Chapter – 5

Holographic quintessence model of dark energy in
Bianchi type-I anisotropic universe

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5.1 Holographic quintessence model of dark energy in Bianchi type-I anisotropic universe

5.1 Introduction

Observations of distant type-Ia supernovae and cosmic microwave background suggest that our universe has entered a phase of accelerated expansion in the recent past [9, 171]. This is attributed to the contribution of an unknown component, dubbed dark energy, which has negative pressure and makes up about three quarters of the total cosmic density. The simplest candidate for dark energy is the cosmological constant ($\Lambda$) with equation of state parameter $\omega = -1$ since it fits the observational data well, but it needs to be extremely fine-tuned to satisfy the present value of dark energy [51]. To solve cosmological constant problem at present epoch, $\Lambda$ with a dynamical character is preferred over a constant $\Lambda$, especially a time dependent $\Lambda$ which has decreased slowly from its large initial value to reach its present small value [80]. To further investigate the properties of dark energy, many dynamical dark energy models have been proposed, such as quintessence [173], phantom [153], tachyon [26, 85, 178], k-essence [34], dilatonic ghost condensate [116], quartessence [60] and so forth. The cosmic viscosity is also an effective quantity as caused mainly by the non-perfect cosmic contents interactions and may play a role as dark energy candidate causing the observed acceleration of the universe [84, 108, 193].

The holographic principle is considered as another alternative to the solution of the dark energy problem. This principle was first put forward by G. 't Hooft [64] in the context of black hole physics. According to the holographic principle, the entropy of a system scales not with its volume, but with its surface area. In the cosmological context,
Fischler and Susskind [190] have proposed a new version of the holographic principle, viz. at any time during cosmological evolution, the gravitational entropy within a closed surface should not be always larger than the particle entropy that passes through the past light-cone of that surface. In the context of the dark energy problem, though the holographic principle proposes a relation between the holographic dark energy density $\rho_A$ and the Hubble parameter $H$ as $\rho_A = H^2$, it does not contribute to the present accelerated expansion of the universe. In [102], Granda and Oliveros proposed a holographic density of the form $\rho_A \approx \alpha H^2 + \beta \dot{H}$ where $H$ is the Hubble parameter and $\alpha, \beta$ are constants which must satisfy the restrictions imposed by the current observational data. They showed that this new model of dark energy represents the accelerated expansion of the universe and is consistent with the current observational data. In [101], they study the correspondence between the quintessence, tachyon, k-essence and dilaton dark energy models with this holographic dark energy model in the flat FRW universe.

But there is a cosmological view that the universe might have been anisotropic and also inhomogeneous in the very early era and that in the course of its evolution these characteristics might have been wiped out under the action of some processes or mechanism, resulting in an isotropic and homogeneous universe. Spatially homogeneous and anisotropic cosmological models play a significant role in the description of large scale behaviour of the universe and such models have been widely studied by many authors in search of a relativistic picture of the early universe. Anisotropic Bianchi type-I, Bianchi type-III, Bianchi type-V dark energy models with the usual perfect fluid have also been extensively studied in the literature [19, 20, 23, 133, 162]. So it will be interesting to study the evolution of holographic dark energy in an anisotropic model like Bianchi type-I.
In this Chapter we consider the holographic dark energy model in the axially symmetric Bianchi type-I model to investigate the correspondence with quintessence models of the universe. We obtain the Equation of state parameter for the holographic dark energy model in axially symmetric Bianchi type-I in section 2. In section 3, we solve the field equations by considering the deceleration parameter to be a constant. The correspondence between the holographic dark energy with quintessence is shown in section 4. We conclude this chapter in section 5.

5.2 Holographic dark energy model in Bianchi type-I universe

The metric for axially symmetric Bianchi type I is
\[ ds^2 = dt^2 - a_1^2(t)dx_1^2 - a_2^2(t)dx_2^2 - a_2^2(t)dx_3^2 \]  

Einstein's field equation is given by
\[ R_{ij} - \frac{1}{2}g_{ij}R = -(T_{ij} + \tau_{ij}) \]  

where \( R_{ij} \) is the Ricci tensor.

The energy momentum tensor for matter and the holographic dark energy are defined as
\[ T_{ij} = \rho_m u_i u_j \] and \[ \tau_{ij} = (\rho_\Lambda + p_\Lambda)u_i u_j + g_{ij}p_\Lambda \]  

where \( \rho_m, \rho_\Lambda \) are the energy densities of matter and the holographic dark energy and \( p_\Lambda \) is the pressure of the holographic dark energy.

The field equations for the axially symmetric Bianchi type-I metric are
\[ \frac{\dot{a}_1}{a_1} \frac{a_2}{a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 = \rho_m + \rho_\Lambda \]  

\[ \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_1}{a_1} \frac{\dot{a}_2}{a_2} = -p_\Lambda \]  

\[ 2\frac{\dot{a}_2}{a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 = -p_\Lambda \]  

The average scale factor \( a \) and the average Hubble's parameter are defined as
\[ a = \left(a_1 a_2^2\right)^{\frac{1}{3}} \]
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\[ H = \frac{1}{3} \left( \frac{a_1}{a_1} + 2 \frac{a_2}{a_2} \right) \]  

(8)

The scalar expansion \( \theta \), deceleration parameter \( q \), shear scalar \( \sigma^2 \) and the average anisotropy parameter \( A_m \) are defined by

\[ \theta = \frac{a_1}{a_1} + 2 \frac{a_2}{a_2} \]  

(9)

\[ q = -\frac{a_1 a_2}{a^2} = -1 - \frac{H}{H^2} \]  

(10)

\[ \sigma^2 = \frac{1}{2} \left( \sum_{i=1}^{3} H_i^2 - \frac{1}{3} \theta^2 \right) \]  

(11)

\[ A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2 \]  

(12)

where \( \Delta H_i = H_i - H \) \((i = 1, 2, 3)\)

The holographic dark energy density is given by

\[ \rho_\Lambda = 3 \left( \alpha H^2 + \beta \dot{H} \right) \]  

(13)

i.e. \( \rho_\Lambda = 3 \left( \alpha H^2 + \beta \dot{H} \right) \) with \( M_P^{-2} = 8\pi G = 1 \) \([101]\).

Combining (4)-(6) the continuity equation can be obtained as

\[ \dot{\rho}_m + \frac{a_1}{a_1} + 2 \frac{a_2}{a_2} \rho_m + \dot{\rho}_\Lambda + \left( \frac{a_1}{a_1} + 2 \frac{a_2}{a_2} \right) (\rho_\Lambda + p_\Lambda) = 0 \]  

(14)

The continuity equation of the matter is

\[ \dot{\rho}_m + \frac{a_1}{a_1} + 2 \frac{a_2}{a_2} \rho_m = 0 \]  

(15)

The continuity equation of the holographic dark energy is

\[ \dot{\rho}_\Lambda + \left( \frac{a_1}{a_1} + 2 \frac{a_2}{a_2} \right) (\rho_\Lambda + p_\Lambda) = 0 \]  

(16)

The barotropic equation of state is

\[ p_\Lambda = \omega_\Lambda \rho_\Lambda \]  

(17)

Using equation (13) and (17) in equation (16) we get

\[ \omega_\Lambda = -1 - \frac{2\alpha H H + \beta \dot{H}}{3H(\alpha H^2 + \beta \dot{H})} \]  

(18)
5.3 Cosmological Solutions

Since the right hand sides of (5) and (6) are identical, we have

$$\frac{a_1}{a_2} + \frac{a_2}{a_1} + \frac{a_1}{a_2} + \frac{a_2}{a_1} = 2 \frac{d_a}{a_1} \left( \frac{d_a}{a_2} \right)^2$$

(19)

Solving equation (19) and using (7) we get,

$$\frac{a_1}{a_2} = D_2 \exp \left( D_1 \int a^{-3} dt \right)$$

(20)

The metric functions, therefore can be read as

$$a_1(t) = D_2^2 a \exp \left( 2D_3 \int a^{-3} dt \right)$$

(21)

$$a_2(t) = D_2^{-1} a \exp \left( -D_3 \int a^{-3} dt \right)$$

(22)

where $D_1$, $D_2$ and $D_3$ are constants and $3D_2 = D_1$

To solve these field equations we assume the average Hubble parameter $H$ to be related to the average scale factor $a$, by the relation following Berman [128]

$$H = ma^{-\frac{1}{m}}$$

(23)

where $m > 0$ is a constant.

Solving the equation (23) we get

$$a(t) = (t + c_1)^m$$

(24)

Using (24) in (21) and (22) we get the metric functions as

$$a_1(t) = D_2^2 (t + c_1)^m \exp \left[ \frac{2D_3}{1-3m} (t + c_1)^{1-3m} \right]$$

(25)

$$a_2(t) = D_2^{-1} (t + c_1)^m \exp \left[ \frac{-D_3}{1-3m} (t + c_1)^{1-3m} \right]$$

(26)

This gives a constant value of the deceleration parameter

$$q = -1 + \frac{1}{m}$$

(27)

Since recent observational data indicates that the universe is accelerating and the value of deceleration parameter lies somewhere in the range $-1 < q < 0$, so we have $m > 1$ for the accelerating universe.
The Hubble parameter, the scalar expansion, shear scalar and the average anisotropy parameter are therefore

\[ H = m(t + c_1)^{-1} \]  \hspace{1cm} (28)
\[ \theta = 3m(t + c_1)^{-1} \]  \hspace{1cm} (29)
\[ \sigma^2 = 3D_3^2(t + c_1)^{-6m} \]  \hspace{1cm} (30)
\[ A_m = 2D_3^2m^{-2}(t + c_1)^{2-6m} \]  \hspace{1cm} (31)

The matter density parameter \( \Omega_m \) and holographic dark energy density parameter \( \Omega_\Lambda \) are given by

\[ \Omega_m = \frac{\rho_m}{3H^2} \text{ and } \Omega_\Lambda = \frac{\rho_\Lambda}{3H^2} \]  \hspace{1cm} (32)

Using (24)-(26), (28) and (32) in (4) we get

\[ \Omega_m + \Omega_\Lambda = 1 - D_3^2m^{-2}(t + c_1)^{2-6m} \]  \hspace{1cm} (33)

Equation (33) shows that the sum of the energy density parameters approaches 1 at late times. So at late times the universe becomes flat. Therefore for sufficiently large time, this model predicts that the anisotropy of the universe will damp out and universe will become isotropic. This result also shows that in the early universe i.e. during the radiation and matter dominated era the universe was anisotropic and the universe approaches to isotropy as dark energy starts to dominate the energy density of the universe. This result is totally different from Pradhan et al.’s result [23] where they found in their model that the universe does not approach isotropy through its whole evolution.

Using (28) in equation (18) we get

\[ \omega_\Lambda = -1 + \frac{2}{3m} \]  \hspace{1cm} (34)

This shows that for \( m > 1 \), we have \(-1 < \omega_\Lambda < -\frac{1}{3}\). In this case the holographic dark energy EOS behaves like quintessence.
5.4 Correspondence between the holographic and quintessence scalar field model of dark energy

To establish the correspondence between the holographic dark energy with quintessence dark energy models, we compare the EOS and the dark energy density for the corresponding models of dark energy.

The action for the quintessence scalar $\varphi$ is given by

$$ s = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right] $$

(35)

The energy density and pressure for the quintessence scalar field are represented by

$$ \rho_\varphi = \frac{1}{2} \dot{\varphi}^2 + V(\varphi) $$

(36)

$$ p_\varphi = \frac{1}{2} \dot{\varphi}^2 - V(\varphi) $$

(37)

where $V(\varphi)$ is the quintessence potential.

The equation of state for the scalar field is given by

$$ \omega_\varphi = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)} $$

(38)

For the accelerated expansion of the universe, the equation of state parameter for quintessence must be less than $-\frac{1}{3}$.

From (34) and (38) we have

$$ -1 + \frac{2}{3m} = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)} $$

(39)

Also comparing equations (13) and (36) one can write

$$ \rho_H = 3 \left( \alpha H^2 + \beta \dot{H} \right) = \frac{1}{2} \dot{\varphi}^2 + V(\varphi) $$

(40)

From equations (38) and (39), the kinetic energy term and the quintessence potential in power law form can be obtained as follows

$$ \varphi - \varphi_0 = \sqrt{2} (m\alpha - \beta) \ln \left( \frac{t + c_1}{t_0 + c_1} \right) $$

(41)

$$ V(\varphi) = (3m - 1)(m\alpha - \beta)(t + c_1)^{-2} $$

(42)
Using (41) in (42) we get the potential in the exponential form as

\[
V(\varphi) = (3m - 1)(ma - \beta)(t_0 + c_1)^{-2} \exp \left( -\frac{1}{\sqrt{2(ma-\beta)}} (\varphi - \varphi_0) \right) \tag{43}
\]

For \( m > 1 \), this type of exponential potential can produce an accelerated expansion of the universe [51]. Thus one can establish a correspondence between the holographic dark energy and quintessence scalar field, and describe holographic dark energy by making use of quintessence.

The dynamics and the potentials of tachyon and k-essence scalar field models of the anisotropic Bianchi type-I universe can be obtained in the same way.

5.5 Conclusion

Though the present day universe appears to be isotropic, there is no evidence that the early universe was also of the same type. There is a cosmological view that the universe might be inhomogeneous and anisotropic in the very early era and that in the course of its evolution these characteristic have been wiped out under the action of some process or mechanism, and finally an isotropic and homogeneous universe had resulted. Observational data also suggest that dark energy is responsible for gearing up the universe some five billion years ago. But at that time the universe need not to be isotropic. So in this chapter we assume the universe to be anisotropic and consider a homogeneous axially symmetric Bianchi type-I universe filled with matter and holographic dark energy. Assuming the deceleration parameter to be a constant, we have obtained an exact solution of Einstein's field equations. Under certain conditions the solution describes the accelerated expansion of the universe. The EOS parameter of the holographic dark energy also behaves like quintessence EOS. Using these results we
have established a correspondence between the holographic dark energy model with the quintessence scalar field dark energy models in the Bianchi type-I universe. Quintessence potential and the dynamics of the quintessence scalar field are reconstructed for this anisotropic accelerating model of the universe. Our result shows that the universe was anisotropic in the early stage and at the late time dynamics anisotropy of the universe damps out and the present day universe becomes isotropic as suggested by different observational data.