CHAPTER - V

SOME FIXED POINT THEOREMS IN HAUSDORFF AND SEMI-HAUSDORFF SPACES
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5.1. Ray [62,63], Jaggi [31], Iseki [29] and Achari [1] have discussed a number of results related to the following theorem of Edelstein [18].

Theorem 5.1.1. Let \( (X,d) \) be a metric space, \( T \) be a contractive self mapping of \( X \). If for some \( x \in X \) the sequence of iterates \( \{T^n x\} \) has a convergent subsequence \( \{T^n x\} \) converging to a point \( x_0 \in X \), then
\[
    x_0 = \lim_{n \to \infty} \{T^n x\}
\]
is a unique fixed point.

These results were further generalized by Bohre and Namdeo [5], Popa [60] in Hausdorff spaces as stated in section 1.6 of Chapter I.

Chugh and Rani [9] improved the above results in the form of the following :

Theorem 5.1.2. Let \( T \) be a continuous mapping of Hausdorff space \( X \) into itself, and let \( f \) be a continuous mapping of \( X \times X \) into the non-negative reals such that
\[
    f(x,y) \neq 0 \quad \text{for all } x \neq y
\]

\[
    \alpha f(y, Ty) [1 + f(x, Tx)]
\]

\[
    f(Tx, Ty) \leq \frac{\alpha f(y, Ty) [1 + f(x, Tx)]}{1 + f(x, x)} + \beta f(x, y)
\]

for all \( x \neq y \), \( \alpha, \beta > 0 \), \( \alpha + \beta < 1 \)

\[
    \frac{1 + f(x, x)}{1 + f(x, y)}
\]

If for some \( x_0 \in X \) the sequence \( x_n = \{T^n x_0\} \) has a convergent subsequence, then \( T \) has a unique fixed point.

The aim of this section is to obtain following fixed point theorems in Hausdorff spaces by generalizing the results of above mentioned authors.

Let \( R^+ \) denotes the set of non-negative real numbers and \( N \) be the set of natural numbers. Let \( H \) denote the family of all functions \( h \) such that \( h : (R^+) \to R^+ \) and \( h \) is non-decreasing in each co-ordinate variable. Also let \( g(t) = h(t,t) \), where \( g : R^+ \to R^+ \).

**Theorem 5.1.3.** Let \( T \) be a continuous mapping of a Hausdorff space \( X \) into itself and let \( f \) be a continuous mapping of \( X \times X \) into \( R^+ \) such that

\[
\begin{align*}
(5.4) & \quad f(x,y) \neq 0 \text{ for all } x \neq y . \\
(5.5) & \quad \text{There is an } h \in H \text{ such that for all } x, y \in X, x \neq y \\
& \quad f(y,Ty) \leq h \left( \frac{f(x,y)}{1+f(x,y)} , f(x,y) \right)
\end{align*}
\]

where \( h \) satisfies \( h(t,t) < t \) for all \( t > 0 \)

\[
(5.6) \quad f(x,y) \geq \frac{[1+f(x,x)]}{1+f(x,y)} f(y,y), \text{ for all } x, y \in X .
\]

If for some \( x_0 \in X \), the sequence \( \{x_n\} = \{T^n x_0\} \) has a convergent subsequence, then \( T \) has a unique fixed point.

**Proof.** By condition (5.5), we have

\[
\begin{align*}
f(x_1,x_2) &= f(Tx_0, Tx_1) \\
&\leq h \left( \frac{f(x_1,Tx_1) [1+f(x_0,Tx_0)]}{1+f(x_0,x_1)} , f(x_0,x_1) \right) \\
&= h \left( \frac{f(x_1,Tx_1) [1+f(x_0,x_1)]}{1+f(x_0,x_1)} , f(x_0,x_1) \right)
\end{align*}
\]
= h(f(x_1, x_2), f(x_0, x_1)).

If f(x_0, x_1) < f(x_1, x_2) then f(x_1, x_2) \leq h(f(x_1, x_2), f(x_1, x_2)) < f(x_1, x_2)

a contradiction. Hence f(x_0, x_1) \geq f(x_1, x_2).

Repeating the above argument, we obtain f(x_0, x_1) \geq f(x_1, x_2) \geq f(x_2, x_3) \geq ....

Thus \{f(x_n, x_{n+1})\} converges with all its subsequences to some real number u being a monotonically decreasing sequence of positive real numbers. Again \{x_n\} has a convergent subsequence \{x_{n_k}\} in X which converges to some x in X. The continuity of T gives

\[ T x = T(\lim x_{n_k}) = \lim T x_{n_k+1}; \quad T^2 x = T(T x) = T(x_{n_{k+1}}) = \lim x_{n_{k+2}}. \]

Now to prove that x is a fixed point of T, we have

\[ f(x, Tx) = f(\lim x_{n_k}, \lim x_{n_{k+1}}) = \lim f(x_{n_k}, x_{n_{k+1}}) = u \]

\[ = \lim f(x_{n_{k+1}}, x_{n_{k+2}}) = f(\lim x_{n_{k+1}}, \lim x_{n_{k+2}}) = f(T x, T^2 x) \quad ....(5.7) \]

If x \neq Tx, then condition (5.5) gives

\[ f(T x, T^2 x)[1+f(x,T x)] \]

\[ f(T x, T^2 x) \leq h \left( \frac{1+f(x,T x)}{1+f(x,T x)}, f(x,T x) \right) \]

\[ = h(f(T x, T^2 x), f(x,T x)) \]

\[ = h(f(T x, T^2 x), f(T x, T^2 x)) \]

\[ < f(T x, T^2 x) \]

which contradicts (5.7). Thus x = Tx.

To prove the uniqueness, let y \neq x be another fixed point of T. Then from (5.5), we obtain

\[ f(x, y) = f(T x, T y) \]

\[ \quad f(y, T y) [1+f(x,T x)] \]

\[ \leq h\left( \frac{1+f(x,y)}{1+f(x,y)}, f(x,y) \right) \]
Now condition (5.6) implies that

\[ f(x,y) \leq h(f(x,y), f(x,y)) < f(x,y) \]

a contradiction. Hence \( x = y \). This completes the proof of theorem.

**Corollary 5.1.1.** Let \( T \) be a continuous mapping of Hausdorff space \( X \) into itself. Let \( f \) be a continuous mapping of \( X \times X \) into \( \mathbb{R}^+ \) satisfying (5.4), (5.6) and

\[
\frac{f(y,Ty) + f(x,Tx)}{1 + f(x,y)} \leq \alpha_1(x,y) f(x,y) + \alpha_2(x,y) \]

for all \( x, y \in X, x \neq y \), where the mappings \( \alpha_i : X \times X \to [0,1] \) have the property

\[
\sum_{i=1}^{2} \alpha_i(x,y) \leq 1
\]

If for some \( x_0 \in X \), the sequence \( \{x_n\} = \{T^n x_0\} \) has a convergent subsequence, then \( T \) has unique fixed point.

**Remark 5.1.1.** If condition 5.5 in theorem 5.1.3 is replaced by

\[
\frac{\alpha f(y,Ty) + f(x,Tx)}{1 + f(x,y)} \leq \alpha f(x,y) + \beta f(x,y)
\]

for all \( x \neq y; \alpha, \beta > 0 ; \alpha + \beta < 1 \), then also \( T \) has a unique fixed point.

We extend the above result for two mappings in the form of the following :

**Theorem 5.1.4.** Let \( T_1 \) and \( T_2 \) be two continuous mapping of a Hausdorff space \( X \) into itself. Let \( f \) be a continuous mapping of \( X \times X \) into \( \mathbb{R}^+ \) satisfying (5.4), (5.6) and

\[
f(x,y) = f(y,x) \quad \text{for all } x, y \in X
\]
(5.9) There is an $h \in H$ such that for all $x, y \in X$: $x \neq y$

$$f(T_1x, T_2y) \leq h \left( \frac{[1+f(x, T_1x)]f(y, T_2y)}{1+f(x, y)} \right)$$

where $h$ satisfies $h(t, t) < t$ for all $t > 0$.

If for some $x_0 \in X$ the sequence $\{x_n\}$ where $T_1x_{2n} = x_{2n+1}$ and $T_2x_{2n+1} = x_{2n+2}$ for $n = 0, 1, 2, \ldots$ has a convergent subsequence of the type $\{x_{(2p+1)n}\}$ where $p \in \mathbb{N}$ is fixed and $n \in \mathbb{N}$, then $T_1$ and $T_2$ have a unique common fixed point.

**Proof.** Condition (5.9) gives

$$f(x_1, x_2) = f(T_1x_0, T_2x_1)$$

$$\leq h \left( \frac{[1+f(x_0, T_1x_0)]f(x_1, T_2x_1)}{1+f(x_0, x_1)} \right)$$

$$= h \left( \frac{[1+f(x_0, x_1)]}{f(x_1, x_2), f(x_0, x_1)} \right)$$

$$= h(f(x_1, x_2), f(x_0, x_1))$$

which implies (as in the proof of theorem 5.1.3) $f(x_0, x_1) \geq f(x_1, x_2)$ and therefore repetition of the above argument gives

$$f(x_0, x_1) \geq f(x_1, x_2) \geq f(x_2, x_3) \geq \ldots$$

and thus the sequence $\{f(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of positive reals and hence converges to some real number $u$.

Again $\{x_n\}$ has a subsequence $\{x_{(2p+1)n}\}$ converges to some $x$ in $X$.

Let $\{x_{(2p+1)2n'}\}$ be a subsequence of $\{x_{(2p+1)n}\}$. Continuity of $T_1$ and $T_2$, then gives

$$T_1x = T_1(\lim x_{(2p+1)2n'}) = \lim x_{(2p+1)2n'+1}$$

$$T_2T_1x = T_2(\lim x_{(2p+1)2n'+1}) = \lim x_{(2p+1)2n'+2}$$
We observe that
\[
\begin{align*}
    f(x,T|x) &= f(\lim_{n \to \infty} x_{(2p+1)2n'}, \lim_{n \to \infty} x_{(2p+1)2n'+1}) \\
    &= \lim_{n \to \infty} f(x_{(2p+1)2n'}, x_{(2p+1)2n'+1}) \\
    &= \lim_{n \to \infty} f(x_{(2p+1)2n'+1}, x_{(2p+1)2n'+2}) = f(T_1x, T_2T_1x).
\end{align*}
\]

If \( x \neq T_1x \), then using condition (5.9) we get
\[
    f(x,T|x) = f(T_1x, T_2T_1x) \leq h \left( \frac{1 + f(x,T|x)}{1 + f(x,T|x)} , f(x,T|x) \right) < f(x,T|x)
\]
a contradiction. Hence \( x = T_1x \).

Similarly, let \( \{x_{(2p+1)2n'+1}\} \) be a subsequence of \( \{x_{(2p+1)2n'}\} \). Then we obtain \( x = T_2x \). Hence \( x \) is a common fixed point of \( T_1 \) and \( T_2 \). Then (5.9) implies
\[
    f(x,y) = f(T_1x, T_2y) \leq h \left( \frac{1 + f(x,T|x)}{1 + f(x,T|x)} , f(x,y) \right) < f(x,y)
\]
which implies (as in the proof of theorem 5.1.2) \( f(x,y) < f(x,y) \) a contradiction. Hence \( x = y \) and this accomplishes the proof of theorem.

**Corollary 5.1.2.** Let \( T_1 \) and \( T_2 \) be two continuous self mappings of a Hausdorff space \( X \) and \( f \) be a continuous mapping of \( X \times X \) into \( \mathbb{R}^+ \) satisfying (5.4), (5.6), (5.8) and the following condition
\[ f(T_1x, T_2y) \leq \frac{f(x, T_1x)}{1+f(x, y)} + \alpha_1(x, y) \frac{f(y, T_2y) + \alpha_2(x, y)f(x, y)}{1+f(x, y)} \]

(5.10)

for all \( x, y \in X, \ x \neq y \), where the mappings \( \alpha_i : X \times X \to [0,1] \) are such that

\[ \sum_{i=1}^{2} \alpha_i(x, y) \leq 1. \]

If for some \( x_0 \in X \), the sequence \( \{x_n\} \) where \( T_1x_{2n} = x_{2n+1} \) and \( T_2x_{2n+1} = x_{2n+2} \) for \( n = 0, 1, 2, \ldots \), has a convergent subsequence of the type \( \{x_{2p+1,n}\} \), where \( p \in \mathbb{N} \) is fixed and \( n \in \mathbb{N} \), then \( T_1 \) and \( T_2 \) have a unique common fixed point.

**Remark 5.1.2.** If condition (5.10) is replaced by the condition

\[
\frac{\alpha f(y, T_2y)[1+f(x, T_1x)]}{1+f(x, y)} \leq f(T_1x, T_2y) + \beta f(x, y)
\]

then also \( T_1 \) and \( T_2 \) have a unique common fixed point.

The above result has been extended for sequence of mappings as under:

**Theorem 5.1.5.** Let \( T_1, T_2, \ldots, T_k \) be continuous mappings of a Hausdorff space \( X \) into itself and \( f \) be a continuous mapping of \( X \times X \) into \( \mathbb{R}^+ \) satisfying (5.4), (5.6), (5.8) and

(5.11)

there is an \( h \in H \) such that for all \( x, y \in X, \ x \neq y \)

\[
\frac{[1+f(x, T_i \cdot x)]}{1+f(x, y)} f(T_i x, T_{i+1} y) \leq h \left( \frac{f(x, T_i \cdot x)}{1+f(x, y)}, f(x, y) \right)
\]

where \( h \) satisfies \( h(t, t) < t \) for all \( t > 0 \) and \( T_{k+1} = T_1 \).

If for some \( x_0 \in X \), the sequence \( \{x_n\} \), where

\[
\begin{align*}
    x_1 &= T_1x_0, \ x_2 = T_2x_1, \ldots, x_k = T_kx_{k-1} \\
    x_{k+1} &= T_1x_k, \ x_{k+2} = T_2x_{k+1}, \ldots, x_{2k} = T_kx_{2k-1}
\end{align*}
\]

..............................................................
\[ x_{nk+1} = T_1 x_{nk}, \quad x_{nk+2} = T_2 x_{nk+1}, \ldots x_{n(n+1)k} = T_k x_{n(n+1)k}. \]

for all \( n = 0, 1, 2, \ldots \) has a convergent subsequence of the type \( \{x_{(mk+1)n}\} \)

where \( m \in \mathbb{N} \) is fixed and \( n \in \mathbb{N} \), then \( T_1, T_2, \ldots, T_k \) have a unique common fixed point.

**Proof.** Condition (5.11) gives

\[
\begin{align*}
    f(x_1, x_2) &= f(T_1 x_0, T_2 x_1) \\
    &\leq h \left( \frac{1 + f(x_0, T_1 x_0)}{1 + f(x_0, x_1)} \right) f(x_1, T_2 x_1, f(x_0, x_1)) \\
    &= h(f(x_1, x_2), f(x_0, x_1))
\end{align*}
\]

which implies

\[ f(x_1, x_2) \leq f(x_0, x_1). \]

Similarly, we have

\[ f(x_2, x_3) \leq f(x_1, x_2) \]

\[ \cdots \cdots \cdots \]

\[ f(x_{k-1}, x_k) \leq f(x_{k-2}, x_{k-1}) \]

now

\[
\begin{align*}
    f(x_k, x_{k+1}) &= f(T_k x_{k-1}, T_1 x_k) \\
    &\leq h \left( \frac{1 + f(x_{k-1}, T_k x_{k-1})}{1 + f(x_{k-1}, x_k)} \right) f(x_k, T_1 x_k, f(x_{k-1}, x_k))
\end{align*}
\]

If \( f(x_k, x_{k+1}) \geq f(x_{k-1}, x_k) \), then

\[ f(x_k, x_{k+1}) \leq g(f(x_k, x_{k+1}) < f(x_k, x_{k+1}) \]

a contradiction. Thus

\[ f(x_k, x_{k+1}) \leq f(x_{k-1}, x_k) \]

and hence in general

\[ f(x_n, x_{n+1}) \leq f(x_{n-1}, x_n), \quad n = 0, 1, 2, \ldots \]

The fact that \( T_1, T_2, \ldots, T_k \) have a unique fixed point follows as in the proof of the Theorem 5.1.4.

**Corollary 5.1.3.** Let \( T_1, T_2, \ldots, T_k \) be continuous mappings of a Hausdorff space \( X \) into itself and \( f \) be a continuous mapping of \( X \times X \) into \( \mathbb{R}^+ \).
satisfying (5.4), (5.6), (5.8) and
\[
\frac{[1+f(x,T;x)]}{1+f(x,y)}
\]
\[
f(T;x, T_{i+1} y) \leq \alpha_i(x,y) \frac{f(y,T_{i+1}y) + \alpha_2(x,y)f(x,y)}{1+f(x,y)}
\]
for all \(x, y \in X, x \neq y\), where the mappings \(\alpha_i : X \times X \rightarrow [0,1]\) have the property
\[
\sum_{i=1}^{2} \alpha_i(x,y) \leq 1.
\]

If for some \(x_0 \in X\), the sequence \(\{x_n\}\) as defined in theorem 5.1.5 has a convergent subsequence of the type \(\{x_{mk+1}\}\) where \(m \in \mathbb{N}\) is fixed and \(n \in \mathbb{N}\), then \(T_1, T_2, ..., T_k\) have a unique fixed point.

**Remark 5.1.3.** If condition (5.11) in theorem 5.1.5 is replaced by condition
\[
\frac{[1+f(x,T;x)]}{1+f(x,y)}
\]
\[
f(T;x, T_{i+1} y) \leq \alpha \frac{f(y,T_{i+1}y) + \beta f(x,y)}{1+f(x,y)}
\]
for all \(x, y \in X, x \neq y\) where \(\alpha + \beta < 1\), then also \(T_1, T_2, ..., T_k\) have a unique fixed point.

Now we present some examples to prove the validity of Theorem 5.1.3 and 5.1.4.

**Example 5.1.1.** Let \(X = \{0, 1/2\}\) and \(\mathcal{J}\) be the discrete topology on \(X\).
Then \((X, \mathcal{J})\) is a Hausdorff space.

Define \(T : X \rightarrow X\) such that \(T(0) = 0, T(1/2) = 0\). Then \(T\) is continuous on \(X\).
\(X \times X = \{(0,0), (0,1/2), (1/2,0), (1/2, 1/2)\}\) and
\(\mathcal{J} \times \mathcal{J} = \{\phi, (0,0), (0,1/2), (1/2,0), (1/2, 1/2)\}\)
are topology on \(X \times X\). Define \(f : X \times X \rightarrow \mathbb{R}^+\) such that
\[
f(x,y) = \frac{3x+6y}{4}
\]
for all \(x, y \in X\).
Then clearly $f$ is continuous on $X \times X$ and $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$h(x, y) = \frac{(x+y)}{3} \quad \text{for all } x, y \in \mathbb{R}^+$$

(a) Here $f(x, y) \neq 0$ for all $x \neq y$, thus condition (5.4) is satisfied.

To check condition (5.5), we have the following cases:

(b) For $x = 0$, $y = 0$ condition (5.5) becomes

$$f(0,0) \leq h\left( \frac{1+f(0,0)}{1+f(0,0)}, f(0,0) \right)$$

$$\Rightarrow 0 < h(0,0) = 0$$

For $x = 1/2$, $y = 1/2$ we have

$$0 = f(0,0) \leq h\left( \frac{3/8}{1 + (3/8)}, \frac{9/8}{9/8} \right) = 31/68$$

Similarly for $x = 0$, $y = 1/2$ we have $0 = f(0,0) \leq h(0, \frac{3}{8}) = 9/28$

and for $x = 1/2$, $y = 0$, we get $0 = f(0,0) \leq h(1/2, 3/8) = 1/8$

Thus condition (5.5) holds.

To check condition (5.6), we have the following cases:

(c) For $x = 0$, $y = 1/2$ we have

$$f(0, 1/2) \geq \frac{1+f(0,0)}{1+f(0,1/2)} f(1/2, 1/2)$$

$$\Rightarrow \frac{3}{4} > \frac{9}{14} \quad \text{which is true.}$$

Similarly for $x = 1/2$, $y = 0$, condition (5.6) becomes

$$3/8 = f(1/2, 0) \geq \frac{1+f(1/2, 1/2)f(0,0)}{(1+f(1/2, 0))} = 0$$

For $x = 0$, $y = 0$ we have $0 = f(0,0) \geq f(0,0)$

and for $x = 1/2$, $y = 1/2$, $f(1/2, 1/2) \geq f(1/2, 1/2)$
open sets $G$ and $H$ in $\mathcal{J}$ are not disjoint. But clearly every convergent sequence in $(X, \mathcal{J})$ has a unique limit. Therefore $(X, \mathcal{J})$ is a semi-Hausdorff space but not Hausdorff. Space $(X, \mathcal{J})$ is clearly $T_1$ since $\{x\}'$ and $\{y\}' \in \mathcal{J}$.

**Orbitally Continuous.** In a semi-Hausdorff space $X$, a self-mapping $T$ is said to be orbitally continuous if for each $x \in X, T(x) \to u \Rightarrow T(T(x)) \to Tu$

In this section, we prove the following fixed point theorems in semi-Hausdorff spaces with a weaker condition of continuity, which extends the results of Hicks-Rhoades [27], Jungck [34] and Sehgal [77] in the new setting.

**Theorem 5.2.1.** Let $S$ and $T$ be two orbitally continuous self maps of a space $X$. Let $f$ be continuous maps of $X \times X \to R^+$ (set of non-negative reals) such that

(5.12) $f(x, y) = 0$ if and only if $x = y$

(5.13) There exists a function $\Phi: R^+ \to R^+$ such that $\Phi$ is non-decreasing and $\Phi(y) < y$, $\Phi(0) = 0$ for each $y > 0$

(5.14) for all $x, y \in X$

$f(Tx, Ty) \leq \Phi(m(x, y))$

where $m(x, y) = \max \{f(Sx, Sy), f(Sx, Tx), f(Sy, Ty), f(Tx, Sy)\}$

(5.15) $T(X) \subseteq S(X)$ and if for some $x_0 \in X$. The sequence $\{x_n\} = \{T^n x_0\}$ has a convergent subsequence, then $S$ and $T$ have a unique common fixed point.

**Proof.** Since $T(X) \subseteq S(X)$ so for every $x_0 \in X, Tx_0 \in X$ and so there exists $x_1 \in X$ such that $Tx_0 = Sx_1$ and in general we can have $x_n$’s such that $Tx_n = Sx_{n+1}$ for $n = 0, 1, 2, \ldots$

Also from $x_n = T^n x_0$ we have $Tx_n = x_{n+1}$
Hence $T_{n} = S_{n+1} = x_{n+1}$ for $n = 0, 1, 2, \ldots$.

Now using (5.14) we have

$$f(x_1, x_2) = f(T_{0}, T_{1}) \leq \Phi(m(x_0, x_1))$$

where $m(x_0, x_1) = \max \{f(S_{x_0}, S_{x_1}), f(S_{x_0}, T_{x_1}), f(S_{x_1}, T_{x_1}), f(T_{x_0}, S_{x_1})\}$

$$= \max \{f(x_0, x_1), f(x_1, x_2), 0\}$$

Now if $m(x_0, x_1) = f(x_1, x_2)$, then

$$f(x_1, x_2) \leq \Phi(f(x_1, x_2)) < f(x_1, x_2),$$

gives a contradiction. Also if $m(x_0, x_1) = 0$, then $f(x_1, x_2) \leq \Phi(0) \Rightarrow x_1 = x_2$

which is not the case. Hence $m(x_0, x_1) = f(x_1, x_2)$ which implies that

$$f(x_1, x_2) \leq \Phi(f(x_0, x_1)) < f(x_0, x_1).$$

Repeating the above arguments, we have

$$f(x_0, x_1) > f(x_1, x_2) > f(x_2, x_3) > \ldots$$

Thus the sequence $\{f(x_n, x_{n+1})\}$ converges to some $u$, being the monotone sequence of positive reals. Again $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some $p$ (say).

Then from orbital continuity of $S$ and $T$, we have

$$T_{p} = T \lim_{k} x_{n} = \lim_{k} T x_{n} = \lim_{k} x_{n+1}$$

$$S T_{p} = S \lim_{k} x_{n+1} = \lim_{k} S x_{n+1} = \lim_{k} x_{n+1} = T_{p}$$

$$T^{2} p = T \lim_{k} x_{n+1} = \lim_{k} T x_{n+1} = \lim_{k} x_{n+2}$$

Now $f(p, T_{p}) = f(\lim_{k} x_{n}, \lim_{k} x_{n+1}) = \lim_{k} f(x_{n}, x_{n+1})$

$$= u$$

$$= \lim_{k} f(x_{n+1}, x_{n+2})$$

$$= f(T_{p}, T^{2} p).$$
Also \( Sx_{n+1} = x_{n+1} \) gives \( \lim Sx_n = \lim x_{n_k} \); \( S \lim x_n = p \)

\[ \text{Or } Sp = p. \]

Now if \( p \neq Tp \), then using \( Sp = p \), we get

\[ f(Tp, T^2p) \leq \Phi(m(p, Tp)) \]

where \( m(p, Tp) = \max \{ f(Sp, STp), f(Sp, Tp), f(STp, TSp), f(Tp, STp) \} \).

\[ = \max \{ f(p, Tp), f(p, Tp), f(Tp, Tp), f(Tp, Tp) \}, \]

which implies that \( m(p, Tp) = f(p, Tp) \)

Then \( f(Tp, T^2p) < \Phi(f(p, Tp)) < f(p, Tp) \) gives a contradiction.

Hence we must have \( p = Tp = Sp \)

uniqueness of common fixed point \( p \) follows from condition (5.14).

We give the following example to prove the validity of our Theorem 5.2.1.

**Example 5.2.2.** If we take \( X = [0, 4] \) with co-countable topology then

\((X, \mathcal{S})\) is a Semi-Hausdorff space. Let us define

\[ T(x) = \frac{x}{2} + 1 \text{ and } S(x) = 4 - x \text{ for all } x, y \in X \]

\[ f(x, y) = |x - y| \text{ for all } x, y \in [0, 4]; \quad \Phi(y) = \frac{3}{4}y \text{ for all } y \in \mathbb{R}^+, \]

then clearly

\[ T(X) = [1, 3] \subseteq S(X) = [0, 4] \]

To verify condition (5.14), we have

\[ f(Tx, Ty) = |\frac{x}{2} + 1 - \frac{y}{2} - 1| = |x - y|/2 \]

\[ m(x, y) = \max \{ |4x - 4 + y|, |4x - (x/2) - 1|, |4 - y - (y/2) - 1|, |x/2 + 1 - 4 + y| \} \]

\[ = \max \{ |x - y|, |3 - (3x/2)|, |3 - (3y/2)|, |x/2 + (y - 3)| \}. \]

Clearly \( |x - y|/2 \leq \Phi(|x - y|) < 3/4 |x - y| \) is true for all \( x, y \in [0, 4] \).

Also for \( x_0 = 0; \quad x_1 = Tx_0 = 1; \quad x_2 = Tx_1 = 3/2 \).

Similarly, \( x_3 = 7/4; \quad x_4 = 15/8; \quad x_5 = 31/16 \) and so on.

Hence the sequence \( \{x_n\} \) is a sequence of positive terms and hence converges to 2. Thus all the conditions of Theorem 5.2.1 are satisfied and clearly \( x = 2 \) is the unique common fixed point of \( S \) and \( T \).
Remarks. Theorem 5.2.1 is general and it has many Corollaries.

(1) If we take $S = \text{Identity mapping}$

\[ f(x,y) = d(x,y) ; \quad \Phi(s) = ks, \quad 0 \leq k < 1 \quad \text{and} \]

\[ m(x,y) = \max \{d(x,y), d(x, Tx), d(y, Ty)\} \]

for $x, y \in X, x \neq y$ we obtain the result of Sehgal [77] in metric spaces.

(2) Also letting $T = h$ and $S = f$:

\[ f(x,y) = d(x,y) \quad \text{and} \]

\[ m(x,y) = \max \{d(fx, fy), d(fx, hx), d(fy, hy)\} \]

we get Theorem 5 of Hicks and Rhoades [27] for $d$-complete topological spaces in this new setting.

(3) Similarly theorem 5.2.1 is a direct generalisation of the results of Rakotch [72], Bianchini [4] and Jungck [34] in complete metric spaces.

Theorem 5.2.1 can be extended for a sequence of maps in the form of following:

**Theorem 5.2.2.** Let $X$ be a Semi-Hausdorff spaces $S$ and $T$ be orbitally continuous self maps of $X$ and \{${A_i}$\} be the sequence of orbitally continuous self maps of $X$, $f$ be a continuous map of $X \times X \to \mathbb{R}^+$ satisfying (5.12), (5.13) and

(5.16) \[ f(A_ix, A_jy) \leq \Phi(m(x,y)) \]

where

\[ m(x,y) = \max \{f(Sx, Ty), f(Sx, A_ix), f(Ty, A_jy), f(A_i x, Ty)\} \]

(5.17) \[ A_i (X) \subseteq S(X) \cap T(X) \quad \text{for all} \quad i \]

(5.18) For some $x_0 \in X$ the sequence \{${x_n}$\} of points in $X$ is such that

\[ A_ix_{2n} = x_{2n+1} \quad \text{and} \quad A_jx_{2n+1} = x_{2n+2} \quad \text{for} \quad n = 0, 1, 2, \ldots, \]

has a convergent subsequence of the type \{${x_{(2p+1)n}}$\} where $p \in \mathbb{N}$ is fixed and $n \in \mathbb{N}$.

Then all the $A_i$'s, $S$ and $T$ have a unique common fixed point in $X$. 
Proof. The above theorem can be proved on the same lines as Theorem 5.2.1.

We furnish the following example in support of our Theorem 5.2.2.

Example 5.2.3. If we take \( X = [1, \infty) \) with co-countable topology \( \mathcal{J} \), then \((X, \mathcal{J})\) is a Semi-Hausdorff space. Let us define

\[ A_i(x) = \frac{((i+3)x-1)}{(i+2)} \text{ for } i = 1, 2, \ldots, x \in [1, \infty) \]

\[ Sx = 8x - 7, \quad Tx = 8x^2 - 7 \text{ for all } x \in X \text{ and} \]

\[ f(x,y) = |x-y| \text{ for all } x, y \in X. \]

Then all the conditions of Theorem 5.2.2 are satisfied and clearly \( x = 1 \) is the unique common fixed point of all the \( A_i \)'s, \( S \) and \( T \).