 5.1.18 are mentioned in [3]. Here we have given the detailed proof of the same for completeness of the chapter.*

**Lemma 5.1.15:** Let G be an abelian group and R be a G-graded prime, graded Goldie ring. Then any nonzero graded ideal I of R contains a non-nilpotent homogeneous element.

**Proof:** Let I be a nonzero graded ideal of R. Let \( x \) be a homogeneous element of I such that \( x^n \neq 0 \), for some \( n = 1, 2, \ldots \) [using lemma 5.1.11].

Let us consider \( m \) such that \( m \deg s + \deg x > 0 \).

Then \( a = s^nx \) is a homogeneous element of I of positive degree which is not nilpotent.
**Lemma 5.1.16:** Let $G$ be an abelian group and $R$ be a $G$-graded prime, graded Goldie ring. Then graded singular ideal of $R$, i.e., $Z(R)$, is nilpotent and hence zero.

**Proof:** Directly follows from lemma 5.1.10.

**Lemma 5.1.17:** Let $G$ be an abelian group and $R$ be a $G$-graded, graded prime, graded Goldie ring. Let $a \in R$ be a homogeneous element such that $aR$ is graded uniform. Then $\text{ann}(a)$ is maximal among annihilators of nonzero homogeneous elements of $R$.

**Proof:** Let $\text{ann}(a) \supseteq J = \text{ann}(b)$, for some homogeneous element $b$ in $R$.

Now $aJ \neq 0$ and since $aR$ is graded uniform, we have $aJ$ is graded essential in $aR$.

Thus $aR/aJ$ is graded singular $R$-module and $aR/aJ \cong R/J \cong bR$. So $bR$ is graded singular and hence by lemma 5.1.13, we must have $b = 0$.

**Theorem 5.1.18:** Let $G$ be an abelian group and $R$ be a $G$-graded prime, graded Goldie ring. Then any essential graded ideal $I$ of $R$ contains a homogeneous regular element.

**Proof:** Using lemma 5.1.12 and the Goldie hypothesis, there exists a non-nilpotent, graded uniform element $a_1 \in I$. By induction, suppose that we have found non-nilpotent, graded uniform elements $a_1, \ldots, a_m \in I$ such that $a_i \notin \bigcap_{1 \leq j \leq i} \text{ann}(a_j)$ for $1 \leq i \leq m$.

If $X = \bigcap_{1 \leq j \leq m} \text{ann}(a_j) \neq 0$, then $I \cap X \neq 0$ and so using lemma 5.1.12, there exists a non-nilpotent, graded uniform element $a_{m+1} \in I \cap X$. Since $a_i \in \text{ann}(a_j) = \text{ann}(a_j^2)$ for $i > j$, it is...
easy to see that $\sum_{i \geq 1} a_i R$ is an internal direct sum of nonzero graded ideals. As the
Goldie dimension of $R$ is finite, there exists some index $n$ with $\bigcap_{i \leq n} \operatorname{ann}(a_i) = 0$. Since $R$
is graded prime and the $a_i$ are not nilpotent, we must have $a_1^2 R a_2^2 R \ldots a_n^2 R \neq 0$.
Thus we may find homogeneous elements $s_1, \ldots, s_n \in R$ such that $a_1^2 s_1 a_2^2 s_2 \ldots a_n^2 s_n \neq 0$. Again using lemma 5.1.12, there exists a homogeneous element $s_n$ such that $c = a_1^2 s_1 a_2^2 s_2 \ldots a_n^2 s_n$ is not nilpotent.
We take $d_i = (a_1 s_1 a_2^2 s_2 \ldots a_{n-1}^2 s_{n-1} a_n^2 s_n) (a_1^2 s_1 \ldots a_2 a_{n-1}^2 s_{n-1} a_n) $ for $i = 1, 2, \ldots, n$
It is to be noted that $d_i$ are subwords of $c^2$ and hence are nonzero.
So by lemma 5.1.14 we have $\operatorname{ann}(d_i) = \operatorname{ann}(a_i)$, for each $i$. Also the sum $\sum_{i = 1}^n d_i R$ is
direct as $d_i R \subseteq a_i R$. Hence $\operatorname{ann}(d_1 + \ldots + d_n) = \bigcap_{i = 1}^n \operatorname{ann}(d_i) = \bigcap_{i = 1}^n \operatorname{ann}(a_i) = 0$. Here it is to
be noted that each $d_i$ is a reordering of the letters of $c$ and so, as $G$ is abelian
$\deg (d_i) = \deg (c)$.
So $d_1 + \ldots + d_n$ is a homogeneous regular element in $I$.

**Lemma 5.1.19** [45]: Let $R$ be a strongly graded ring where order of $G = n$. If $R$ has no n-torsion then $Z_{Re}(M) = Z_R(M)$. [$Z_{Re}(M)$ is singular submodule as an $R_e$-module]

**Proof**: It is obvious that $Z_{Re}(M) \subseteq Z_R(M)$. To prove the other inclusion we take
any element $x \in Z_R(M)$. Then $I = \operatorname{ann}_R(x)$ is an essential ideal of $R$ and hence $I$ is
essential as a $R_e$-module. So we have $J = I \cap R_e$ as an essential left ideal of $R_e$.

Clearly $J x = 0$ implies that $x \in Z_{Re}(M)$.
5.2 Basic results

Following the same line of proof as in lemma 1.1,[16] we get

Lemma 5.2.1: Let $M$ be graded right $R$-module. Let $m$ be a nonzero homogeneous element of $M$. Let $N$ be a graded essential submodule of $M$, then there is an essential graded right ideal $I$ of $R$ such that $mI \neq 0$ and $mI \subseteq N$.

Proof: We take $I = \{ r_g \in R_g / m r_g \in N, \text{for some } g \in G \}$.

We claim that $I = \bigoplus I_g$ is a graded ideal of $R$.

Let $r \in I$ such that $r = r_{g_1} + \ldots + r_{g_k}$ with $r_{g_i} \neq 0$.

We want to show that $r_{g_i} \in I$, for each $i$ where $i \in \{1, \ldots, k\}$

Now, $r \in I$

$\Rightarrow mr \in N$

$\Rightarrow m (r_{g_1} + \ldots + r_{g_k}) \in N$

$\Rightarrow mr_{g_1} + \ldots + mr_{g_k} \in N$

$\Rightarrow mr_{g_i} \in N$, since $N$ is graded submodule.

$\Rightarrow r_{g_i} \in I_g$

$\Rightarrow r_{g_i} \in I$

So $I$ is a graded ideal of $R$.

Since $N$ is graded essential submodule of $M$, we must have

$mR \cap N \neq 0$. 

For some \( r \in R \) we must have \( 0 \neq mr \in N \). Thus we have \( r \in I \) so that \( mr \neq 0 \) and hence \( mI \neq 0 \).

Let \( L \) be a nonzero graded ideal of \( R \). We need to show that \( I \cap L \neq 0 \).

If \( mL = 0 \). Then \( mL \subseteq N \).

Then from definition of \( I \), we must have \( L \subseteq I \) and hence we obtain the required result.

Let \( mL \neq 0 \). Since \( N \) is graded essential submodule of \( M \), we must have \( mL \cap N \neq 0 \). For some \( 0 \neq x \in L \) we have \( 0 \neq mx \in N \) which implies that \( x \in I \).

Thus we have \( I \cap N \neq 0 \).

**Lemma 5.2.2**: Let \( R \) be a graded ring with finite Goldie dimension and \( c \) be a regular homogeneous element of \( R \). Then \( cR \) is graded essential ideal of \( R \).

**Proof**: It is clear that \( cR \) is graded ideal of \( R \).

To prove the essentiality of \( cR \), first we assume a nonzero graded ideal \( I \) of \( R \).

Suppose, if possible \( I \cap cR = 0 \). Then the sum \( I + cI + c^2I + \ldots \) is direct and since \( R \) has finite Goldie dimension, there exists some index \( n \) such that \( c^n I = 0 \) which implies that \( I \subseteq \text{ann}(c^n) \) and hence \( I \subseteq \text{ann}(c) = 0 \) as \( c \) is regular homogeneous element. So we have \( I = 0 \), which is a contradiction. So \( I \cap cR \neq 0 \) implies that \( cR \) is graded essential ideal of \( R \).
Lemma 5.2.3: The torsion submodule $T_g(M)$ is a graded submodule of $M$

Proof: Let $x, y \in T_g(M)$

Then $cx = 0$, $dy = 0$, for some $c, d$ regular homogeneous elements of positive degree.

Now $Rc \leq_e R$, $Rd \leq_e R$

Therefore, $Rc \cap Rd \leq_e R$.

This implies $Rc \cap Rd$ contains a regular homogeneous element $e$, say, with $\text{deg } e > 0$.

Then $e(x - y) = ex - ey = 0$ which implies $x - y \in T_g(M)$

Also, let $r \in h(R)$. Then $\exists \ r' \in h(R), s' \in C_0(0)$ such that $s'r = r'c$

Now $(s'r)x = (r'c)x = r' (cx) = 0 \Rightarrow s'(rx) = 0$ implies $rx \in T_g(M)$.

Thus $T_g(M)$ is a submodule of $M$.

Next we show $T_g(M)$ is a graded submodule of $M$.

Let $0 \neq x \in T_g(M)$ such that $x = x_{\sigma_1} + \ldots + x_{\sigma_n}$

where $x_{\sigma_i} \in M_{\sigma_i}$, $x_{\sigma_i} \neq 0$ and $\sigma_1 < \ldots < \sigma_n$.

Then for some regular homogeneous element $c$ we have

$cx = 0$ which implies that $c \in \text{Ann}_R(x)$

Thus $c \in \text{Ann}_R(x_{\sigma_1})$ [since $\text{Ann}_R(x) \subset \text{Ann}_R(x_{\sigma_1})$].
This gives $c \sigma_n x_n = 0$

Hence $x_\sigma n \in T_g(M)$

For the element $x - x_\sigma_n$, repeating the same argument we obtain by induction that $x \sigma_1, \ldots, x \sigma_n \in T_g(M)$.

Note: $T_g(M)$ is called graded torsion submodule of $M$.

As in the ungraded case proposition 1.3 [27] we get the following:

**Lemma 5.2.4:** Let $Y$ be a graded submodule of a graded $R$-module $X$ and $Y_1$ be graded submodule of $X$ such that $Y_1$ is maximal with respect to the property $Y \cap Y_1 = 0$. Then $Y \oplus Y_1 \leq X$. [Such submodules $Y_1$ always exist by Zom's Lemma and we call $Y_1$ graded relative complement for $Y$]

**Lemma 5.2.5:** Let $M$ be a non-zero right graded $R$-module. If $M$ has finite Goldie dimension then each non-zero graded submodule of $M$ contains a graded uniform submodule and there is a finite number of graded uniform submodules of $M$ whose sum is direct and is a graded essential submodule of $M$.

**Proof:** As in ungraded case [proposition 1.9, [16]].

**Lemma 5.2.6:** Let $N$ be a graded $R$-submodule of $M$ and $N \leq M$.

Then $\dim N = \dim M$.

**Proof:** Let $\dim N = k$. 

Then \( \exists \) uniform submodules \( u_1, u_2, \ldots, u_k \) such that \( u_1 + u_2 + \ldots + u_k \) is direct and \( u_1 + u_2 + \ldots + u_k \leq N \)

Since \( N \leq M \) so \( u_1 + u_2 + \ldots + u_k \leq N \leq M \) \( \Rightarrow u_1 + u_2 + \ldots + u_k \leq M \)

Thus, \( \exists \) graded uniform submodules \( u_1, u_2, \ldots, u_k \) of \( M \) such that \( u_1 + u_2 + \ldots + u_k \) is direct and \( u_1 + u_2 + \ldots + u_k \leq M \)

\( \Rightarrow \) \( \dim M = k \). So \( \dim N = \dim M \).

5.3 Main Results:

We now prove our main results.

**Theorem 5.3.1:** Let \( G \) be an abelian group and \( R \) be a \( G \)-graded prime, graded Goldie ring. Let \( M \) be a graded \( R \)-module. Then the graded torsion submodule of \( M \) coincides with the graded singular submodule of \( M \), that is \( T_g(M) = Z_g(M) \).

**Proof:** Let \( x \in Z_g(M) \).

Then \( Kx = 0, K \leq R \), for some essential graded ideal \( K \) of \( R \).

Then \( K \) contains a regular homogeneous element, \( c \) (say).

Then \( c \cdot x = 0 \Rightarrow x \in T_g(M) \Rightarrow Z_g(M) \subseteq T_g(M) \).

Conversely, let \( x \in T_g(M) \)

Then, \( cx = 0 \), where \( c \) is a regular homogeneous element.

Thus we have \( \text{ann } c = 0 \).

So \( Rc \) is an essential graded left ideal of \( R \).
Thus $cx=0 \Rightarrow Rcx=0$. Also $Rc \subseteq R \Rightarrow x \in Z_g(M)$. Hence $T_g(M) \subseteq Z_g(M)$. This proves the result.

**Theorem 5.3.2:** Let $R$ be a semiprime graded Goldie ring and $X$ is a graded $R$-module of finite Goldie dimension with a graded submodule $Y$ such that $X/Y$ is torsion-free, then $\dim X = \dim Y + \dim(X/Y)$.

**Proof:** $X$ is a graded $R$-module and $Y$ is a graded $R$-submodule of $X$.

Then by lemma 5.2.4,3 a graded submodule $Y_1$ of $X$ such that $Y \cap Y_1 = 0$ and $Y \oplus Y_1 \leq X$.

Let $x \in h(X)$ such that $x \notin h(Y)$.

Then $\exists$ an essential graded ideal $I$ such that $xI \subseteq Y \oplus Y_1$. [Lemma 5.2.1]

Now $I$ contains a regular homogeneous element $c$ [lemma 5.1.18]

Then $xc \in Y \oplus Y_1$

If $xc \in Y$, then $xc \in T(X/Y)$, which contradicts $X/Y$ is torsion-free.

Let $S$ be a graded submodule of $X$ such that $S \cap (Y \oplus Y_1) = Y$

Let $x \in S$ such that $x \notin Y$.

Then $xc \in S$ such that $xc \notin Y$.

Also, $xc \in Y \oplus Y_1$

Thus $xc \in S \cap (Y \oplus Y_1) = Y$, a contradiction.
Hence $Y = S$

Now $Y \oplus Y_1 \leq X \Rightarrow (Y \oplus Y_1)/Y \leq X/Y$

This implies

$$\dim(X/Y) = \dim(Y_1) = \dim(Y_1/Y) = \dim(Y_1)/0 = \dim(Y_1)$$

Since $Y$ and $Y_1$ are graded $R$-submodules of finite Goldie dimension,

So, $\dim(Y \oplus Y_1) = \dim Y + \dim Y_1$

Also $Y \oplus Y_1 \leq X \Rightarrow \dim X = \dim Y \oplus Y_1 = \dim Y + \dim Y_1 = \dim Y + \dim(X/Y)$

**Theorem 5.3.3**: Let $R$ be a semiprime graded Goldie ring. Let $M$ be a graded right $R$-module with finite Goldie dimension and $K$, a graded submodule of $M$.

Then $\rho_R(M) = \rho_R(K) + \rho_R(M/K)$

**Proof**: Let $L$ be a graded submodule of $M$ such that $K \subseteq L$ and $L/K = Z_R(M/K)$

Clearly $L/K$ is a graded $R$-module.

We first show that $K + Z_R(M) \leq L$

Let $0 \neq y \in L$

Now $y + K \in Z_R(M/K)$

$\Rightarrow y + K \in T_R(M/K)$

$\Rightarrow d(y + K) = K$ for some regular homogeneous element $d$

$\Rightarrow dy + K = K$
$\Rightarrow dy \in K$

Assume $dy \neq 0, dy \in Ry, dy \in K \Rightarrow dy \in K + Z g(M)$

Thus $0 \neq dy \in Ry \cap K + Z g(M)$.

This implies $K + Z g(M) \leq L$

If $dy = 0$ then $y \in T g(M) \Rightarrow y \in Z g(M) \Rightarrow y \in K + Z g(M)$

Also $y \in Ry$. Thus $y \in Ry \cap K + Z g(M)$. This implies $Ry \cap K + Z g(M) \neq 0$

As a consequence, we get $K + Z R(M) \leq L$. This gives $\dim L = \dim (K + Z R(M))$

$M/L \cong (M/K)/(L/K) = (M/K)/(Z R(M/K))$ so that $M/L$ is torsion free.

Therefore, $\dim (M/L) = \dim M - \dim L$

As a consequence we get

$s-R(M/K) = \dim (M/K)/(Z R(M/K)) = \dim (M/K)/(L/K) = \dim (M/L) = \dim M - \dim L$

$= \dim M - \dim Z R(M) - (\dim L - \dim Z R(M))$

$s-R(K) = \dim (K/Z R(K))$

$= \dim (K/K \cap Z R(M)) = \dim ((K + Z R(M))/Z R(M)) = \dim ((K + Z R(M))/Z R(M)) - \dim Z R(M)$

[since $(K + Z R(M))/Z R(M)$ is torsion free]

Thus $s-R(K) = \dim L - \dim Z R(M)$ and

hence $s-R(M/K) = \dim M - \dim Z R(M) - s-R(K)$

$= \dim M/Z R(M) - s-R(M) = s-R(M) - s-R(K)$.
This proves \( s-\rho_R(M) = s-\rho_R(K) + s-\rho_R(M/K) \).

As in ungraded case - theorem2.5,[50] we get the following result:

**Theorem 5.3.4**: Let \( M \) be a graded \( R \)-module with finite Goldie dimension. \( N_1 \) and \( N_2 \) are two graded \( R \)-submodules of \( M \) such that \( N=N_1 \cap N_2 \) is graded relative complement in \( M \). Then \( \dim N_1 + \dim N_2 = \dim (N_1 + N_2) + \dim N \).

**Corollary 5.3.5**: Let \( M \) be a graded \( R \)-module with finite Goldie dimension. \( N_1 \) and \( N_2 \) are two graded \( R \)-submodules of \( M \) strictly containing \( Z_R(M) \) such that \( N=N_1 \cap N_2 \) is graded relative complement in \( M \).

Then \( s-\rho_R(N_1 + N_2 ) = s-\rho_R(N_1 ) + s-\rho_R(N_2 ) - s-\rho_R(N ) \).

**Proof**: From definition of \( s \)-rank of a graded \( R \)-module, we must have

\[
s-\rho_R(N_1 + N_2 ) = \dim (N_1 + N_2 / Z_R(M ))
\]

\[
= \dim (N_1 + N_2 ) - \dim (Z_R(M )) , \text{[using Theorem 5.3.2]}
\]

\[
= \dim N_1 + \dim N_2 - \dim N - \dim (Z_R(M )) , \text{[using Theorem5.3.4]}
\]

\[
= \dim N_1 - \dim (Z_R(M )) + \dim N_2 - \dim (Z_R(M )) - \dim N + \dim (Z_R(M )) ,
\]

\[
= \dim (N_1 / Z_R(M )) + \dim (N_2 / Z_R(M )) - \dim (N / Z_R(M ))
\]

\[
= s-\rho_R(N_1 ) + s-\rho_R(N_2 ) - s-\rho_R(N ).
\]

**Theorem 5.3.6**: Let \( G \) be an abelian group and \( R \) be a \( G \)-graded prime, graded Goldie ring and \( M \), a graded \( R \)-module. Then
(i) If $M$ is torsion, then $s-\rho_R(M)=0$.

(ii) If $M$ is torsion-free, then $s-\rho_R(M)=\rho_R(M)$.

**Proof:** (i) As $M$ is torsion, we have $T_g(M)=M$.

Then $s-\rho_R(M)=\dim (M/ Z_R(M))$

$$= \dim (M/ T_g(M)), \text{[using theorem 5.3.1]}$$

$$= 0$$

(ii) As $M$ is torsion-free, we have $T_g(M)=0$.

Then $s-\rho_R(M)=\dim (M/ Z_R(M))$

$$= \dim (M/ T_g(M)), \text{[using theorem 5.3.1]}$$

$$= \dim (M/(0))=\dim(M)=\rho_R(M)$.

**Theorem 5.3.7:** Let $\omega(G)=n$ and $R$ be strongly $G$-graded ring without $n$-torsion.

Then $s-\rho_{Re}(M)=s-\rho_R(M)$

**Proof:** As $R$ has no $n$-torsion, by lemma we have $Z_{Re}(M)=Z_R(M)$. Let us suppose that $s-\rho_{Re}(M)=m$, then there exist non-zero $R_e$-submodules $A_1/Z_{Re}(M)$ such that the sum $A_1/Z_{Re}(M)+\ldots+A_m/Z_{Re}(M)$ is direct modulo $Z_{Re}(M)$.

This implies that the sum $A_1/Z_R(M)+\ldots+A_m/Z_R(M)$ is direct modulo $Z_R(M)$.

Thus we have $\dim M/Z_R(M)=m$ which gives that $s-\rho_R(M)=m$.

This proves the required result.
In the lemma 5.3.8 and the theorem 5.3.9, we consider $R$ to be almost strongly graded ring graded by a finite group $G$.

**Lemma 5.3.8:** Let $M$ be an $R$-module. Then $M$ contains an $R_e$-submodule $Z_{Re}(N) = N_e$ (say) such that $N_e$ is maximal with respect to $N_e = Z_R(M)$.

**Proof:** We set $Z_{Re}(N) = N_e$ such that $N_e$ is maximal with respect to $N_e = Z_R(M)$ for each $i$.

We claim that $(\bigcup_{i \in I} N_e) = Z_R(M)$.

If $(\bigcup_{i \in I} N_e) \neq Z_R(M)$ then there exist an $x \in (\bigcup_{i \in I} N_e)^\ast$ such that $\text{ann}(x)$ is not essential in $R$. We take $x = r_g x_{e_i}$ where $r_g \in R_e$ and $x_{e_i} \in N_e$. For some nonzero graded ideal $I_e$ of $R_e$ we must have a nonzero graded ideal $nR_g I_e$ of $R$ such that $nR_g I_e \cap \text{ann}(x) \neq 0 \Rightarrow nR_g I_e \cap \text{ann}(r_g x_{e_i}) \neq 0 \Rightarrow I_e \cap \text{ann}(r_g x_{e_i}) \neq 0$

$\Rightarrow r_g x_{e_i} \notin N_{e_i}$ which is a contradiction. So we must have $(\bigcup_{i \in I} N_e)^\ast = Z_R(M)$.

Now by Zorn's lemma, we can conclude that there exists an $R_e$-submodule $Z_{Re}(N) = N_e$ (say) such that $N_e$ is maximal with respect to $N_e = Z_R(M)$.

**Theorem 5.3.9:** Let $M$ be an $R$-module such that $s_{-\rho_{Re}(M)} = m$.

If $Z_{Re}(M) = N$ (say) is an $R_e$-submodule such that $N$ is maximal with respect to $N = Z_R(M)$, then $s_{-\rho_{Re}(M)} \leq m$.

**Proof:** Let $A_1 / Z_{Re}(M), \ldots, A_d / Z_{Re}(M)$ be $R_e$-submodules of $M / Z_{Re}(M)$ such
that the sum $A_i / Z_{Re}(M) + \ldots + A_t / Z_{Re}(M)$ is direct modulo $Z_{Re}(M)$. This gives
the sum $A_1 / Z_R(M) + \ldots + A_t / Z_R(M)$ is direct modulo $Z_R(M)$.

Now, for each $i$, $1 \leq i \leq t$, we have $A_i^* / Z_{R}(M) \neq Z_R(M)$.

If $t > m$, then for some $i$, we must have

$\sum_{j<i} A_j^* / Z_R(M) \cap A_i^* / Z_R(M) \neq Z_R(M)$

which gives

$\left( \sum_{j<i} A_j^* / Z_R(M) \cap A_i^* / Z_R(M) \right)^* \neq Z_R(M)$

as

$\left( \sum_{j<i} A_j^* / Z_R(M) \cap A_i^* / Z_R(M) \right)^* \supseteq \sum_{j<i} A_j^* / Z_R(M) \cap A_i^* / Z_R(M)$.

So we have $\sum_{j<i} A_j^* / Z_R(M) \cap A_i^* / Z_R(M) \neq Z_{Re}(M)$ and hence we obtain

$\sum_{j<i} A_j^* / Z_{Re}(M) \cap A_i^* / Z_{Re}(M) \neq Z_{Re}(M)$ which contradicts our assumption.

Therefore $s - \rho_{Re}(M) \leq m.$

In [24], for any ring $R$, the authors have considered $R/Z(R)$ as a right $R$-
module and have defined $G(R)$ to be the right ideal of $R$ containing $Z(R)$
considering $G(R)/Z(R)=Z(R/Z(R)).$ As in ungraded case, we have seen a similar
result for a graded ring also.

**Theorem 5.3.10:** For every graded ring $R, G(R)$ is a graded ideal of $R$ such that

$Z_R(R/G(R))=0$ and $Z(R/G(R))=0.$

**Proof:** Let $G(R) \subseteq R$ such that $a \in G(R)$ is homogeneous element of positive degree.

Then $a + Z(R) \in G(R)/Z(R)=Z_R(R/Z(R))$
Thus \( \text{ann}_R(a + Z(R)) \subseteq R \).

Also \( \text{ann}_R(a + Z(R)) = \{ b \in h(R) / ab + Z(R) = Z(R) \} = \{ b \in h(R) / ab \in Z(R) \} \)

For any \( r \in h(R) \), \( \text{ann}_R(a + Z(R)) \subseteq \text{ann}_R(ra + Z(R)) \),

for if \( x \in \text{ann}_R(a + Z(R)) \), and \( x \) is homogeneous,

Then we have \( (a + Z(R))x = Z(R) \) which implies \( ax + Z(R) = Z(R) \).

This gives \( ax \in Z(R) \)

Thus \( ra \in rZ(R) = Z(R) \) gives \( ra + Z(R) = Z(R) \).

This implies that \( (ra + Z(R))x = Z(R) \)

Therefore \( x \in \text{ann}_R(ra + Z(R)) \)

So \( ra \in G(R) \)

This proves that \( G(R) \) is an ideal of \( R \).

Next we show \( G(R) \) is, in fact, a graded ideal .

Let \( 0 \neq x \in G(R) \) such that \( x = x_{\sigma_1} + \ldots + x_{\sigma_n} \)

where \( x_{\sigma_i} \in R \), \( x_{\sigma} \neq 0 \) and \( \sigma_1 < \ldots < \sigma_n \).

Then \( \text{ann}_R(x + Z(R)) = \{ y \in h(R) / yx \in Z(R) \} \subseteq R \)

Also we have \( \text{ann}_R(x + Z(R)) \subseteq \text{ann}_R(x_{\sigma_n} + Z(R)) \)

which implies that \( \text{ann}_R(x_{\sigma_n} + Z(R)) \subseteq R \)

So we claim that \( x_{\sigma_n} \in G(R) \).
Suppose $x^\sigma_n \notin G(R)$.

Then $x^\sigma_n \notin Z(R)$. So $0 \neq x^\sigma_n + Z(R) \in R/Z(R)$ such that $\text{ann}_R(x^\sigma_n + Z(R)) \subseteq R$.

So $0 \neq x^\sigma_n + Z(R) \in Z_R(R/Z(R))$

which is a contradiction since $x^\sigma_n + Z(R) \notin G(R)/Z(R)$.

So $x^\sigma_n \in G(R)$.

Repeating the same argument, for the element $x - x^\sigma_n$ by induction we obtain that

$x^\sigma_1, \ldots, x^\sigma_n \in G(R)$.

Let $a + G(R) \in Z_R(R/G(R))$ and $0 \neq H$ be a graded right ideal of $R$.

If $H \cap Z(R) \neq 0$ and $H \cap Z(R) \subseteq Z(R)$, then $a(H \cap Z(R)) \subseteq aZ(R) Z(R)$.

So $\exists 0 \neq b \in H$ such that $ab \in Z(R)$

Thus $b \in \{ b \in h(R)/ab \in Z(R) \} = \text{ann}_R(a + Z(R))$

This implies $\text{ann}_R(a + Z(R)) \cap H \neq 0$

Suppose $H \cap Z(R) = 0$ and let $0 \neq b \in H$ such that $ab \in G(R)$, so $b \notin Z(R)$

So there exists a nonzero right graded ideal $I$ of $R$ such that $\text{ann}_R(b) \cap I = 0$

Also $ab \in G(R)$ and $G(R)/Z(R) = Z_R(R/Z(R))$

Therefore $\text{ann}_R(ab + Z(R)) \cap I \neq 0$. So there exists $0 \neq c \in I$ such that $c \in \text{ann}_R(ab + Z(R)) \Rightarrow (ab + Z(R))c = Z(R) \Rightarrow abc \in Z(R)$
Now $0 \neq b \in H$, $0 \neq c \in I \implies 0 \neq bc \in H$

Now $abc \in Z(R) \implies abc+Z(R) = Z(R) \implies (a+Z(R))bc = Z(R) \implies bc \in \text{ann}_R(a+Z(R))$

Thus $\text{ann}_R(a+Z(R)) \cap H \neq 0$

So, $\text{ann}_R(a+Z(R)) \leq c R$

The above implies $a+Z(R) \in Z_R(R/ Z(R)) = G(R)/ Z(R)$.

Thus $a \in G(R)$

This implies $Z_R(R/ G(R)) = 0$

Let $a+G(R) \in Z(R/ G(R))$

Then $\text{ann}_{R/G(R)}(a+G(R)) \leq c R/G(R)$

Let $\text{ann}_{R/G(R)}(a+G(R)) = J/G(R)$

Then $J/G(R) \leq c R/G(R) \implies J \leq c R \implies \text{ann}_{R/G(R)}(a+G(R)) \leq c R$

$\implies (a+G(R)) \in Z_R(R/ G(R)) = 0$

$\implies (a+G(R)) = 0$

Hence $Z(R/ G(R)) = 0$