CHAPTER - IV

SOME FIXED POINT THEOREMS RELATED TO ASYMPTOTIC REGULARITY
CHAPTER - IV

SOME FIXED POINT THEOREMS RELATED TO ASYMPTOTIC REGULARITY

4.1. Pathak-Kang-Baek [53,55] introduced the concept of weak compatible mappings of type (A) in menger and 2-metric space, which is also more general than that of weak commutativity as studied by Sessa [74]. This concept is also equivalent to compatible and compatible of type (A) under some conditions.

Analogues to weak compatible mappings of type (A) in menger and 2-metric space, we can deduce the following definition and propositions in ordinary metric space.

Throughout this chapter \((X,d)\) denote a complete metric space.

**Definition 4.1.** The pair of mappings \(\{A,S\}\) is said to be weak compatible of type (A) if
\[
\lim d(ASx_n, SSx_n) \leq \lim d(SAx_n, SSx_n) \quad \text{and} \\
\lim d(SAx_n, AAx_n) \leq \lim d(ASx_n, AAx_n)
\]
whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim Sx_n = \lim Ax_n = t\) for some \(t \in X\).

Now we give some propositions to discuss the relation between compatible, compatible of type (A) and weak compatible of type (A) mappings.

**Proposition 4.1.** Every pair of compatible mappings of type (A) is weak compatible of type (A).

**Proof.** Suppose that \(A\) and \(S\) are compatible of type (A) mappings then we have
\[
0 = \lim d(ASx_n, SSx_n) \leq \lim d(SAx_n, SSx_n) \quad \text{and} \\
0 = \lim d(SAx_n, AAx_n) \leq \lim d(ASx_n, AAx_n)
\]
which shows that the pair \(\{A,S\}\) is weak compatible of type (A).
Proposition 4.2. Let \( A \) and \( S \) are continuous mappings of a metric space \((X,d)\) into itself. If \( A \) and \( S \) are weak compatible of type \((A)\), then they are compatible of type \((A)\).

Proof. Suppose that \( A \) and \( S \) are weak compatible of type \((A)\). Let \( \{x_n\} \) be a sequence in \( X \) such that \( \lim Sx_n = \lim Ax_n = t \) for some \( t \in X \). Since \( A \) and \( S \) are continuous mappings, then we have

\[
\lim d(ASx_n, SSx_n) \leq \lim d(SAx_n, SSx_n) = d(St, St) = 0 \quad \text{and} \quad \lim d(SAx_n, AAx_n) \leq d(ASx_n, AAx_n) = d(At, At) = 0
\]

Therefore, \( A \) and \( S \) are compatible of mappings of type \((A)\). This completes the proof.

Proposition 4.3. Let \( A \) and \( S \) be weak compatible mappings of type \((A)\) from a metric space \((X, d)\) into itself. If one of \( A \) and \( S \) is continuous, then \( A \) and \( S \) are compatible.

Proof. Without loss of generality suppose that \( S \) is continuous. Let \( \{x_n\} \) be a sequence in \( X \) such that

\[
\lim Ax_n = \lim Sx_n = t \quad \text{for some} \quad t \in X
\]

Since \( S \) is continuous, we have

\[
\lim SAx_n = St = \lim SSx_n.
\]

Thus, by triangle inequality we have

\[
d(SAx_n, ASx_n) \leq d(SAx_n, SSx_n) + d(SSx_n, ASx_n)
\]

\[
\leq 0 + d(SSx_n, ASx_n)
\]

Since \( \{A,S\} \) are weak compatible of type \((A)\). Therefore, we have

\[
\lim d(SAx_n, ASx_n) \leq \lim d(ASx_n, SSx_n) \leq \lim d(SSx_n, SSx_n) \leq 0
\]

Therefore, \( A \) and \( S \) are compatible.

Proposition 4.4. [38] Let \( A \) and \( S \) be continuous mappings of \((X,d)\) into itself. If \( A \) and \( S \) are compatible, then they are compatible of type \((A)\).
As a direct consequence of Proposition 4.1 and Proposition 4.4, we have the following:

**Proposition 4.5.** Let $A$ and $S$ be continuous mapping from a metric space $(X,d)$ into itself. If $A$ and $S$ are compatible, then they are weak compatible of type (A).

Next, we give some properties of weak compatible mappings of type (A) for our main theorems.

**Proposition 4.6.** Let $A$ and $S$ are weak compatible maps of type (A) from $(X,d)$ into itself and let $\lim Ax_n = \lim Sx_n = t$ for some $t \in X$ then we have the following:

1. $\lim ASx_n = St$ if $S$ is continuous at $t$
2. $\lim SAx_n = At$ if $A$ is continuous at $t$
3. $SA_t = ASt$ and $At = St$ if $S$ and $A$ are continuous at $t$.

**Proof.** Immediate from Proposition 2.9 and 2.10 of Pathak-Kang-Baek [55].

In Jungck [34] and in all generalization of Jungck's theorem, a family of commuting mapping have been considered. Hardy-Rogers [26], Mukhrjee [44] and Fisher [20] extended the result of Jungck [34] using different contractive condition. Rhoades-Sessa-Khan-Khan [69] generalized the above result for three weakly commuting mappings in the form of the following using Hardy-Rogers type contractive condition.

**Theorem 4.1.1.** Let $A$, $S$ and $T$ be self maps of a complete metric space $(X,d)$ satisfying

$$d(Ax,Ay) \leq a_1d(Sx,Ax) + a_2d(Tx,Ax) + a_3d(Sy,Ay) + a_4d(Ty,Ay)$$
$$a_5d(Sx,Ay) + a_6d(Tx,Ay) + a_7d(Sy,Ax) + a_8d(Ty,Ax)$$
$$a_9d(Sx,Ty) + a_{10}d(Sy,Tx)$$

where $a_i = a_i(x,y)$ are non-negative functions of $x$ and $y$ satisfying
(4.2)\ Max. $\{\sup_{x,y \in X} (b_3 + b'_3 + b_4 + b_5 + b'_5),$
$\sup_{x,y \in X} (b'_1 + b_2 + b_3 + b'_4 + b_5 + b'_5),$
$\sup_{x,y \in X} (b'_1 + b'_2 + b_3 + b_4)\} < 1$

and $b_1, b_2$ are bounded, the $b_i, b'_i$ as defined below

$2b_1(x,y) = a_1(x,y) + a_3(y,x); \ 2b_2(x,y) = a_2(x,y)+a_4(y,x),$

$2b_3(x,y) = a_5(x,y)+a_7(y,x); 2b_4(x,y) = a_6(x,y)+a_8(y,x),$

$2b_5(x,y) = a_9(x,y)+a_{10}(y,x) \quad \text{and } b_1(y,x) = b'_1(x,y)$

(4.3) \ \text{S and T are continuous.}

(4.4) \ \{A,S\} \ \text{and} \ \{A,T\} \ \text{are weakly commuting pairs and}

(4.5) \ \text{there exists an asymptotically A-regular sequence with respect to both S and T.}

Then A, S and T have a unique common fixed point. Further A is continuous at the fixed point if \ \sup_{x,y \in X} (b_1 + b_2 + b'_3 + b'_4) < 1

We generalize the above result by replacing the more general concept of weak compatible mapping of Type (A) as introduced by Pathak-Kang-Baek [53] for four mappings by proving the following:

\textbf{Theorem 4.1.2.} \ Let A, B, S and T be four self maps of complete metric space $(X,d)$ satisfying

(4.6) \ \begin{align*}
&d(Ax,By) \leq a_1d(Sx,Ax)+a_2d(Tx,Ax)+a_3d(Sy,Ax) + a_4d(Ty,Ax) \\
&\quad + a_5d(Sx,Ty)+a_6d(Sx,Ty)+a_7d(Sx,By) + a_8d(Ty,By) \\
&\quad + a_9d(Sy,By)+a_{10}(Ty,By)+a_{11}d(Sy,Sy)+a_{12}d(Sx,Tx) \\
&\quad + a_{13}d(Sy,Ty)+a_{14}d(Tx,Ty)
\end{align*}

for all $x, y$ in $X$ if $a_i \geq 0$ for any $i = 1, 2, 3, \ldots, 14$, where $a_i = a_i(x,y)$

(4.7) \ Max \ \{\sup_{x,y \in X} (a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_1 + a_1 + a_4)),

$\sup_{x,y \in X} (a_2 + a_4 + a_5 + a_6 + a_7 + a_9 + a_{12} + a_{13}),$

$\sup_{x,y \in X} (a_1 + a_2 + a_3 + a_4), \sup_{x,y \in X} (a_7 + a_8 + a_9 + a_{10})\} < 1$
(4.8) \( S \) and \( T \) are continuous.

(4.9) \( \{A,S\} \) and \( \{B,T\} \) are weak compatible of type (A).

(4.10) There exists an asymptotically \( S \)-regular sequence \( \{x_n\} \) with respect to \( A \) and \( B \), and \( T \)-regular with respect to \( B \).

Then \( A,B,S \) and \( T \) have a unique common fixed point.

**Proof.** Let \( \{x_n\} \) satisfy (4.10). Then from (4.6)

\[
d(S_{x_n}, S_{x_m}) \leq d(S_{x_n}, A_{x_n}) + d(A_{x_n}, B_{x_m}) + d(B_{x_m}, S_{x_m})
\]

\[
\leq d(S_{x_n}, A_{x_n}) + a_1 d(S_{x_n}, A_{x_n}) + a_2 d(T_{x_n}, A_{x_n}) + a_3 d(S_{x_m}, A_{x_n})
\]

\[
+ a_4 d(T_{x_m}, A_{x_n}) + a_5 d(S_{x_n}, T_{x_m}) + a_6 d(S_{x_m}, T_{x_m}) + a_7 d(S_{x_n}, B_{x_m})
\]

\[
+ a_8 d(T_{x_n}, B_{x_m}) + a_9 d(S_{x_m}, B_{x_m}) + a_{10} d(T_{x_m}, B_{x_m}) + a_{11} d(S_{x_n}, S_{x_m})
\]

\[
+ a_{12} d(S_{x_n}, T_{x_n}) + a_{13} d(S_{x_m}, T_{x_m}) + a_{14} d(T_{x_n}, T_{x_m}) + d(B_{x_m}, S_{x_m})
\]

\[
\leq d(S_{x_n}, A_{x_n}) + a_1 d(S_{x_n}, A_{x_n}) + a_2 [d(T_{x_n}, B_{x_m}) + d(B_{x_m}, S_{x_n})]
\]

\[
+ d(S_{x_n}, A_{x_n}) + a_3 [d(S_{x_m}, S_{x_n}) + d(S_{x_n}, A_{x_n})] + a_4 [d(T_{x_m}, B_{x_m})
\]

\[
+ d(B_{x_m}, S_{x_m}) + d(S_{x_m}, S_{x_m}) + d(S_{x_n}, A_{x_n})] + a_5 [d(S_{x_n}, S_{x_m})
\]

\[
+ d(S_{x_m}, B_{x_m}) + d(B_{x_m}, T_{x_m})] + a_6 [d(S_{x_m}, S_{x_n}) + d(S_{x_n}, B_{x_n})
\]

\[
+ d(B_{x_n}, T_{x_n})] + a_7 [d(S_{x_n}, S_{x_n}) + d(S_{x_m}, B_{x_m})] + a_8 [d(T_{x_n}, B_{x_n})
\]

\[
+ d(B_{x_n}, B_{x_n}) + d(S_{x_n}, S_{x_n}) + d(S_{x_m}, B_{x_m}) + a_9 d(S_{x_m}, B_{x_m})
\]

\[
+ a_{10} d(T_{x_m}, B_{x_m}) + a_{11} d(S_{x_n}, S_{x_m}) + a_{12} [d(S_{x_n}, B_{x_m}) + d(B_{x_n}, T_{x_n})]
\]

\[
+ a_{13} [d(S_{x_m}, B_{x_m}) + d(B_{x_m}, T_{x_m})] + a_{14} [d(T_{x_n}, B_{x_n}) + d(B_{x_n}, S_{x_n})
\]

\[
+ d(S_{x_n}, S_{x_m}) + d(S_{x_m}, B_{x_m}) + d(B_{x_m}, T_{x_m})] + d(B_{x_m}, S_{x_m})
\]

where \( a_i = a_i(x_n, x_m) \) for any \( i = 1, 2, 3, \ldots, 14 \). Then

\[
[1 - a_3 - a_4 - a_5 - a_6 - a_7 - a_8 - a_{11} - a_{14}] d(S_{x_n}, S_{x_m})
\]

\[
\leq [1 + a_1 + a_2 + a_3 + a_4] d(A_{x_n}, S_{x_n}) + [a_2 + a_6 + a_8 + a_{12} + a_{14}] d(B_{x_n}, S_{x_n})
\]

\[
+ [a_4 + a_5 + a_7 + a_8 + a_9 + a_{13} + a_{14}] d(B_{x_m}, S_{x_m}) + [a_2 + a_6 + a_8 + a_{12} + a_{14}] d(T_{x_n}, B_{x_n})
\]

\[
+ [a_4 + a_5 + a_{10} + a_{13} + a_{14}] d(T_{x_m}, B_{x_m})
\]
Now by (4.7), (4.10) and taking the limit when \( m, n \to \infty \), we deduce that \( Sx_n \) is a Cauchy sequence and hence convergent, let \( Sx_n \to z \).

\[
d(Ax_nz) \leq d(Ax_n,Sx_n) + d(Sx_n,z)
\]

We see that the sequence \( (Ax_n) \) also converges to \( z \). Similarly, it can be proved that the sequence \( \{Bx_n\} \) and \( \{Tx_n\} \) also converges to \( z \). By condition (4.8) the sequences \( \{SA_n\} \), \( \{SSX_n\} \), \( \{STx_n\} \) converges to \( Sz \) and sequences \( \{TBx_n\} \), \( \{TTx_n\} \), \( \{TSx_n\} \) converges to \( Tz \). By condition (4.9), the sequence \( \{ASx_n\} \) converges to \( Sz \) and \( \{BTx_n\} \) converges to \( Tz \). Again by using (4.6), we have

\[
d(ASx_n,BTx_n) \leq a_1d(SSx_n,ASx_n) + a_2d(TSx_n,ASx_n) + a_3d(STx_n,ASx_n)
\]
\[
+ a_4d(TTx_n,ASx_n) + a_5d(SSx_n,TTx_n) + a_6d(STx_n,TTx_n)
\]
\[
+ a_7d(SSx_n,BTx_n) + a_8d(TSx_n,BTx_n) + a_9d(STx_n,BTx_n)
\]
\[
+ a_{10}d(TTx_n,BTx_n) + a_{11}d(SSx_n,STx_n) + a_{12}d(SSx_n,TSx_n)
\]
\[
+ a_{13}d(STx_n,TTx_n) + a_{14}d(TSx_n,TTx_n)
\]

where \( a_i = a_i(Sx_n,x_n) \) for any \( i = 1, 2, 3, \ldots, 14 \).

Hence

\[
d(Sz,Tz) = \lim \sup d(ASx_n,BTx_n)
\]

\[
\leq \lim \sup (a_2 + a_4 + a_5 + a_6 + a_7 + a_9 + a_{12} + a_{13})d(Sz,Tz)
\]

\[
\leq \sup_{x,y \in X}(a_2 + a_4 + a_5 + a_6 + a_7 + a_9 + a_{12} + a_{13})d(Sz,Tz)
\]

whence \( Sz = Tz \) by (4.7). Due to (4.6), we obtain

\[
d(ASx_n,Bx_n) \leq a_1d(SSx_n,ASx_n) + a_2d(TSx_n,ASx_n) + a_3d(Sx_n,ASx_n)
\]
\[
+ a_4d(Tx_n,ASx_n) + a_5d(SSx_n,Tx_n) + a_6d(Sx_n,TSx_n)
\]
\[
+ a_7d(SSx_n,Bx_n) + a_8d(TSx_n,Bx_n) + a_9d(Sx_n,Bx_n)
\]
\[
+ a_{10}d(Tx_n,Bx_n) + a_{11}d(SSx_n,Sx_n) + a_{12}d(SSx_n,TSx_n)
\]
\[
+ a_{13}d(Sx_n,Tx_n) + a_{14}d(TSx_n,Tx_n)
\]

where \( a_i = a_i(Sx_n,x_n) \) for any \( i = 1, 2, 3, \ldots, 14 \).
Then
\[ d(Sz,z) = \lim \sup d(ASx, Bx) \]
\[ \leq \lim \sup (a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_{14}) d(Sz,z) \]
\[ \leq \sup_{x,y \in X} (a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_{14}) d(Sz,z) \]
which implies \( Sz = z \) by (4.7).

Again by using (4.6), we have
\[ d(Az, BTx) \leq a_1 d(Sz, Az) + a_2 d(Tz, Az) + a_3 d(STx, Az) + a_4 d(TTx, Az) + a_5 d(Sz, TTx) + a_6 d(STx, TTx) + a_7 d(St, BTx) + a_8 d(Tz, BTx) + a_9 d(STx, BTx) + a_{10} d(TTx, BTx) + a_{11} d(Sz, STx) + a_{12} d(Sz, Tz) + a_{13} d(STx, TTx) + a_{14} d(Tz, TTx) \]
where \( a_i = a_i(z, Tx) \) for any \( i = 1, 2, 3, \ldots, 14 \)
\[ d(Az, Tz) = \lim \sup d(Az, BTx) \]
\[ \leq \lim \sup (a_1 + a_2 + a_3 + a_4) d(Az, Tz) \]
\[ \leq \sup_{x,y \in X} (a_1 + a_2 + a_3 + a_4) d(Az, Tz) \]
whence \( Az = Tz \) by (4.7). Further by (4.6), we obtain
\[ d(Ax, Bz) \leq a_1 d(Sx, Ax) + a_2 d(Tx, Ax) + a_3 d(Sz, Ax) + a_4 d(Tz, Ax) + a_5 d(Sx, Tz) + a_6 d(Sz, Tx) + a_7 d(Sx, Bz) + a_8 d(Tx, Bz) + a_9 d(Sz, Bz) + a_{10} d(Tz, Bz) + a_{11} d(x, Bz) + a_{12} d(x, Tx) + a_{13} d(Sz, Tz) + a_{14} d(Tx, Tz) \]
where \( a_i = a_i(x, z) \) for any \( i = 1, 2, 3, \ldots, 14 \).
\[ d(z, Bz) = \lim \sup d(Ax, Bz) \]
\[ \leq \lim \sup (a_7 + a_8 + a_9 + a_{10}) d(z, Bz) \]
\[ \leq \sup_{x,y \in X} (a_7 + a_8 + a_9 + a_{10}) d(z, Bz) \]
which implies that \( z = Bz \) by (4.7) i.e., \( z \) is a fixed point of \( A, B, S \) and \( T \).

For unicity, suppose \( w \) be another common fixed point of \( A, B, S \) and \( T \).
Then by condition (4.6) we have
\[ d(w, z) = d(Aw, Bz) \leq a_1 d(Sw, Aw) + a_2 d(Tw, Aw) + a_3 d(Sz, Aw) + a_4 d(Tz, Aw) + a_5 d(Sw, Tw) + a_6 d(Tw, Sw) + a_7 d(Sz, Sw) + a_8 d(Sz, Tz) + a_9 d(Tw, Bz) + a_{10} d(Sw, Bz) \]
\[ + a_{11} d(Tz, Sw) + a_{12} d(Sz, Sz) + a_{13} d(Tz, Tz) + a_{14} d(Tw, Tz) \]
\[ \leq \sup_{x, y \in X} (a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_{11} + a_{14}) d(w, z) \]
so \( z = w \) by (4.7). Hence \( z \) is the unique common fixed point of \( A, B, S \) and \( T \).

This completes the proof of the theorem.

**Remark 4.1.1.** Theorem 4.1.2 is a direct generalization of Theorem 4.1.1 of Rhoades-Sessa-Khan-Khan [69] if we put \( B = A \) and \( a_{11} = a_{12} = a_{13} = a_{14} = 0 \).

We give the following example in support of our theorem.

**Example 4.1.1.** Let \( X = [0, 1] \) with Euclidean metric \( d \) and define

\[ Sx = x/5 \] for all \( x \) in \( X \);

\[ Bx = \begin{cases} x/10 & \text{if } x \neq 1/2 \\ 1/10 & \text{if } x = 1/2 \end{cases} \]

\[ Ax = \begin{cases} x/6 & \text{if } x \neq 1/2 \\ 1/6 & \text{if } x = 1/2 \end{cases} \]

\[ Tx = 2x/3 \] for all \( x \) in \( X \).

Further for any sequence \( \{x_n\} \) converges to zero as \( n \to \infty \) we have
\[ d(ASx_n, SSx_n) = x_n/150 \to 0 \text{ as } n \to \infty \]
\[ d(SAx_n, AAx_n) = x_n/180 \to 0 \text{ as } n \to \infty \]

Also
\[ d(BTx_n, TTx_n) = 17x_n/45 \to 0 \text{ as } n \to \infty \]
\[ d(TBx_n, BBx_n) = 17x_n/300 \to 0 \text{ as } n \to \infty . \]

Thus the \( \{A, S\} \) and \( \{B, T\} \) are compatible of type (A) and hence by Proposition 4.1 are weak compatible of type (A). Further we have
\[ d(Ax_n, Sx_n) \to 0 \text{ as } x_n \to 0 \; ; \; d(Bx_n, Sx_n) \to 0 \text{ as } x_n \to 0 \]
and \( d(Tx_n, Bx_n) \to 0 \) as \( x_n \to 0 \)

and hence satisfying condition (4.10).

Now to check the condition (4.6), we have the following cases:

**Case I.** At \( x = \frac{1}{2}; \quad y = \frac{1}{2} \)

\[
d(Ax, By) = \frac{1}{15} \leq a_8 \quad \frac{7}{30} = a_8 \quad d(Tx, By)
\]

so the condition (4.6) satisfies for \( a_8 = \frac{1}{4} \) and \( a_i = 0 \) when \( i = 1, 2, ..., 7, 9, 10, ..., 14 \).

**Case II.** At \( x = \frac{1}{2}; \quad y \neq \frac{1}{2} \)

\[
d(Ax, By) = |(5-3y)/30| \leq \frac{4}{5} |(3y-5)/15| = a_8 \quad d(Sy, Tx)
\]

therefore

condition (4.6) satisfies when \( a_i = 0 \) except \( a_8 = \frac{4}{5}; \quad i = 1, 2, 3, 4, ..., 14 \).

**Case III.** At \( x \neq \frac{1}{2}; \quad y = \frac{1}{2} \) we get

\[
d(Ax, By) = |(5-3x)/30| \leq a_5 \quad |(5-3x)/15| = a_5 d(Sx, Ty)
\]

is true for \( \frac{1}{2} \leq a_5 < 1 \) and \( a_i = 0 \) for all \( i \)'s except \( i = 5 \).

**Case IV.** When \( x \neq \frac{1}{2} \) and \( y \neq \frac{1}{2} \) then we have two subcases:

(i) When \( 5x > 3y \), then \( 2x > y \) and taking \( a_i = 0 \) for all \( i \)'s except \( i = 7 \) and \( a_7 = \frac{5}{6} \) condition (4.6) becomes

\[
d(Ax, By) = |(5x-3y)/30| \leq \frac{5}{6} |(2x-y)/10| = a_7 \quad d(Sx, By)
\]

i.e., \( 5x-3y < 5x - (5/2)y \) implies \( y/2 \geq 0 \). which is true.

(ii) When \( 5x < 3y \), then \( 5x < 6y \) and taking \( a_3 = \frac{1}{2} \) and all other \( a_i \)'s = 0

condition (4.6) becomes

\[
d(Ax, By) = |(5x-3y)/30| \leq \frac{1}{2} |(6y-5x)/30| = a_3 \quad d(Sy, Ax)
\]

\( 3y-5x < 3y - (5/2)x \) which is true.

Thus all assumptions of theorem (4.1.2) are satisfied and hence zero is a common fixed point of \( A, B, S \) and \( T \).
4.2. Now in the following result, we use four weak compatible mappings of type (A) for \( \phi \)-contractive condition which is defined by Singh and Meade [79], Matkowski J. [43].

This result also generalize and improves the result of Rhoades-Sessa-Khan-Khan [69] for three mappings in a new setting.

**Theorem 4.2.1.** Let \( A, B, S \) and \( T \) be four self maps of complete metric space \((X, d)\) satisfying the following conditions

\[
\begin{align*}
\text{(4.11)} & \quad d(Ax, By) \leq \phi(d(Sx, Ax), d(Tx, Ax), d(Sy, Ax), d(Ty, Ax), d(Sx, Ty), \\
& \quad \quad d(Sy, Tx), d(Sx, By), d(Tx, By), d(Sy, By), d(Ty, By), \\
& \quad \quad d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Tx, Ty))
\end{align*}
\]

for all \( x, y \in X \) where \( \phi : [0, \infty] \rightarrow [0, \infty) \) is upper semi-continuous and non-decreasing in each co-ordinate variable and also satisfies the following:

\[
\begin{align*}
\text{(4.12)} & \quad \psi(t) = \max\{ \phi(0, 0, t, t, t, t, t, t, t, t, t), \phi(0, t, 0, t, t, t, t, t, 0, 0, 0), \\
& \quad \quad \phi(t, t, t, t, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \phi(0, 0, 0, 0, 0, 0, 0, t, t, t, 0, 0, 0, 0) \} < t
\end{align*}
\]

for \( t > 0 \)

\[
\begin{align*}
\text{(4.13)} & \quad S \text{ and } T \text{ are continuous} \\
\text{(4.14)} & \quad \{A, S\} \text{ and } \{B, T\} \text{ are weak compatible of type (A).} \\
\text{(4.15)} & \quad \text{There exists an asymptotically } S\text{-regular sequence } \{x_n\} \text{ with respect to } A \text{ and } B, \text{ and } T\text{-regular with respect to } B.
\end{align*}
\]

Then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof.** Let \( \{x_n\} \) satisfy (4.15). Then from (4.11)

\[
\begin{align*}
d(Sx_n, Sx_m) & \leq d(Sx_n, Ax_n) + d(Ax_n, Bx_m) + d(Bx_m, Sx_m) \\
& \leq d(Sx_n, Ax_n) + \phi(d(Sx_n, Ax_n), d(Tx_n, Ax_n), \\
& \quad d(Sx_m, Ax_n), d(Tx_m, Ax_m), d(Sx_n, Tx_m), d(Sx_m, Tx_n), d(Sx_n, Bx_m), \\
& \quad d(Tx_n, Bx_m), d(Sx_m, Bx_m), d(Tx_m, Bx_m), d(Sx_n, Sx_m), d(Sx_n, Tx_n), \\
& \quad d(Sx_m, Tx_m), d(Tx_n, Tx_m)) + d(Bx_m, Sx_m)
\end{align*}
\]
\[
\leq d(S_{x_n}, A_{x_n}) + \phi(d(S_{x_n}, A_{x_n}), d(T_{x_n}, B_{x_n}) + d(B_{x_n}, S_{x_n}) + d(S_{x_n}, A_{x_n}) + d(T_{x_n}, B_{x_n}) + d(B_{x_n}, S_{x_n}) + d(S_{x_n}, S_{x_n}) + d(S_{x_n}, B_{x_n}) + d(B_{x_n}, S_{x_n}) + d(S_{x_n}, S_{x_n}) + d(S_{x_n}, B_{x_n}) + d(B_{x_n}, S_{x_n}) + d(S_{x_n}, S_{x_n}) + d(S_{x_n}, B_{x_n}) + d(B_{x_n}, S_{x_n}) + d(S_{x_n}, S_{x_n}) + d(S_{x_n}, B_{x_n}) + d(B_{x_n}, S_{x_n}) + d(S_{x_n}, S_{x_n}) + d(S_{x_n}, B_{x_n}) + d(B_{x_n}, S_{x_n}) + d(S_{x_n}, S_{x_n}) + d(S_{x_n}, B_{x_n}) + d(B_{x_n}, S_{x_n}) + d(S_{x_n}, S_{x_n}) + d(S_{x_n}, B_{x_n}) + d(B_{x_n}, S_{x_n}) + d(S_{x_n}, S_{x_n}) + d(S_{x_n}, B_{x_n}) + d(B_{x_n}, S_{x_n})) + d(B_{x_n}, S_{x_n})
\]

By using upper semi-continuity of \(\phi\), condition (4.15), (4.12) taking limit as \(m, n \to \infty\) we get
\[t < \phi(0, 0, t, t, t, t, t, 0, 0, t) < t\]
where \[\lim_{m, n \to \infty} d(S_{x_n}, S_{x_m}) = t\]
gives a contradiction.
Thus \(\{S_{x_n}\}\) is a Cauchy sequence and hence converges to \(z\) (say). As \(d(A_{x_n}, z) \leq d(A_{x_n}, S_{x_n}) + d(S_{x_n}, z)\), we see that \(\{A_{x_n}\}\) also converges to \(z\).
Similarly \(\{B_{x_n}\}\) and \(\{T_{x_n}\}\) converges to \(z\). By condition (4.13) the sequence \(\{S_{A_{x_n}}\}, \{S_{S_{x_n}}\}, \{S_{T_{x_n}}\}\) converges to \(S_z\) and \(\{T_{B_{x_n}}\}, \{T_{T_{x_n}}\}, \{T_{S_{x_n}}\}\) converges to \(T_z\). Also by condition (4.14) the sequence \(\{A_{S_{x_n}}\}\) converges to \(S_z\) and \(\{B_{T_{x_n}}\}\) to \(T_z\). Again by using condition (4.11) we have
\[d(A_{S_{x_n}}, B_{T_{x_n}}) \leq \phi(d(S_{S_{x_n}}, A_{S_{x_n}}), d(T_{S_{x_n}}, A_{S_{x_n}}), d(ST_{S_{x_n}}, A_{S_{x_n}}), d(TT_{S_{x_n}}, A_{S_{x_n}}), d(S_{S_{x_n}}, T_{T_{x_n}}), d(ST_{S_{x_n}}, T_{T_{x_n}}), d(S_{T_{x_n}}, T_{T_{x_n}}), d(ST_{S_{x_n}}, S_{T_{x_n}}), d(S_{S_{x_n}}, S_{T_{x_n}}), d(ST_{S_{x_n}}, S_{T_{x_n}}), d(S_{T_{x_n}}, S_{T_{x_n}}), d(ST_{S_{x_n}}, S_{T_{x_n}}), d(S_{S_{x_n}}, S_{T_{x_n}}), d(ST_{S_{x_n}}, S_{T_{x_n}}), d(S_{T_{x_n}}, S_{T_{x_n}}), d(ST_{S_{x_n}}, S_{T_{x_n}}), d(ST_{S_{x_n}}, S_{T_{x_n}}))\]
Taking limit when \(n \to \infty\), we have
\[d(S_z, T_z) \leq \phi(0, d(S_z, T_z), 0, d(T_z, S_z), d(S_z, T_z), d(S_z, T_z), 0, d(S_z, T_z), 0, 0, d(S_z, T_z), d(S_z, T_z), 0)\]
whence \(S_z = T_z\) by (4.12).
Due to (4.11) we obtain
\[ d(AS_{x_n}, B_{x_n}) \leq \phi(d(SS_{x_n}, AS_{x_n}), d(TS_{x_n}, AS_{x_n}), d(S_{x_n}, AS_{x_n}), d(T_{x_n}, AS_{x_n}),
\]
\[ d(SS_{x_n}, T_{x_n}), d(S_{x_n}, TS_{x_n}), d(SS_{x_n}, B_{x_n}), d(TS_{x_n}, B_{x_n}), d(S_{x_n}, B_{x_n}),
\]
\[ d(T_{x_n}, B_{x_n}), d(SS_{x_n}, S_{x_n}), d(SS_{x_n}, TS_{x_n}), d(S_{x_n}, T_{x_n}), d(TS_{x_n}, T_{x_n})) \]

Taking limit when \( n \to \infty \), we get
\[ d(S, z) \leq \phi(0, 0, d(z, z), d(z, S), d(z, S), d(z, z), 0, 0, d(z, z)) \]
which implies \( S = z \) by (4.12). Again by using condition (4.11), we have
\[ d(A, B_{T_{x_n}}) \leq \phi(d(S, A), d(T, A), d(S, T), d(T, A), d(T, A), 0, 0, 0, 0, 0, 0, 0, 0, 0) \]
whence \( A = T \) by (4.12). Further by (4.11), we obtain
\[ d(A_{x_n}, B_z) \leq \phi(d(S_{x_n}, A_{x_n}), d(T_{x_n}, A_{x_n}), d(S_{x_n}, A_{x_n}), d(T_{x_n}, A_{x_n}), d(S_{x_n}, T_z),
\]
\[ d(S_{x_n}, T_{x_n}), d(S_{x_n}, B_z), d(T_{x_n}, B_z), d(S_z, B_z), d(T_z, B_z), d(S_{x_n}, S_z),
\]
\[ (S_{x_n}, T_z), d(S_z, T_z), d(T_z, T_z)) \]
Taking limit when \( n \to \infty \) we get
\[ d(z, B_z) \leq \phi(0, 0, 0, 0, 0, 0, 0, d(z, B_z), d(z, B_z), d(z, B_z), 0, 0, 0, 0, 0) \]
Hence \( z = B_z \) by condition (4.12).

Therefore \( z \) is a common fixed point of \( A, B, S \) and \( T \). Unicity can be easily proved from condition (4.11).

This completes the proof of theorem.
Remark 4.2.1. If we define \( \phi(t_1,t_2,t_3,...,t_{14}) = a_1t_1 + a_2t_2 + a_3t_3 + ... + a_{14}t_{14} \) in condition (4.11) of Theorem (4.2.1), where \( a_i \geq 0 \). Then from (4.12), we get
\[
\psi(1) = \text{Max. } \{ (a_3+a_4+a_5+a_6+a_7+a_8+a_{11}+a_{14}), (a_2+a_4+a_5+a_6+a_7+a_9+a_{12}+a_{13}),
(a_1+a_2+a_3+a_4), (a_7+a_8+a_9+a_{10}) \} < 1
\]
Then we get the result of Rhoades-Sessa-Khan-Khan [69] by putting \( B = A \) and \( a_{11} = a_{12} = a_{13} = a_{14} = 0 \).

We give the following example in support of our theorem.

Example 4.2.1. Let \( X = [0,1] \) with Euclidean metric \( d \) and define
\[
Sx = \begin{cases} 
\frac{x}{6} & \text{if } x \neq \frac{1}{2} \\
\frac{1}{6} & \text{if } x = \frac{1}{2} 
\end{cases}
\]
\[
Bx = \begin{cases} 
\frac{x}{10} & \text{if } x \neq \frac{1}{2} \\
\frac{1}{10} & \text{if } x = \frac{1}{2} 
\end{cases}
\]
\[
A_{x} = \begin{cases} 
\frac{x}{5} & \text{for all } x \in X; \quad T_{x} = \frac{2x}{3} \text{ for all } x \in X.
\end{cases}
\]
Thus \( \{A,S\} \) and \( \{B,T\} \) are compatible of type (A) and hence are weakly compatible of type (A) and also satisfy condition (4.15), as in example 4.1.1.

Now to check the condition (4.11), we define \( \phi(t_1,t_2,...,t_{14}) = h \text{ max. } \{t_1,t_2,...,t_{14} \} \) where \( 0 \leq h < 1 \).

Case I. When \( x = \frac{1}{2}, y = \frac{1}{2} \), then taking \( 2/7 \leq h \leq 1 \), we get
\[
d(Ax,By) = \frac{1}{15} \leq h \cdot \frac{7}{30} = h \cdot d(Tx,By)
\]
and so condition (4.11) is satisfied.

Case II. When \( x = \frac{1}{2}, y \neq \frac{1}{2} \), then taking \( \frac{1}{2} \leq h \leq 1 \)
\[
d(Ax,By) = \left| \frac{5-3y}{30} \right| \leq h \left| \frac{3y-5}{15} \right| = hd(Sy,Tx)
\]
and so condition (4.11) is satisfied.

Thus \( \{A,S\} \) and \( \{B,T\} \) are compatible of type (A) and hence are weakly compatible of type (A) and also satisfy condition (4.15), as in example 4.1.1.
Case III. At $x \neq \frac{1}{2}; y = \frac{1}{2}$ then taking $\frac{1}{2} \leq h < 1$, we obtain
\[ d(Ax, By) = \frac{|5-3x|}{30} \leq h \frac{|5-3x|}{15} = h d(Sx, Ty) \]
Thus (4.11) is satisfied.

Case IV. When $x \neq 12; y \neq \frac{1}{2}$, then we have two subcases:

(i) When $5x > 3y$, then $2x > y$, taking $\frac{5}{6} \leq h < 1$, we get
\[ d(Ax, By) = \frac{|5x-3y|}{30} \leq h \frac{|2x-y|}{10} = h d(Sx, By) \]
\[ \leq \phi(d(Sx, Ax), ..., d(Tx, By), ..., d(Tx, Ty)) \text{ is true.} \]

(ii) When $5x < 3y$, then $5x < 6y$, taking $\frac{1}{2} \leq h < 1$
\[ d(Ax, By) = \frac{|5x-3y|}{30} \leq h \frac{|6y-5x|}{30} = h d(Sy, Ax) \]
and hence condition (4.11) is satisfied in all cases. Thus all the assumptions of Theorem 4.2.1 are satisfied and $0$ is the common fixed point of $A$, $B$, $S$ and $T$. 