CHAPTER - I

ELASTIC WAVE DIFFRACTION IN A MEDIUM CONTAINING A FINITE CRACK AND A RIGID ELLIPTIC INCLUSION (2-D CASE)

1.1 INTRODUCTION

The behaviour of elastic bodies under static or dynamic loading has been a topic of long term interest to researchers in many applied fields such as geophysics, NDT, structural designs, composite materials etc. In earlier times the attention was confined to simple albeit very useful models or to explain certain observations such as those in geophysical problems. For basic literature on such developments one can refer to the books of Love (1926), Timoshenko and Goodier (1951), Ewing, Jardetzky and Press (1957) and Achenbach (1973) for instance.

In recent years, the increasing demand for special applications in engineering practice and allied sciences, as well as for a more realistic understanding of the processes occurring in nature, has generated the need to examine a variety of new models and better techniques to handle them. Infact, as a result of the same, many branches of special applied interests have come to combine the analytical, numerical and empirical techniques, some or all of which had earlier been in vogue only in certain
exclusive fields. For a recent review of such interaction see Varadan and Varadan (1979).

One of the very important areas of increasing activity is that of stress concentration in bodies of various shapes which contain internal defects such as cracks, voids (cavities) and inclusions. Both static and dynamic problems have been studied by many authors. The background of crack problems can be found in Sneddon and Lowengurb (1969) apart from other innumerable publications from time to time. Studies on isolated cracks, interface cracks, array of cracks etc. have been carried out in the literature see for instance, Achenbach and Brind (1981 a,b), Achenbach, Gautesan and Mc-Maken (1978, 1982), Achenbach, Keer and Mendelsohn (1980), Das (1985), Gubernatis and Domany (1979), Hills and Comninou (1985), Ni and Nemat-Nasser (1991) and Freund (1976).

In addition to the cracks, other internal defects such as voids (cavity) or rigid/elastic interactions can also considerably affect the distribution of stresses and the overall material behaviour. Various dynamic and static models have been attempted. See, for example, Datta (1985), Olsson, Datta and Bostram (1990), Olsson (1986), Ting and Yan (1991), Hwu and Ting (1989), Keer (1992), Selvadurai,
Singh and Att (1991), Rajapakse (1990), Folias (1991). One further notices the use of a variety of techniques (analytical and numerical) in all these studies.

The interaction between dissimilar internal flaws constitutes another area of useful study as in many situations one encounters the presence of cracks, voids and other inclusions. These are very common in modern metallurgical processes as well as in the development of composite materials for specified applications.

Sharma and Viswanathan (1990) and Viswanathan and Sharma (1987, 1988, 1992) have studied the stress concentration and diffraction of elastic waves in a medium containing a combination of a crack and a thin rigid ribbon. They considered both finite and semi-infinite cracks but assumed the rigid ribbon to be thin and of finite length. In order to solve the above problem they had employed the Chebyshev polynomial expansion technique. So and Huang (1988) have considered the case of two arbitrarily inclined cracks.

In the present chapter we propose to study a more useful and improved model where the inclusion is not necessarily thin. We consider the stress-wave diffraction, in two dimensions, between a finite crack and a nearby
rigid elliptic inclusion. The problem is studied with the help of the Green's functions and reduced to a set of coupled singular integral equations. These are solved with the help of Chebyshev polynomial expansions. The final results are numerically computed to evaluate the nature of stress concentration at the crack tips, displacement jumps across the crack faces and also the values of stresses around the rigid inclusion.

1.2 FORMULATION OF THE PROBLEM

We consider an infinite, homogeneous, isotropic and perfectly elastic medium which contains a finite crack of length $c$ along the $x$-axis, viz.,

$$-c < x < c, \quad y = 0$$

and a rigid elliptic inclusion, of semi-axes $a, b$, in the region

$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

where $(x_0, 0)$ denotes the location of the centre of the inclusion. (See fig. 1.1). In equation (1.1) the parameter $l$ defines the distance of the rigid tip of the crack from the origin.

In the present study we deal with SH-wave diffraction and accordingly the problem is essentially two dimensional
with the elastic displacement non-zero only in the z-direction transverse to the plane of reference. Denote this displacement by $W(x)$ where

$$x = (x, y)$$

(1.3)

denotes the position vector of any point in the medium. The time dependence of $W(x)$ is implied.

The elastic stresses applicable to our problem are then given by the shear stresses

$$t_{xz} = \mu \frac{\partial W}{\partial x}$$

$$t_{yz} = \mu \frac{\partial W}{\partial y}$$

(1.4)

Where $\mu$ is the shear modulus of the elastic material.

Across any curved surface (with normal $\hat{n}$) we can, in place of the cartesian components in eqn. (1.4), deal with the tangential stress given by

$$t_{n-z} = \mu \frac{\partial W}{\partial n}$$

(1.5)

The equation of motion governing the elastic deformation is given by (Achenbach 1973):

$$\frac{\partial}{\partial x} (t_{xz}) + \frac{\partial}{\partial y} (t_{yz}) = \frac{\partial^2 W}{\partial t^2}$$

(1.6)

where no body forces are assumed to act on the system. $f$ is
the density of the elastic material. By using eqn. (1.4) in eqn. (1.6) we get the wave equation (for a constant parameter \( \mu \)).

\[
\nabla^2 W(x) = \left( \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} \right) W(x) = \frac{1}{V_s^2} \frac{\delta^2 W}{\delta t^2}
\]

(1.7)

where

\[ V_s = (\mu/\rho)^{1/2} \]  

(1.8)

is the shear-wave velocity, (\( \nabla^2 \) is the well-known Laplace operator).

We confine our study to harmonic time dependence of all the quantities of interest. Thus we assume that the displacement \( W \) and the shear stresses \( t_{xz}, t_{yz} \) (or any \( t_{nz} \)) are all proportional to the harmonic exponential factor \( e^{i\omega t} \), where \( \omega \) is the harmonic frequency. This factor, in future, will not be explicitly repeated but assumed wherever necessary.

In view of the above, eqn. (1.7) reduces to the Helmholtz equation

\[
(\nabla^2 + k^2) W = 0
\]

(1.9)

where

\[ k = \omega/V_s \]  

(1.10)

is the wave number.
We assume finally that the above system is excited by a plane wave given by the incident field displacement

\[ W_o(x) = A \exp \left\{ i \omega t - ik(x \cos \theta_0 + y \sin \theta_0) \right\} \quad (1.11) \]

where \( A \) refers to the amplitude of the incident displacement and \( \theta_0 \) is the angle of incidence w.r.t. the x-axis.

The total displacement \( W \) can now be expressed in terms of the incident (\( W_o \)) and scattered (\( W^{sc} \)) parts as

\[ W(x) = W_o(x) + W^{sc}(x) \quad (1.12) \]

The problem is thus to find the total field in eqn. (1.12) which satisfies eqn. (1.9) as well as the boundary conditions to be described in the following section.

1.3 BOUNDARY CONDITIONS

We have two types of boundaries viz. the crack (1.1) and the surface \( S \) of the elliptic inclusion (1.2). The elastic wave in eqn. (1.11) will be scattered by both these boundaries. Hence appropriate conditions must be specified on these two boundaries. We assume the following conditions to hold good:

1) On the crack, for the general case, (where there can also be locally applied tractions) we take for the stress in the total field,
where $P_0(x)$ is the net applied tangent load on the crack at any point.

ii) On the surface of the rigid elliptic inclusion we take for the total field displacement,

$$W(x) = \Delta_0, \, x < S$$

(1.14)

where $\Delta_0$ is a constant rigid body displacement. Both $P_0(x)$ and $\Delta_0$ contain the harmonic time factor $e^{i\omega t}$ for consistency.

In terms of the scattered fields $w_{SC}(x)$ and $t_{ij}^{SC}(x)$ the above conditions can be reexpressed as

$$\mu \frac{\partial \xi}{\partial y} \bigg|_{y \to \pm 0} = - \mu \frac{\partial \xi_0}{\partial y} \bigg|_{y \to \pm 0} + \frac{1}{2} P_0(x)$$

$$= - t_{yz}^0(x,0) \mp \frac{1}{2} P_0(x), \quad x \in \mathcal{C} \subset \mathcal{S}$$

(1.15)

and

$$w_{SC}(x) = - w_0(x) + A \Delta_0, \quad x \in C \subset \mathcal{S}$$. 

(1.16)

In addition, the scattered field should die out at infinity in the $x$-$y$ plane of reference (i.e. far away from
the scatterers) which is expressed by the following statements for the 2-D case considered.

\[ e^{i k r} w_{sc}(x) \leq 0 \left( \frac{1}{r^{3/2}} \right), \ r \to \infty \]  

\[ e^{i k r} t_{sc}^{\perp}(x) \leq 0 \left( \frac{1}{r^{3/2}} \right), \ r \to \infty \]  

(\[ r = \sqrt{x^2 + y^2} \])  

The symbol \( \leq \) means 'less than or equal to' the order of the terms following the sign.

In eqn. (1.15), \( t_{yz}^0(x,0) \) is explicitly known from the incident field in eqn. (1.11). On using eqn. (1.4) we thus obtain (omitting the \( e^{iwt} \) factor):

\[ t_{yz}^0(x,0) = \mu \left. \frac{\partial w}{\partial y} \right|_{y \to 0} \]  

\[ = -\mu Aik \sin \theta_o \exp \left\{ -ik(x \cos \theta_o + y \sin \theta_o) \right\} \]  

(1.18)

1.4 REPRESENTATION OF THE SCATTERED FIELD USING GREEN'S FUNCTIONS

We have to solve eqn. (1.9) which satisfies the conditions (1.15) - (1.17). This is carried out by first
transforming eqn. (1.9) for $w^{sc}$ into an integral equation based on the Green's function (see Achenbach 1973).

$$w^{sc}(x) = \int_{-\ell-c}^{\ell} \gamma(x') \left\{ \mu \frac{\partial G(x,x')}{\partial y'} \right\} _{y'=0} \, dx'$$

$$- \int_{-\ell-c}^{\ell} \nu_0(x') \, G(x,x') \, \gamma(x') \, dy' \, dx'$$

$$+ \int_S \left[ w^{sc}(x') \left\{ \mu \frac{\partial G}{\partial n'}, (x,x') \right\} _{y'=0} \right] \, dS(x') \, dx'$$

$$- \mu \frac{\partial w^{sc}}{\partial n'} (x') G(x,x') \, dS(x') \, (1.19)$$

Here

$$\gamma(x) \equiv \Delta W(x), \, y = 0, \, -\ell-c \leq x \leq \ell \, (1.20)$$

is the unknown displacement jump across the crack face.

The operator $\partial/\partial n'$ refers to the normal derivative on the surface $S$ of the elliptic inclusion (in the $n'$ system) and $G(x,x')$ represents the Green's function solution associated with the Helmholtz equation (1.9). This is given by

$$G(x,x') = \frac{1}{2\pi \mu} H^{(2)}_0(k|x-x'|) \, (1.21)$$

where $(i = \sqrt{-1})$ and $|x-x'|$ denotes the distance between the two points $x$ and $x'$. Also, $H^{(2)}_n(z)$ represents the Hankel function of the second kind and order $n$. 
In what follows we will deal with the case when there is no locally applied traction on the crack faces. Thus we take
\[ P_0(x) = 0. \quad (1.22) \]
Then eqn. (1.19) becomes
\[
W_{sc}(x) = \int_{-l_c}^{l_c} \gamma(x') \left\{ \mu \frac{\partial G(x', x')}{\partial y'} \right\} dx' y'=0
- \int_S \left\{ \omega_S(x') - A\Delta_o \right\} \mu \frac{\partial G(x', x')}{\partial n'} (x', x') dS(x')
\]
\[
-\int_S t_S(x') G(x', x') dS(x') \quad (1.23)
\]
where for the first term in the last integral in eqn. (1.19) we have made use of the boundary condition in eqn. (1.16), and use the following additional notations:
\[
\omega_S^0(x') = W_0(x') \bigg|_{x' \subset S} \quad (1.24)
\]
is the incident field displacement on \( S \), and
\[
t_S(x') = t_{sc}^0(x') \bigg|_{n'z \subset S} \equiv \mu \frac{\partial W_{sc}^0(x')}{\partial n'} \bigg|_{x' \subset S} \quad (1.25)
\]
is the unknown shear stress on the elliptic boundary \( S \) due to the scattered field.
1.5 **SOLUTION OF THE INTEGRAL EQUATION (1.23)**

We have therefore to solve for the two unknown functions \( y(x) \) and \( t_s(x) \) on the crack face and the elliptic boundary respectively by applying the conditions in eqn. (1.15) and eqn. (1.16) to the solution represented in eqn. (1.23). By the choice of outgoing waves made for the Green's function (1.21), the conditions (1.17) are also ensured.

To begin with, we note that the Green's function in eqn. (1.21) is singular because of the Hankel function \( H_o^{(2)}(|x-x'|) \) which has a logarithmic singularity. It is known that

\[
H_o^{(2)}(z) = J_0(z) - iY_0(z)
\]

\[
= J_0(z) \left[ 1 - \frac{2i}{\pi} \left( \gamma + \log \frac{z}{2} \right) \right] + \frac{2i}{\pi} s(z)
\]

where

\[
s(z) = \sum_{m=1}^{\infty} (-1)^m \left( \frac{z}{2} \right)^{2m} \frac{p_m}{(m!)^2}
\]

\[
p_m = 1 + \frac{1}{2} + \ldots + \frac{1}{m}
\]

\[
\gamma = 0.5772
\]
and

\[
J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{2m}}{(m!)^2} ;
\]

we have used the definition

\[
\gamma_0(z) = \frac{2}{\pi} J_0(z) (1 + \log \frac{z}{2} + \gamma) - S(z) .
\]

In view of the above, we apply the conditions in eqn. (1.15) (with \( P_0(X) = 0 \)) and eqn. (1.16), to (1.23) and obtain the following singular coupled integral equations for \( \gamma(x) \) and \( t_s(x) \):

\[
\left[ \frac{k}{2i} \int_{-\ell}^{\ell} \gamma(x') \left[ \frac{\partial^2}{\partial y \partial y'} H_0^{(2)}(kr') \right] \bigg|_{y' = 0} dx' \right.
\]

\[
- \frac{1}{2i\mu} \int_{S} t_s(x') H_o^{(2)}(kr') dS(x') \right]
\]

\[
= \left\{ A\Delta_o - \omega_s^o(x) \right\} + \frac{1}{2i} \int_{S} \left\{ \omega_s^o(x') - A \Delta_o \right\} \frac{\partial H_0^{(2)}(k|x-x'|)}{\partial y} dS(x')
\]

\[
= \begin{cases} \end{cases} \begin{array}{l} (R' = |x-x'|, \quad R'' = |x-x'''|), \quad x_0 \subset S \end{array} \quad (1.31)
\]

and

\[
\left[ \frac{k}{2i} \int_{-\ell}^{\ell} \gamma(x''') \left[ \frac{\partial^2}{\partial y \partial y'''} H_0^{(2)}(kr'''') \right] \bigg|_{y''' = 0} dx''' \right.
\]

\[
- \frac{1}{2i} \int_{S} t_s(x') \frac{\partial}{\partial y} H_o^{(2)}(kr') dS(x') \right]
\]

\[
= -t_{yz}^o(x,0) + \frac{k}{2i} \int_{S} \left\{ \omega_s^o(x') - A\Delta_o \right\} \frac{\partial^2}{\partial y \partial n'} H_0^{(2)}(kr') dS(x')
\]
The derivatives of the Hankel functions obviously will make the above system singular in the respective domains on the crack face or on the surface $S$ of the inclusion. The solution of these equations is handled better by first standardising the range of the variables to $(-1,1)$. This we carry out in the following section.

1.6 TRANSFORMATION OF THE VARIABLES

For the points on the crack-face $(-l-c < x < -l)$, let us introduce the variable $\eta$ by the relation

$$x = -l + \frac{1}{2} (\eta-1) c$$

where $-1 \leq \eta \leq 1$, (1.33)

Similarly for points on the elliptic surface $S$ we can introduce a variable $\beta$ such that

$$x = x_0 + a\beta$$

$$y = \pm b \sqrt{1-\beta^2} \quad \text{on} \quad S^\pm,$$ (1.34)
where $-1 \leq \beta \leq 1$. In this context it will also be convenient to identify the upper and lower parts of $S$ by the symbols $S^+$ (for $\gamma > 0$) and $S^-$ (for $\gamma \leq 0$) respectively.

In the same manner, the variable $x''$ on the crack occurring in eqns. (1.31) and (1.32) can be replaced in terms of $\eta''$ as in eqn. (1.33) and the coordinates $x'$ on $S$ can be replaced in terms of $\beta'$ as in eqn. (1.34).

Thus by making use of the above, the eqns. (1.31) and (1.32) can be expressed in terms of the new variables in the fixed range $(-1,1)$. The various derivatives of the Hankel functions can also be worked out in detail using eqns. (1.20) to (1.30). The singular and non-singular terms can be separated for convenience.

Before deriving the transformed equations we note from the standard literature on cracks, that the displacement jump function $\gamma(x)$ across the crack can be assumed to have an expected behaviour proportional to the square root of the distance from either crack tip:

$$\gamma(x) = \mathcal{G}(x) \sqrt{(x+c)(-x-c)} \quad \text{(say)} \quad (1.35)$$

$$(-l-c \leq x \leq -l)$$

where $\mathcal{G}(x)$ is an analytic function.

But on employing the transformation in eqn. (1.33), this can be put in the new form
\[ \gamma(x) = \frac{c}{2} (1 - \eta^2)^{1/2} \phi(\eta) \]

\[ (-1 \leq \eta \leq 1) \quad (1.36) \]

where \( c \) denotes the length of the crack, and \( \phi(\eta) \) is an analytic function.

1.7 THE INTEGRAL EQUATIONS IN THE REVISED COORDINATES

Using the transformation of the variables and the eqn. (1.36) of the previous section, the integral equations (1.31) and (1.32) finally transform into the following equivalent equations:

i) On the surfaces \( S \) of the elliptic inclusion we have the following relations for \( x \in S^\pm \):

\[ \pm \int_{-1}^{1} \phi(\eta''') (1 - \eta''^2)^{1/2} K_{1}(\eta''', \beta') \, d\eta''' \]

\[ - \int_{-1}^{1} t^{(1)}(\beta') \begin{cases} K_{1}(\beta, \beta'), \text{ for } x \in S^- \, \check{\sim} \\ - \frac{1}{\pi \mu} \log |\beta - \beta'| + K_{2}(\beta, \beta') \end{cases} \]

\[ \text{for } x \in S^+ \check{\sim} \]

\[ \cdot C(\beta') d\beta' \]

\[ - \int_{-1}^{1} t^{(2)}(\beta') \begin{cases} - \frac{1}{\pi \mu} \log |\beta - \beta'| + K_{2}(\beta, \beta'), \text{ for } x \in S^- \check{\sim} \\ K_{1}(\beta, \beta'), \text{ for } x \in S^+ \check{\sim} \\ \cdot C(\beta') d\beta' \end{cases} \]
\[ F^t = \left\{ A_u - w_o S^t (\beta) \right\} + F^+ (w) + F^- (w) \; ; \quad \text{where} \]

\[ F^s = \int_{-1}^{1} \left\{ w_o S^s (\beta') - A_u \right\} \cdot \left\{ \begin{aligned}
K_3 (\beta, \beta'), & \text{for } x \subset S^s \\
\frac{K_4 (\beta, \beta')}{(\beta' - \beta)} + K_5 (\beta, \beta'), & \text{for } x \subset S^s 
\end{aligned} \right. \]

\[ \cdot C (\beta') d\beta' \]

\[ (-1 \leq \beta \leq 1) \quad (1.37) \]

These are actually to be interpreted as two-part equations such that we can separately equate the parts applicable for \( S^+ \) only and \( S^- \) only where the bifurcated expressions are given. (The remaining terms are applicable on both \( S^\pm \)).

ii) On the crack face, we have the relation

\[ \int_{-1}^{1} \left[ t^{(1)} (\beta') - t^{(2)} (\beta') \right] E_1 (\eta, \beta') C (\beta') d\beta' \]

\[ = - t^{0}_{yz} (\eta) \]

\[ + \int_{-1}^{1} \left[ w^o_s (\beta') - w^o_s (\beta') \right] E_2 (\eta, \beta') \]

\[ \cdot C (\beta') d\beta' \quad (1.38) \]
for \((-1 \leq \eta \leq 1)\).

In eqns. (1.37) and (1.38) we have introduced the following notations:

\[ W_s^0(\beta) = \text{displacement} \quad W^0(x) \text{ of the incident field for } x \text{ on } S^+ \text{ and } S^- \text{ respectively (after using eqn. (1.34))} \]

\[ [t^{(1)}(\beta), t^{(2)}(\beta)] = \text{unknown shear stresses (viz. t}_S(x) \text{ in eqn. (1.25)) on } S^+ \text{ and } S^- \text{ respectively.} \quad (1.39) \]

\[ C(\beta) = (a^2 + \frac{b^2\beta^2}{1-\beta^2})^{1/2} \quad (1.40) \]

\[ A_1 = \mu \]

\[ A_2 = -\frac{k^2\mu c^2}{8} \]

All other functions are regular and these are defined in Appendix -B.

1.8 **CHEBYSHEV POLYNOMIAL EXPANSIONS**

In order to solve the equations (1.37) and (1.38) we have employed a procedure based on the Chebyshev polynomial expansions. These are well established
techniques and there is a vast published literature on this topic. However, very useful basic properties are discussed in the work of Fox and Parker (1968). Many relevant applications and useful results can also be found in the works of Erdogan (1967) and Erdogan and Gupta (1972) and many other publications from time to time.

The Chebyshev polynomials \( T_n(z) \) and \( U_n(z) \) are expressed (implicitly) by the definitions

\[
T_n(z) = \cos (n\theta)
\]

\[
U_n(z) = \frac{\sin ((n+1)\theta)}{\sin \theta}, \quad (z = \cos \theta)
\]  

and have many interesting relationships and integration formulae listed in Appendix-C.

Accordingly we attempt to express the unknown functions \( \phi(\eta) \), \( t^{(1)}(\beta) \) and \( t^{(2)}(\beta) \) occurring in eqn. (1.37) and eqn. (1.38) in terms of the following expansions.

\[
\phi(\eta) = \sum_{m=0}^{\infty} \phi_m U_m(\eta)
\]

\[
t^{(1)}(\beta) = \sum_{m=0}^{\infty} t^{(1)}_m T_m(\beta)
\]  

\[
t^{(2)}(\beta) = \sum_{m=0}^{\infty} t^{(2)}_m T_m(\beta)
\]  

(1.43)
where the expansion coefficients \( \phi_m, t_m^{(1)}, t_m^{(2)} \)
are yet to be found. The various regular functions
\( K_1(\beta, \beta'), K_2(\beta, \beta'), \ldots, K_7(\eta', \beta'), E_1(\eta, \beta'), E_2(\eta, \beta') \)
and other accompanying functions in (1.37) and (1.38) are
also expanded in each of the variables in terms of
Chebyshev polynomials as indicated in Appendix-A.

Substituting these in eqns. (1.37) and (1.38), and
making use of the integrations from Appendix-C, and com­
paring the like terms on both sides of each equation, we
finally obtain the following system of linear equations
for the expansion coefficients of eqution (1.43).

\[
A_{ij}^{(1)} t_j^{(1)} + C_{ij}^{(1)} t_j^{(2)} + B_{ij}^{(1)} \phi_j = (G_i^{(1)}+ - \Delta \phi_i^{(3)})
\]

\[
A_{ij}^{(2)} t_j^{(2)} + C_{ij}^{(1)} t_j^{(1)} - B_{ij}^{(1)} \phi_j = (G_i^{(1)}- - \Delta \phi_i^{(3)})
\]

(from conditions on \( S^+ \) and \( S^- \)), \hspace{1cm} (1.44)

and

\[
A_{ij}^{(2)} (t_j^{(2)} - t_j^{(1)}) + B_{ij}^{(2)} \phi_j = G_i^{(2)}
\]

(from the condition on the crack) \hspace{1cm} (1.45)
where the various symbols are defined in Appendix-A.

The above system is solved by truncating the summation upto a finite number of terms \((\zeta, j \leq M)\) taking advantage of the fast convergence of the Chebyshev expansions.

1.9 EVALUATION OF PHYSICAL QUANTITIES.

Once the functions \(\phi(\eta), t^{(1)}(\beta)\) and \(t^{(2)}(\beta)\) are determined from eqn. (1.43) we can easily compute the various physical quantities of interest:

i) **The displacement jump across the crack face**:

This is given by

\[
W(x) = \gamma(x)
\]

\[
= \frac{c}{2} (1-\eta^2)^{1/2} \phi(\eta), \text{(vide 1.36)}
\]

\[
\eta = 1 + 2 \frac{(x+1)}{c}, \quad -1 \leq \eta \leq 1. \quad (1.45)
\]

ii) **The stress intensity factor at the crack tips**:

The stress intensity at any crack tip \(x = x_i\) (where \(x_i = -l-c\) or \(-l\)) is defined by the following relation
\[ K_1 = \lim_{x \to x_1} \sqrt{2\pi} \frac{1}{|x-x_1|^{1/2}} t_{yz}(x,0) \]  
(Here \( x \to x_1 \) from outside the crack) \hspace{1cm} (1.47)

We note from our earlier derivations of eqn. (1.38) that the leading terms of \( t_{yz}(x,0) \) have the behaviour

\[ t_{yz}(x,0) \approx \int_{-1}^{1} \phi(\eta'')(1-\eta^2)^{1/2} \frac{A_1}{\pi|\eta-\eta''|^2} d\eta'', \quad (A_1=\mu) \]

\[ \approx + \frac{\mu \eta \phi(\eta)}{(\eta^2-1)^{1/2}}, \text{ as } \eta \to \begin{cases} -1 \to 0 & \text{or} \\ +1 \to 0 & \end{cases} \]  
(1.48)

corresponding to

\[ x \to x_1 = -l-\ell \text{ or } -l \]  
(1.49)

Thus from the above observations it follows that

\[ (K_1) = (K_L) = \mu \left( \frac{\pi c}{2} \right)^{1/2} \phi(\mp 1) \]  
(1.50)

corresponding to the left side or right side of the crack-tip.

To calculate \( \phi(\mp 1) \) in eqn. (1.50) we have used eqn. (1.43) which gives
\[ \phi(\pm 1) = \sum_{m=0}^{M} \phi_m \cdot (m+1) \]
\[ \phi(-1) = \sum_{m=0}^{M} \phi_m \cdot (m+1)(-1)^m \]

### iii) Stress distribution around the Inclusion:

The rigid inclusion will have generally different stresses on \( S^+ \) and \( S^- \). As far as the scattered part is concerned these are given by \( t^{(1)}(\beta) \) and \( t^{(2)}(\beta) \) vide eqns. (1.43).

### iv) Formal solution of eqns. (1.44) and (1.45)

We now briefly outline the solution of eqn. (1.44) and (1.45) for the coefficients \( t_m^{(1)} \), \( t_m^{(2)} \) and \( \phi_m \) which are eventually used in eqns. (1.43), (1.50) and (1.51).

From eqn. (1.45) we get

\[ \phi_i = \left[ B_{ik}^{(2)} \right]^{-1} G_k^{(2)} - \left[ B_{ij}^{(2)} \right]^{-1} A_{jl}^{(2)} \left[ t_{j}^{(2)} - t_{j}^{(1)} \right]. \]

Substituting this in eqn. (1.44) we get

\[ M_{ij} t_j^{(1)} + N_{ij} t_j^{(2)} = p_i^{(1)} - \Delta_o Q_i^{(1)} \]

\[ N_{ij} t_j^{(1)} + M_{ij} t_j^{(2)} = p_i^{(2)} - \Delta_o Q_i^{(2)} \]
where

\[
\begin{align*}
[M_{ij}] &= [A_{ij}^{(1)}] + [B_{ik}^{(1)}] [B_{kj}^{(2)}]^{-1} [A_{lj}^{(2)}] \\
[N_{ij}] &= [C_{ij}^{(1)}] - [B_{ik}^{(1)}] [B_{kj}^{(2)}]^{-1} [A_{lj}^{(2)}] \\
[P^{(1)}_i] &= [G_i^{(1)}]^+ - [B_{ij}^{(1)}] [B_{jk}^{(2)}]^{-1} [G_k^{(2)}] \\
[P^{(2)}_i] &= [G_i^{(1)}]^\text{-} + [B_{ij}^{(1)}] [B_{jk}^{(2)}]^{-1} [G_k^{(2)}]
\end{align*}
\]

and

\[
\begin{align*}
[Q_i^{(1)}] &= [G_i^{(3)^+}] \\
[Q_i^{(2)}] &= [G_i^{(3)^-}]
\end{align*}
\]

We note in passing that

\[
G_i^{(3)} = G_i^{(3)^-}
\]

so that

\[
[Q_i^{(1)}] = [Q_i^{(2)}].
\]

Adding and subtracting the relations in eqn. (1.53) we get
\(( [M] \pm [N]) ([t^{(1)}] \pm [t^{(2)}]) = ([P^{(1)}] \pm [P^{(2)}]) \)

\(- \Delta_o ([Q^{(1)}] \pm [Q^{(2)}]) \)

or

\([t^{(1)}] + [t^{(2)}] = ([M] \pm [N])^{-1} ([P^{(1)}] \pm [P^{(2)}]) \)

\(- \Delta_o ([Q^{(1)}] \pm [Q^{(2)}]) \)

\([V^{(1)\pm}] = ([M] \pm [N])^{-1} ([P^{(1)}] \pm [P^{(2)}]) \)

\([V^{(2)\pm}] = ([M] \pm [N])^{-1} ([Q^{(1)}] \pm [Q^{(2)}]) \)

from which we can easily obtain \( t_m^{(1)} \) and \( t_m^{(2)} \). Thus scattered field stresses \( t^{(1)}(\beta) \) and \( t^{(2)}(\beta) \) on \( S^+ \) and \( S^- \) can be calculated by using eqn. (1.43) with the help of the above solution for the coefficients. To get the total surface stresses one must add the incident field contribution \( t_{nz}^o \) arising from eqns. (1.5) and (1.11) on \( S^\pm \).
The working formula for $t_{nz}$ is given by

$$t_{nz} = \mu \frac{\partial w}{\partial n} = \mu (\frac{\partial x}{\partial n} \frac{\partial w}{\partial x} + \frac{\partial y}{\partial n} \frac{\partial w}{\partial y}) \quad (1.58)$$

where the derivatives $(\partial X/\partial n, \partial Y/\partial n)$ are easily found on $S$. In fact we note that

$$\frac{\partial x}{\partial n} = \frac{x-x_0}{a^2 T(x)}, \quad \frac{\partial y}{\partial n} = \frac{y}{b^2 T(x)}, \quad \text{for } (x,y) \in S \quad (1.58)$$

where $T(x) = \left\{(x-x_0)^2/a^4 + y^2/b^4\right\}^{1/2} \quad (1.58)$

The coefficients $\phi_m$ are finally found from (1.52) on making use of (1.57)b. This helps the computation of (1.50) and (1.51) for the crack-tip stress intensity factors. We also note that due to eqn. (1.56)b, $[V^{(2)}] = 0$ in (1.57)c.

1.10 **DISCUSSION OF THE RIGID BODY DISPLACEMENT OF THE INCLUSION**

The rigid body displacement factor $\Delta_o$ assumed for the elliptic inclusion as in eqn. (1.14) needs more discussion. There are two possibilities viz., that $\Delta_o$ is arbitrarily imposed from external sources (usually at $|z| = \infty$) so that for all practical purposes the two dimensional assumption can be retained. Alternatively
one can consider \( \Delta_0 \) as part of the solution to be determined in terms of the effects of the incident field, eqn. (1.11), and the elastic stresses developed in the medium especially on the surfaces \( S^\pm \) of the inclusion. The latter are given by \( t^{(1)}(\beta) \) and \( t^{(2)}(\beta) \) of equation (1.43) in addition to the incident field terms. We choose the second alternative in our following discussions. Accordingly, taking the mass of the elliptic inclusion per unit width (in the z-direction) to be \( M_0 \), the equation governing the motion of this mass can be seen to be

\[
M_0 \frac{d^2}{dt^2} \left\{ A \Delta_0(t) \right\} = \int_{S^+} t^{(1)}(\beta) - t^{(2)}(\beta) + \Delta t_0(\beta) \, dS(\beta)
\]

\[
\equiv \sigma_{M_0}, \quad \text{say}
\]

where \( \sigma_{M_0} \) is the net shear traction acting on the inclusion in the z-direction per unit length. The \( \Delta t_0(\beta) \) term corresponds to the incident field contribution. As all physical quantities are taken proportional to the harmonic time factor \( e^{i\omega t} \), eqn. (1.59) can be written as

\[
(-M_0 \omega^2) A \Delta_0 \equiv (\alpha_{\text{inc}} + \alpha_{\text{sc}}) \equiv \alpha \text{ (say)}
\]
where we readily see that

\[ \alpha_{sc} = \int_{s^+} [t^{(1)}(\beta) - t^{(2)}(\beta)] d\delta(\beta) \]

\[ = \int \left( \sum_{i=0}^{N} V^{(1)}_i - T^{(1)}_i(\beta) \right) C(\beta) d\beta \]

(vide (1.43) and (1.57)).

(1.61a)

\[ \alpha_{inc} = \int_{-1}^{1} [t^+_{on}(\beta) - t^-_{on}(\beta)] C(\beta) d\beta \]

(1.61b)

with

\[ C(\beta) = \left[ a^2 + \frac{b^2 \beta^2}{1-\beta^2} \right]^{1/2} \].

(1.61c)

In effect, eqn. (1.60) gives us the required value of \( \Delta_0 \) as

\[ \Delta_0 = - \left( \frac{\alpha}{\omega \omega^2} \right) \].

(1.62)

Our focus is in approximations for small frequencies and eqn. (1.62) can be used to obtain \( \Delta_0 \) without much difficulty. When \( M_0 \to \infty \), clearly \( \Delta_0 \to 0 \) as expected. We also note that we can write (1.62) in another useful form viz.,

\[ \Delta_0 = -\left( \frac{\alpha}{\omega \omega^2} \right) \frac{A(\pi \omega)(\rho_0/\rho)k^2}{A(nab)(\rho_0/\rho)k^2} \]

(1.63)

\( \alpha = \alpha_{sc} + \alpha_{inc} \)
Note that \( k = (\omega/V_s) \) is the wave number defined in eqn. (1.10), and also \( v_s^2 = \mu/\rho \) by definition. One can thus calculate \( \Delta_0 \) for various density ratios and area of the elliptic inclusion.
1.11. NUMERICAL DISCUSSION AND CONCLUSIONS

The simultaneous equations (1.44) and (1.45) are first solved as in (1.52) - (1.57), taking various parametric combinations for the wave number \(k\) and the angle of incidence \(\theta_0\) of the plane wave. There are also other parameters like crack length \(c\), elliptic axes \((a,b)\), and the distances \((L, x_0)\) associated with the right crack tip and centre of the inclusion respectively from the origin of the coordinates. Obviously we are dealing with a large variety of parameters on the one hand and with a number of physical quantities to be computed in terms of these, on the other. For convenience we have restricted our numerical work only to a variation in some of them, e.g. angle of incidence of the plane wave, and the wave number \(ka\) (non-dimensionalised w.r.t. the semi major axis \(a\) of the inclusion). All other parameters have been fixed for our calculations. Any variation in them can obviously be included without difficulty.

Some of the fixed parameter-values chosen are given below

\[
\begin{align*}
\frac{b}{a} &= 0.5 & \frac{c}{a} &= 1.0 \\
\frac{L}{a} &= -0.75 & \frac{x_o}{a} &= 2.0
\end{align*}
\]

For all our calculations we take \(A = 1\) in the incident field.
As remarked in an earlier section we use the truncation of series method because the expansions in Chebyshev polynomials are rapidly convergent. (We take normally up to eleven terms).

The physical quantities of interest are listed below:

i) Stress Intensity Factors at the two crack tips \((y = 0, x = -\ell)\) and \((y = 0, x = -(-c))\) for the following cases
   - only due to the incident field interacting with crack;
   - effect of the rigid inclusion only as secondary field;
   - combined case;

ii) Displacement jump across the crack faces

iii) Stress-differences across the inclusion, and

iv) Effect of \(\Delta_0\) on stress around the inclusion, etc.

The results obtained are summarised below:

**Displacement jumps across the Crack**

Figs. 1.2 and 1.3 show the absolute values of
the displacement jump across the crack due to an isolated crack only for $\Theta_0 = 30^\circ$ and $60^\circ$ respectively. The value increases with $ka$ in both the cases but the values are higher at $60^\circ$ as compared to $30^\circ$. The absolute values of displacement jumps across the crack due to the secondary fields created by the inclusion for $\Theta_0 = 30^\circ$, $60^\circ$ respectively are shown in Figs. 1.4 and 1.5, each covering $ka = 0.2$ to $1.4$. These are comparatively much smaller than the isolated crack case. For $\Theta_0 = 30^\circ$ and $ka = .6$ and $1.4$ the values are fairly close. The values are high at $ka = 1.0$. Again for $\Theta_0 = 60^\circ$ and $ka = 1.0$ the values are higher than $ka = 1.4$. In both cases while the value increases with $ka = .2$ to $1.0$ it decreases for $ka > 1.0$. The values for $\Theta_0 = 60^\circ$ are basically higher than those for $\Theta_0 = 30^\circ$.

In Figs. 1.6 and 1.7 the displacement jumps due to combined effect across the crack are plotted for $\Theta_0 = 30^\circ$ and $60^\circ$ ($ka = 0.2$ to $1.4$). The values of displacement jumps are lower than the corresponding case for isolated crack (Fig. 1.2 and 1.3). This indicates that the fields caused by the crack scattering and the secondary effect from the inclusion mutually do not reinforce at least for the low frequency range considered.
In the combined model, the case for $\theta_o = 60^\circ$ has higher values compared to $\theta_o = 30^\circ$ case.

**Stress Intensity factors at the crack tips**

The SIF at the two tips due to pure crack vs. $ka$ for different $\theta_o$ are shown in Figs. 1.8 and 1.9. Basically, the values are comparable at both the tips except for $\theta_o = 30^\circ$ where there is some noticeable difference. The SIFs generally increase with $ka$ as well as with $\theta_o$ up to $90^\circ$. After $90^\circ$ the values again fall. Only absolute values have been plotted for illustration.

Figs. 1.10 and 1.11 show the SIFs at the crack tips due to the secondary fields from the rigid inclusion. The SIF values generally increase with $\theta_o$ till $90^\circ$ and then decrease with reference to $ka$ variation. The secondary effect is again basically very small compared to the isolated crack case.

Combined values of SIFs vs. $ka$ for different $\theta_o$'s are plotted in Figs. 1.12 and 1.13. The interaction between direct and secondary fields is mainly seen in an overall increase in the value of SIFL at $\theta_o = 60^\circ$ and $120^\circ$ while it is reduced for $\theta_o = 90^\circ$. SIFRs have very distinct behaviour for $\theta_o = 30^\circ$, $60^\circ$, and $90^\circ$. The combined value is much reduced, while
for \( \Theta_o = 120^\circ \), the value is larger than the pure crack or secondary effect only.

In Fig. 1.14 we illustrate the typical plot of SIFL vs. \( \Theta_o \) at \( x = - \ell - c, y = 0 \) for convenience, keeping the values of \( k \alpha \) fixed for each separate curve. The conclusions drawn earlier apply for these curves, as such.

**Stresses on Elliptic Boundary**

Figs. 1.15 to 1.20 describe the stresses on \( S^+ \) and \( S^- \) \((t^1 \text{ and } t^2)\) and the stress difference across the inclusion under various conditions such as:

a) due to incident wave only

b) due to scattered field (without the rigid body motion).

c) effect of rigid body motion on stresses, etc.

It can be observed that the stress differences are generally much smaller than the stresses themselves. The stress difference in case of scattered field shows conspicuous oscillations across the span of the major axis of the inclusion as compared to the incident field (See Figs. 1.15 to 1.17).
Figs. 1.18 and 1.19 show the stresses on $S^+$ and $S^-$ (without $\Delta_0$ part) due to scattered fields. Fig. 1.20 shows the scattered stress on $S^+$ including $\Delta_0$ part. The value on $S^-$ is almost identical and hence not shown separately.

For the sake of illustration the absolute values of the rigid body displacement function $\Delta_0$ calculated for some values of the parameters held in our study are given in Table 1.1.

<table>
<thead>
<tr>
<th>ka</th>
<th>$30^\circ$</th>
<th>$60^\circ$</th>
<th>$90^\circ$</th>
<th>$120^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.361</td>
<td>33.57</td>
<td>6.201</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>2.054</td>
<td>4.895</td>
<td>2.084</td>
<td>3.321</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8095</td>
<td>2.309</td>
<td>1.253</td>
<td>2.313</td>
</tr>
<tr>
<td>1.4</td>
<td>0.7069</td>
<td>1.6882</td>
<td>0.8376</td>
<td>1.960</td>
</tr>
</tbody>
</table>
APPENDIX - A

A brief description of the coefficient matrices occurring in eqns. (1.44) and (1.45) are given below

\[
\begin{align*}
A_{ij}^{(1)} & = - \frac{\pi}{2} \gamma_1 e_{ij}^{(4)} \mu_j y_j + q_{ij} \\
B_{ij}^{(1)} & = \frac{\pi}{2} \gamma_1 e_{ij}^{(7)} \\
C_{ij}^{(1)} & = - \frac{\pi}{2} \gamma_1 y_j \mu_j e_{ij}^{(3)} \\
G_{i}^{(1)+} & = - \gamma_1 d_{i}^{(k)} + \pi \sum_{m=1}^{M} C_{m}^{(k)} \gamma_{m} s_{i}^{(m)} \\
& \quad + \gamma_1 \sum_{n=0}^{M} d_{in}^{(5)} \gamma_{n} D_{n}^{(k)} \in_{n} \\
& \quad + \gamma_1 \sum_{n=0}^{M} d_{in}^{(3)} \gamma_{n} t_{n}^{(k)} \in_{n} \\
G_{i}^{(2)} & = - g_{i}^{(1)} y_{i} + g_{i}^{(2)} y_{i} \\
A_{ij}^{(2)} & = \frac{\pi}{2} \gamma_1 e_{ij}^{(1)}
\end{align*}
\]
\[
B_{ij}^{(2)} = \frac{A_2}{\pi} p_i \delta_{ij} - \frac{A_2}{\pi} \delta_{i-2,j} q_{i-2}^H(i-2)
\]

\[
+ \frac{A_2}{\pi} \delta_{i_0} r_{ij}^e + \frac{\pi}{2} \gamma_1 k_{ji}^{(6)}
\]

\[
+ A_1 \sum_{N=0}^{\infty} \lambda_{N,i} \delta_{i+2N,j}
\]

where

\[
\gamma_0 = 0.5, \gamma_1 = 1, i \geq 1
\]

\[
\mu_0 = 2, \mu_j = 1, j \geq 1
\]

\[
\epsilon_0 = \pi, \epsilon_n = \frac{\pi}{2} (n \geq 1)
\]

\[
\delta_{ij} = \begin{cases} 
1, & i = j \\
0, & i \neq j
\end{cases}
\]

\[
\lambda_{N,m} = (m+2N+1) \times \begin{cases} 
1, & m = 0 \\
2, & m \geq 1
\end{cases}
\]

\[
A_1 = \mu, A_2 = -\frac{K^2 \mu e^2}{8}
\]

\[
\bar{q}_{ij} = \sum_{k=0}^{\infty} A_{ik} h_{kj}
\]
\[ A_{ik} = \frac{\alpha_i}{\pi} \delta_{ik} - \frac{\delta_{i0}}{\pi} \beta_k \]

\[ \alpha_o = -\pi \log_2, \quad \alpha_m = \frac{\pi}{m}, \quad (m \geq 1) \]

\[ p_n^e = \begin{cases} \frac{2\pi}{n} (-1)^{n/2}, & n = 2, 4, 6, \ldots \\ 0, & n = 0, 1, 3, 5, \ldots \end{cases} \quad (A.4) \]

\[ p_n = \frac{\pi}{2n}, \quad q_n = \frac{\pi}{2(n+2)} \quad (n \neq 0) \]

\[ p_o = -\frac{\pi}{2} (1+\log 2), \quad q_o = \frac{\pi}{4} \]

\[ H(N) = \begin{cases} 1, & N \geq 0 \\ 0, & N < 0 \end{cases} \]

\[ \gamma_n^e = \begin{cases} \frac{n+2}{2} (-1)^{\frac{n}{2}}, & n = 2, 4, 6, \ldots \\ \frac{2\pi(n+1)}{(n+1)^2-1}, & n = 2, 4, 6, \ldots \end{cases} \]

\[ \tau_{(1)}^n = D_{(1)}^n, \quad \tau_{(2)}^n = D_{(2)}^n \]

\[ s_{n}^{(m)} = 0, \quad n \geq m \]
\[ s_n^{(m)} = \frac{1}{m} \sum_{N=0}^{\infty} \lambda_{N,m}^{n+2N} \delta_{n+2N, m-1}, \quad (m \geq 1) \quad (A.5) \]

\[ C_i^{(3)+} = G_i^{(3)-} = -\delta_{i0} + \pi\gamma_1 [d_i^{(5)} + d_i^{(3)}] \]

\[ = \pi[C_{i-1}^{(4)}] \text{ for } i \geq 1 \quad (A.6) \]

(with \( C_{-1}^{(4)} = 0 \))

where \( C_{-1}^{(4)} \) are the expansion coefficients associated with the function \( C_4(\beta') \) defined below.

\[ C_4(\beta') = \frac{K_4(\beta', \beta') C(\beta')}{\sqrt{1-\beta'^2}} \]

\[ = \sum_{m=0}^{N} C_m^{(4)} U_m(\beta') \quad \text{(say)} \quad (A.7) \]

where

\[ C_m^{(4)} = \frac{2}{N+1} \sum_{i=1}^{N} C_4(\beta_i) U_m(\beta_i) (1-\beta_i^2) \]

\[ \beta_i = \cos\left(\frac{i\pi}{N+1}\right), \quad i = 1, 2, \ldots, N \quad (A.8) \]
Some of the useful expansions are next given below. The various functions occurring in these have been described in Appendix B.

\[ D^{(1,2)}_n \] are the expansion coefficients in

\[ w^o_{s^\pm}(\beta) = \sum_{m=0}^{M} \gamma_m D^{(1,2)}_m T_m(\beta) \quad (A.9) \]

which can be calculated from

\[ D^{(1,2)}_m = 2^M \sum_{i=1}^{M} w^o_{s^\pm}(\eta_i) T_m(\eta_i) \quad (A.10) \]

where \( \eta_i = \cos \left( \frac{(2i-1)\pi}{2M} \right) \).

Similarly \( e^{(N)}_{ij} \) are defined in terms of the expansion coefficients (in two coordinates) of the functions

\[
\begin{bmatrix}
  E_1(\eta, \beta') \\
  E_2(\eta, \beta') \\
  K_1(\beta, \beta') \\
  K_2(\beta, \beta')
\end{bmatrix}
\cdot 
\begin{bmatrix}
  C(\beta')(1-\beta'^2)^{1/2}
\end{bmatrix}
\equiv
\begin{bmatrix}
  e_1(\eta, \beta') \\
  e_2(\eta, \beta') \\
  e_3(\beta, \beta') \\
  e_4(\beta, \beta')
\end{bmatrix}
\quad (A.11)
\]
These are expanded in the form

\[ e_k(x, y) = \sum_{n=0}^{M} \sum_{m=0}^{M} \gamma_n \gamma_m e^{(k)}_{mn} T_m(x) T_n(y) \]

\[ e^{(k)}_{mn} = (\frac{2}{N+1})^{2} \sum_{i=1}^{M} \sum_{j=1}^{M} e_k(\eta_i, \eta_j) T_m(\eta_i) T_n(\eta_j) \cdot (A.12) \]

\[ K_{ij}^{(6)} \text{ and } K_{ij}^{(7)} \text{ are defined in similar manner:} \]

\[ K_6(\eta, \eta^{''}) = \sum_{m=0}^{N} \sum_{n=0}^{M} \gamma_n K_{mn}^{(6)} U_m(\eta^{''}) T_n(\eta) \cdot (A.13) \]

\[ K_7(\eta^{''}, \beta) = \sum_{m=0}^{N} \sum_{n=0}^{M} \gamma_n K_{mn}^{(7)} U_m(\eta^{''}) T_n(\beta) \]

\[ K_{mn}^{(6)} = (\frac{2}{N+1})^{2} (\frac{2}{M})^{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \{ K_6(\eta_i, \beta_j) U_m(\beta_i) \cdot T_n(\eta_j) \cdot (1-\beta_i^2) \} \]

\[ K_{mn}^{(7)} = (\frac{2}{N+1})^{2} (\frac{2}{M})^{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \{ K_7(\beta_i, \eta_j) U_m(\beta_i) \cdot T_n(\eta_j) \cdot (1-\beta_i^2) \} \cdot (A.14) \]
where \( \beta_1 = \cos\left(\frac{\pi i}{N+1}\right) \), \( i = 1, 2, \ldots, N \) while \( \eta_i \) is defined in (A.10). \( C_m^{(k)} \) in (A.2) arise from expanding the following

\[
C^{(k)}(\beta') = w^\circ(\beta') \mathcal{K}_4(\beta', \beta') C(\beta')(1-\beta^2)\frac{1}{2}, \quad (k = 1, 2)
\]

\[
= \sum_{m=0}^{M} \gamma_m T_m(\beta') C_m^{(k)} \quad (A.15)
\]

so that

\[
C_m^{(k)} = \sum_{i=1}^{M} 2 M \sum_{m=1} M C^{(k)}(\eta^1_i) T_m(\eta^1_i) \quad (A.16)
\]

Finally \( g_1^{(1)}, g_1^{(2)} \) occurring in (A.2) are defined as follows in terms of the incident field contributions

\[
g_1(\eta) = t^\circ_{yz}(\eta) \text{ on the crack}
\]

\[
= \sum_{m=0}^{M} \gamma_m g^{(1)}_m T_m(\eta) \quad (A.17)
\]

so that

\[
g^{(1)}_m = \sum_{i=1}^{M} \sum_{i=1}^{M} g_1(\eta^1_i) T_m(\eta^1_i) \quad (A.18)
\]
similarly, if

\[ g_2(\eta) = \int_{-1}^{1} \left[ W^{o}(\beta') - W^{o}_{-}(\beta') \right] E_2(\eta, \beta') \cdot \right. \\
\left. \cdot C(\beta') d\beta' \right]

\[ = \sum_{m=0}^{M} \gamma_m g_m^{(2)} T_m(\eta) \]  

(A.19)

then

\[ g_m^{(2)} = \frac{2}{M} \sum_{i=1}^{M} g_2(\eta_i) T_m(\eta_i) \]  

(A.20)

It is easily shown that if

\[ f_{w}(\beta') \equiv \frac{W^{o}(\beta') - W^{o}_{-}(\beta')}{S^+ - S^-} \]

\[ = \sum_{m=0}^{M} \gamma_m \alpha_m^{(1)} T_m(\beta') \]  

(A.21)

so that

\[ \alpha_m^{(1)} = \frac{2}{M} \sum_{i=1}^{M} f_{w}(\eta_i) T_m(\eta_i) \]  

(A.22)

then

\[ g_m^{(2)} = \sum_{n=0}^{M} \gamma_n \alpha_n^{(1)} e_n e_{mn} ^{(2)} \]  

(A.23)
The symbols $d^{(p)}_{mn}$ occurring in (A.2) are defined below:

Let

$$\gamma 1 - \beta, 2 K_p (\beta, \beta') C(\beta') = d_p (\beta, \beta') \quad (A.24)$$

($p = 3$ or $5$). Then we expand $d_p (\beta, \beta')$ as follows:

$$d_p (\beta, \beta') = \sum_{m=0}^{M} \sum_{n=0}^{M} d^{(p)}_{mn} \gamma_n T_m (\beta) t_n (\beta') \quad (A.25)$$

so that

$$d^{(p)}_{mn} = (\frac{2\gamma}{M})^2 \sum_{i=1}^{M} \sum_{j=1}^{M} d_p (\eta_i, \eta_j) T_m (\eta_i) T_n (\eta_j) \quad (A.26)$$

$$\eta_i = \frac{(2i-1)\pi}{2M}; \quad (i = 1, 2, \ldots, M)$$

($p = 3$ or $5$)

In a similar manner the symbol $h_{kj}$ occurring in the definition of $\bar{d}_{ij}$ in (A.4) is explained as follows:

$$h_{mn} = \frac{2\gamma_n}{M} \sum_{i=1}^{M} t_n (\eta_i) T_m (\eta_i) f(\eta_i) \quad (A.27)$$

where

$$f(\eta) = \frac{1}{\mu} C(\eta) V 1 - \eta^2 \quad (A.28)$$
Some of the functions occurring in the text are given below

\[ K_1(\beta, \beta') = \frac{1}{2i\mu} \left\{ J_0(kR) \left[ 1 - \frac{2i}{\pi} \left( y + \log \frac{kR}{2} \right) \right] + \frac{2i}{\pi} S(kR) \right\} \]

\[ K_2(\beta, \beta') = -\frac{1}{\mu\pi} \log(aV_{21}) \]

\[ K_3(\beta, \beta') = -\frac{k}{2iT} \left[ \frac{X_1'((\beta'-\beta)}{aR} - \frac{D|Y'|}{bR} \right] \cdot \tilde{H}_1^{(2)}(kR) \]

\[ K_4(\beta, \beta') = -\frac{k}{2iT} \left[ \frac{X_2'}{a^2} \chi_2 + \frac{|Y'|}{b^2} \chi_3 \right] \frac{2i}{\pi kaV_{21}} \]

\[ K_5(\beta, \beta') = -\frac{k}{2iT} \left[ \frac{X_3'}{a^2} \chi_2 + \frac{|Y'|}{b^2} \chi_3 \right] \cdot F_2(ka|\beta-\beta'|V_{21}) \]

\[ K_4(\beta, \beta') - K_4(\beta', \beta') \begin{cases} \begin{array}{ll} S_{\beta-\beta'} & |\beta-\beta'| \end{array} \end{cases} \]
\[ K_\delta(\eta, \eta') = \frac{k^2 \mu g^2}{8i} \left[ -\frac{1}{\pi} \log \left( \frac{k\xi}{2} \right) + F_4(\eta, \eta') \right] \]

\[ K_\gamma(\eta', \beta) = \frac{k}{2i} \left( \frac{\xi}{2} \right)^2 \Gamma_{11} \cdot \Gamma_{12} \]

\[ E_1(\eta, \beta') = \frac{k}{2i} \frac{b(1-\beta'^2)^{1/2}}{R_3} H_1(2)(kR_3) \]

\[ E_2(\eta, \beta') = -\frac{k}{2i} \left[ \frac{x'b(1-\beta'^2)^{1/2}}{a^2 T} \right] \left\{ -\frac{kH_1'(2)(kR_3)}{R_3^2} \right. \]

\[ + \left. \frac{H_1(2)(kR_3)}{R_3^3} \right\} (x'^2 - x^2) + \frac{(1-\beta'^2)}{bT} \]

\[ \times \left\{ \frac{kH_1'(2)(kR_3) b^2(1-\beta'^2)}{R_3^2} - \frac{H_1(2)(kR_3)}{R_3} \right. \]

\[ + \left. \frac{H_1(2)(kR_3)}{R_3^3} \cdot (1-\beta'^2)b^2 \right\} \]  \hspace{1cm} \text{(8.2)}

where \( J_n(z) \) is the Bessel function of the first kind, and \( Y = 0.5772 \)

\[ R = [a^2(\beta-\beta')^2 + b^2 \left\{ (1-\beta^2)^{1/2} + (1-\beta'^2)^{1/2} \right\}^2]^{1/2} \]
\[ S(z) = \sum_{m=1}^{\infty} (-1)^m \left( \frac{z}{2} \right)^{2m} \frac{2m}{(m!)^2} \]

\[ P_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \]

\[ V_{21} = \left[ 1 + \frac{b^2}{a^2} \cdot \frac{D^2}{|\beta - \beta'|^2} \right]^{1/2} \]

\[ D = (1 - \beta^2)^{1/2} - (1 - \beta'^2)^{1/2} \]

\[ F_1(z) = \frac{1}{21\mu \pi} \left[ - \frac{2i}{\mu} \log k + J_0(z) \left\{ 1 - \frac{2i}{\mu} (\gamma - \log z) \right\} \right. \]

\[ + \frac{2i}{\pi} S(z) - \frac{2i}{\pi} (J_0(z) - 1) \log z \]

\[ F_2(z) = J_1(z) \left\{ 1 - \frac{2i}{\mu} (\gamma + \log (z/2)) \right\} \]

\[ + \frac{2i}{\pi} \frac{J_0(z)}{z} - \frac{2i}{\pi} S'(z) \]

\[ F_4(\eta, \eta') = \sum \left( \frac{k c}{2} \left| \eta - \eta' \right| \right) \]

\[ \Omega(z) = \left( \frac{J_1(z)}{z} - 1 \right) \left( - \frac{2i}{\mu} \log z \right) \]

\[ + \frac{J_1(z)}{z} \left\{ 1 - \frac{2i}{\mu} (\gamma - \log 2) \right\} \]

\[ - \frac{2i}{\pi} \frac{S'(z)}{z} + \frac{2i}{\pi z^2} (J_0(z) - 1) \]
\[ V_{11} = |\beta|/R, \quad V_{12} = H_1^{(2)}(kR_1) \]

\[ R_1 = \left[ (x-x')^2 + y^2 \right]^{1/2}, \]

with

\[ x' = x_0 + a\beta \]
\[ y = \pm b(1-\beta^2)^{1/2} \quad \text{on } \mathbb{S}^+ \]

\[ R_3 = \left[ (x-x')^2 + b^2(1-\beta^2) \right]^{1/2} \]

with

\[ x' = x_0 + \gamma \beta' \quad , \quad x' = a\beta' \]

\[ x = -\ell + \frac{c}{2}(\eta'-1) \]
\[ T = \left\{ \frac{\beta^2}{a^2} + \frac{(1-\beta^2)^2}{b^2} \right\}^{1/2} \]

\[ \chi_2 = \frac{\operatorname{sign}(\beta'-\beta)}{V_21(\beta, \beta')} \]

\[ \chi_3 = \frac{b}{a} \left[ \frac{(1-\beta^2)^{1/2}-(1-\beta'^2)^{1/2}}{|\beta-\beta'|} \right] \frac{V_21(\beta, \beta')}{V_21(\beta', \beta')} \]

Finally, \( S'(z) \), \( H_n^{(2)}(z) \) etc. denote the derivatives with respect to argument.
In numerical analysis one often needs to represent a continuous function in terms of expansions using simple power series or other base-polynomials. It is known from literature that of all the polynomials of degree \( n \), with leading coefficient unity, only Chebyshev polynomial leads to the minimax property of errors. This means that the maximum value in the range \((-1 < x < 1)\) for a polynomial \( P_n(x) \) is minimum only when \( P_n(x) \) is a suitable multiple of \( T_n(x) \), the Chebyshev polynomial. In addition, of all the expansions in terms of ultra-spherical polynomials, the Chebyshev series will generally have the fastest rate of convergence while the Taylor's series has the slowest rate. Such an expansion also has the minimum error in the least square sense. There are both continuous as well as discrete least square approximations for the expansion coefficients. We use the latter one to be described later in eqns. (C.9) and (C.10). For more information one may refer to Fox and Parker (1968).

We define the chebyshev polynomials \( T_n(x) \) and \( U_n(x) \) by the formulae
\[ T_n(x) = \cos(n\theta) \quad \text{and} \quad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad (x = \cos\theta) \quad (C.1) \]

In detail, we get

- \[ T_0(x) = 1 \]
- \[ T_1(x) = x \]
- \[ T_2(x) = 2x^2 - 1 \quad (C.2) \]
- \[ U_0(x) = 1 \]
- \[ U_1(x) = 2x \]
- \[ U_2(x) = 4x^2 - 1, \ldots \quad (C.3) \]

These satisfy the following differential equations

- \[ (1-x^2) y''' - xy'' + y = 0 \quad (C.4) \]
for \( y = T_n(x) \), and

- \[ (1-x^2) y''' - 3xy' + n(n+2)y = 0 \quad (C.5) \]
for \( y = U_n(x) \).

These polynomials also satisfy the recurrence relations

- \[ T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x) \quad (C.6) \]
- \[ T_0(x) = 1, \quad T_1(x) = x \]
and

$$U_{n+1}(x) = 2x U_n(x) - U_{n-1}(x) \quad (C.7)$$

$$U_0(x) = 1, \quad U_1(x) = 2x$$

We also have the relations

$$T'_x(x) = n U_{n-1}(x)$$

$$(1-x^2)U'_n(x) = x U_n(x) - (n+1)T_{n+1}(x)$$

$$U_m(x) = \begin{cases} 2 \left[ T_1(x) + T_3(x) + \cdots + T_m(x) \right], & \text{if } m \text{ is odd} \\ T_0(x) + 2 \left[ T_2(x) + T_4(x) + \cdots + T_m(x) \right], & \text{if } m \text{ is even} \end{cases} \quad (C.8)$$

**EXPANSION OF FUNCTIONS:**

We also make use of the following expansion formulae

$$f(x) \approx p_n(x) = \sum_{r=0}^{\infty} C_r T_r(x) \quad (C.9)$$

with expansion coefficients given by the discrete form

$$C_r = \frac{2}{N+1} \sum_{k=0}^{N} f(x_k) T_n(x_k) \quad (C.10)$$

$$x_k = \cos \left( \frac{(2k+1)\pi}{2(N+1)} \right)$$
The prime in (C.9) over the summation sign denotes that an extra fraction of 1/2 is present at the leading term for \( r = 0 \).

The above approximation satisfies the discrete least square criterion

\[
S = \sum_{k=0}^{N} e_n^2(x_k) = \text{minimum}, \quad (e_n = |f - p_n|); \quad (C.11)
\]

with

\[
S = \sum_{k=0}^{N} \left\{ f^2(x_k) - \sum_{r=0}^{N} c_r^2 U_r^2(x_k) \right\}. \quad (C.12)
\]

For a proper interpolation formula we take \( n = N \) in equation (C.9). This then means that \( P_n(x) \) will fit \( f(x) \) exactly at the points \( x_k \), with \( k = 0,1,2,3,\ldots,N \).

In a similar manner we also have expansions in terms of the Chebyshev polynomials of the second kind \( U_n(x) \). These are given by

\[
f(x) \approx q_n(x) = \sum_{r=0}^{n} c_r U_r(x) \quad (C.13)
\]

\[
c_r = (2/N+1) \sum_{i=1}^{N} U_r(\beta_i)(1-\beta_i^2) \quad (C.14)
\]
where

\[ \beta_i = \cos \left( \frac{i\pi}{N+1} \right), \quad i = 1, \ldots, N \quad (C.15) \]

We also use double expansions based on two independent variables, such as

\[ f(x,y) = \sum c_i \beta_j T_i(x) T_j(y) \]

\[ g(x,y) = \sum d_{is} T_i(x) U_s(y) \quad (C.16) \]

which can be easily carried out on the basis of the above discussion. Some of the relevant results can be seen in Appendix-A.

**SOME INTEGRALS INVOLVING CHEBYSHEV POLYNOMIALS**

Finally, we have made use of certain well known integrals involving the Chebyshev polynomials. These are summarised below.

\[ \int_{-1}^{1} U_m(x) U_n(x)(1-x^2)^{1/2} dx = \begin{cases} \pi/2, & m = n \\ 0, & m \neq n \end{cases} \]
\[
\int_{-1}^{1} \frac{T_m(x)T_n(x)\,dx}{\sqrt{1-x^2}} = \begin{cases} 
0 & , \ m \neq n \\
\pi & , \ m = n = 0 \\
\pi/2 & , \ m = n = 0
\end{cases}
\]

\[
\int_{-1}^{1} \frac{U_m(x)\sqrt{1-x^2}}{(x-x)} \, dx = \pi T_{m+1}(x)
\]

\[
\int_{-1}^{1} \frac{T_m(x)\,dx}{\sqrt{(1-x^2)(x-x)}} = \begin{cases} 
0 & , \ n = 0 \\
-\pi U_{n-1}(x) & , \ n > 1
\end{cases}
\]

\[
\int_{-1}^{1} U_m(x)\sqrt{1-x^2} \log |x-x| \, dx = -\left(\frac{\pi}{2}\right)(x^2+1/2+\log 2), \ m=0
\]

\[
\frac{\pi}{2} \left( \frac{T_m(x)}{m} - \frac{T_{m+2}(x)}{m+2} \right), \ m=1,3,5,\ldots
\]

\[
\frac{\pi}{2} \left( \frac{T_m(x)}{m} - \frac{T_{m+2}(x)}{m+2} \right) + \frac{(-1)^{(m+2)/2}}{2\pi(m+1)} \cdot \frac{(m+1)^2}{(m+1)^2-1}, \ m=2,4,6,\ldots
\] 

(c. 21)
and

\[
\int_{-1}^{1} \frac{T_m(x) \log |x-x_0| \, dx}{\sqrt{1-x^2}} = \begin{cases} 
-\pi \log 2, & m = 0 \\
\frac{\pi}{m} T_m(x), & m=1, 3, 5, \ldots \\
\frac{\pi}{m} T_m(x) - \frac{(2\pi/m)(-1)^{m/2}}{2}, & m=2, 4, \ldots 
\end{cases}
\]

(\mathcal{C.22})
Fig. 1.1

(Geometry of the problem)
Fig. 1.2
Displacement jump (absolute value) across crack faces due to plane wave incident on isolated crack only.
($x = x + \frac{\ell}{c}$) ($\theta_o = 30^\circ$, $ka = 0.2, \ldots, 1.4$).
Same as in Fig. 1.2 for $\theta_o = 60^\circ$. 

Fig. 1.3
Displacement jump on crack faces due to secondary field created by the inclusion \( \theta_0 = 30^\circ \)

\( ( \bar{x} = \frac{x + \ell + c}{c} ) \) \( \quad (ka = 0.2, \ldots, 1.4) \).
Fig. 1.5

Same as in Fig. 1.4 for $\theta_o = 60^\circ$. 
Displacement jump (absolute value) across crack faces due to the combined effect ($\bar{x} = \frac{x + \ell}{c}$)

$\phi_0 = 30^\circ$, $ka = 0.2$ to $1.4$. 

Fig. 1.6
Fig. 1.7

Same as in Fig. 1.6 for $\theta_0 = 60^\circ$. 

\[ \frac{|f(x)|}{a} \rightarrow \]

\[ k_\alpha = 1.4 \]

\[ 1.0 \]

\[ 0.6 \]

\[ 0.2 \]

\[ 0.0 \]

\[ 0 \]

\[ 0.1 \]

\[ 0.2 \]

\[ 0.3 \]

\[ 0.4 \]

\[ 0.5 \]

\[ 0.6 \]

\[ 0.7 \]

\[ 0.8 \]

\[ 0.9 \]

\[ 1 \]
Stress Intensity Factor (at left tip) (absolute value) vs ka due to pure crack also (at \( x = - \ell - c \)) (\( \theta_0 = 30^\circ \) to \( 120^\circ \)).
Stress Intensity Factor (at right tip, absolute value) vs ka due to pure crack only (at x = -l)

\( \theta_0 = 30^\circ \) to \( 120^\circ \).
Stress Intensity Factor (absolute value) at $x = -\ell - c$ vs $ka$ due to secondary field for the inclusion ($\theta_o = 30^\circ$ to $120^\circ$).
Fig. 1.11
Stress Intensity Factor (absolute value) at $x = -L$ vs $ka$ due to secondary field for the inclusion. Other parameters as in Fig. 1.10.
Fig. 1.12
Combined value of Stress Intensity Factor at
$x = -l -c, y = 0$ vs $ka$ for $\theta_0 = 30^\circ$ to $120^\circ$. 
Combined value of Stress Intensity factor at $x = -l$, $y = 0$ vs $ka$ for $\Theta_0$ = 30° to 120°.
Fig. 1.14

Combined values of Stress Intensity Factor at $x = -L_c, y = 0$ vs $\theta_0$ for $ka = 0.2$ to $1.4$
Stress difference magnitudes due to incident field across the inclusion (no scattering) \((X = x - x_0)\)
for \(\theta_0 = 30^\circ\).
Same as in Fig. 1.15 for $\theta_o = 60^\circ$. 
Scattered-field stress-difference (magnitudes) across the inclusion (without rigid motion) for $\theta_o = 30^\circ$.
Stress on \( S^+ \) (magnitude) due to scattered field (without \( \Delta_0 \)) for \( \Theta_0 = 30^\circ \).
Stress on $S^-$ (magnitude) for same conditions as in Fig. 1.18 for $\theta_o = 30^\circ$. 

Fig. 1.19
Stress (magnitude) on $S^+$ due to total scattered field (including $\Delta_\theta$ effect) for $\theta_o = 30^\circ$ (values on $S^-$ are almost similar).