CHAPTER IV

STATIC DEFORMATION OF A TRANSVERSELY ISOTROPIC MULTILAYERED HALF-SPACE BY TWO-DIMENSIONAL SURFACE LOADS*

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4.1. Introduction

The assumption that the physical properties at a point of an elastic medium vary with direction characterizes the medium as being anisotropic. Lekhnitskii (1963) extended Muskhelishvili's work (1953) in the plane theory of isotropic elasticity to the anisotropic case. The propagation of elastic waves in anisotropic media has been considered, amongst others, by Musgrave (1954, 1961), Buchwald (1959) and Crampin (1977). Evidence of anisotropy in the upper mantle of the Earth has been given by Crampin (1977), Bamford and Crampin (1977), and others.

There are several special types of anisotropy that have some interest in Seismology. The simplest departure from isotropy is that for which the elastic properties are the same in all directions in planes perpendicular to one particular direction (known as axis of symmetry). Such a medium is called transversely isotropic. The propagation of surface waves in transversely isotropic layered half-space has been studied by several authors (Anderson, 1961, 1962, 1966; Harkrider and Anderson, 1962; Payton, 1983). Abubakar (1961, 1962) obtained exact closed algebraic expressions for the surface displacements as functions of time and horizontal distance due to a buried impulsive line source in a homogeneous, elastic, transversely isotropic half-space. Lekhnitskii (1963) discussed
the problem of the static deformation of a transversely isotropic uniform half-space under the influence of symmetric normal loads and has obtained closed form expressions for stresses due to a concentrated force. Singh (1986) studied the axially-symmetric deformation of a transversely isotropic, multilayered, elastic half-space by surface loads.

In chapter III, we have discussed the two-dimensional problem of the static deformation of an isotropic, multilayered, half-space by surface loads. In the present chapter, we have formulated the corresponding problem of the static deformation of a transversely isotropic, multilayered, half-space by surface loads. Both plane strain and antiplane strain cases are considered. The Thomson-Haskell matrix method is used to obtain the required field. The particular cases of a normal line load and a shear line load are considered in detail. It is shown that in the case of a transversely isotropic uniform half-space the integrals giving the stresses can be evaluated analytically. The present formulation avoids the cumbersome nature of the problem and is quite convenient for numerical computation.

4.2. Basic Equations

For a transversely isotropic medium, whose axis of symmetry coincides with the z-axis, the stress-strain relations
are given by the matrix equation (Payton, 1983; p.3)

\[
\begin{bmatrix}
p_{11} \\
p_{22} \\
p_{33} \\
p_{23} \\
p_{15} \\
p_{12}
\end{bmatrix}
= \begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{66}
\end{bmatrix}
\begin{bmatrix}
e_{11} \\
e_{22} \\
e_{33} \\
e_{23} \\
e_{13} \\
e_{12}
\end{bmatrix},
\]

(4.1)

with

\[
c_{66} = \frac{1}{2}(c_{11} - c_{12}).
\]

(4.2)

An isotropic solid is a special case of a transversely isotropic solid for which

\[
c_{12} = c_{15} = \lambda, \quad c_{11} = c_{33} = \lambda + 2\mu, \quad c_{44} = c_{66} = \mu,
\]

(4.3)

where \(\lambda, \mu\) are the Lame' constants.

Thus there are only two material constants \(\lambda, \mu\) needed to specify an isotropic solid, whereas a transversely isotropic solid requires the specification of five material constants \(c_{11}, c_{12}, c_{13}, c_{33}, \text{ and } c_{44}\) for its description.
Makeing use of equation (4.1), we note that in the case of a transversely isotropic medium the plane strain problem \((u_1 = 0)\) and the antiplane strain problem \((u_2 = u_3 = 0)\) get decoupled. However, for anisotropic media of a most general type plane strain and antiplane strain problems may not get decoupled.

4.3. Antiplane Strain Problem

For the antiplane strain problem \([\text{cf., equation (1.7)}]\)

\[
\begin{align*}
  u_1 &= u_1(y,z) , \quad u_2 = u_3 = 0 .
\end{align*}
\]

(4.4)

The non-zero stresses, using (4.1) and (1.5), are

\[
\begin{align*}
  p_{12} &= 2 c_{66} e_{12} = c_{66} \frac{\partial u_1}{\partial y} , \\
  p_{13} &= 2 c_{44} e_{13} = c_{44} \frac{\partial u_1}{\partial z} .
\end{align*}
\]

(4.5)  \quad (4.6)

Equilibrium equations (1.2) and (1.3) are identically satisfied. Using (4.5) and (4.6), (1.1) takes the form (for zero body force)

\[
\left( s^2 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u_1 = 0 ,
\]

(4.7)

where

\[
s = \left( \frac{c_{66}}{c_{44}} \right)^{1/2} .
\]

(4.8)
A solution of (4.7) is of the form

\[
u_1 = \int_0^\infty (A e^{-skz} + B e^{skz}) (\sin ky) \, dk,
\]
where \(A, B\) are functions of \(k\) only. From (4.6) and (4.9), we have

\[
p_{13} = \bar{s} \int_0^\infty (-A e^{-skz} + B e^{skz})(\sin ky)(\cos ky) \, dk,
\]
where

\[
\bar{s} = (c_{44} c_{66})^{1/2}.
\]

Equations (4.9) and (4.10) may be written as

\[
u_1 = \int_0^\infty U(\sin ky) \, dk,
\]
\[
p_{13} = \int_0^\infty T(\sin ky) k \, dk,
\]
where

\[
U = \begin{bmatrix}
\cosh(skz) & -\sinh(skz) \\
\bar{s} \sinh(skz) & -\bar{s} \cosh(skz)
\end{bmatrix}
\[
T = \begin{bmatrix}
A + B \\
A - B
\end{bmatrix}.
\]

In the case of an isotropic medium \(s = 1, \bar{s} = \mu\) and equations (4.9) and (4.10) coincide with equations (3.39) and
(3.40) for an isotropic medium obtained in Section (3.2.2).

4.4. Plane Strain Problem

For the plane strain problem

\[ u_2 = u_2(y,z), \quad u_3 = u_3(y,z), \quad u_1 = 0, \tag{4.15} \]

and the non-zero strains are

\[ e_{22} = \frac{\partial u_2}{\partial y}, \tag{4.16} \]

\[ e_{23} = \frac{1}{2} \left( \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} \right), \tag{4.17} \]

\[ e_{33} = \frac{\partial u_3}{\partial z}. \tag{4.18} \]

Equation (4.1) then yields

\[ p_{11} = c_{12} e_{22} + c_{13} e_{33}, \tag{4.19} \]

\[ p_{22} = c_{11} e_{22} + c_{13} e_{33}, \tag{4.20} \]

\[ p_{33} = c_{13} e_{22} + c_{33} e_{33}, \tag{4.21} \]

\[ p_{23} = 2c_{44} e_{23}, \tag{4.22} \]

\[ p_{13} = 0, \tag{4.23} \]

\[ p_{12} = 0. \tag{4.24} \]
Using (4.23) and (4.24), we find that equation (1.1) is identically satisfied and equations (1.2) and (1.3) take the form (for zero body force)

\[
\frac{\partial p_{22}}{\partial y} + \frac{\partial p_{23}}{\partial z} = 0 , \tag{4.25}
\]

\[
\frac{\partial p_{23}}{\partial y} + \frac{\partial p_{33}}{\partial z} = 0 . \tag{4.26}
\]

Therefore, there exists an Airy stress function \( F(y, z) \) such that

\[
P_{22} = \frac{\partial^2 F}{\partial z^2} , \quad P_{23} = \frac{\partial^2 F}{\partial y \partial z} , \quad P_{33} = \frac{\partial^2 F}{\partial y^2} . \tag{4.27}
\]

The compatibility equation, given by (1.78), is

\[
\frac{\partial^2 e_{22}}{\partial z^2} + \frac{\partial^2 e_{33}}{\partial y^2} = 2 \frac{\partial^2 e_{23}}{\partial y \partial z} . \tag{4.28}
\]

Eliminating \( e_{22} \), \( e_{33} \) and \( e_{23} \) from (4.20) - (4.22) and (4.28) and then using (4.27), we obtain

\[
\left[ a \frac{\partial^4}{\partial y^4} + (ac-2b-b^2) \frac{\partial^4}{\partial y^2 \partial z^2} + c \frac{\partial^4}{\partial z^4} \right] F = 0 , \tag{4.29}
\]

where

\[
a = \frac{c_{11}}{c_{44}} , \quad b = \frac{c_{13}}{c_{44}} , \quad c = \frac{c_{33}}{c_{44}} . \tag{4.30}
\]
we may write (4.29) as

\[
\left( \alpha^2 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left( \beta^2 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) p = 0 ,
\]

(4.31)

where \( \alpha \) and \( \beta \) are given by the relations

\[
\alpha^2 + \beta^2 = \frac{ac - 2b - b^2}{c} , \quad \alpha \beta = \frac{a}{c} .
\]

(4.32)

In the case of an isotropic medium, using (4.3), we find

\[
a = c = \frac{\lambda}{\mu} + 2 , \quad b = \frac{\lambda}{\mu} , \quad \alpha = \beta = 1 ,
\]

(4.33)

and equation (4.31) reduces to (1.83).

A solution of (4.31) is of the type (assuming \( \alpha \neq \beta \))

\[
P = \int_0^\infty \left( A e^{-\alpha k z} + B e^{\alpha k z} + C e^{-\beta k z} + D e^{\beta k z} \right) \sin ky \, dk ,
\]

(4.34)

where \( A, B, C, D \) may be functions of \( k \).

Using (4.27) and (4.34), the stresses are found to be

\[
P_{22} = \int_0^\infty \left[ A \alpha^2 e^{-\alpha k z} + B \alpha^2 e^{\alpha k z} + C \beta^2 e^{-\beta k z} + D \beta^2 e^{\beta k z} \right] \frac{\sin ky}{\cos ky} k^2 \, dk ,
\]

(4.35)

\[
P_{33} = \int_0^\infty \left[ -A e^{-\alpha k z} - B e^{\alpha k z} - C e^{-\beta k z} - D e^{\beta k z} \right] \frac{\sin ky}{\cos ky} k^2 \, dk ,
\]

(4.36)
The expressions for the displacements can be obtained by integrating the stress-displacement relations. From (4.16) – (4.18) and (4.20) – (4.22), we have

\[ p_{22} = c_{11} \frac{\partial u_2}{\partial y} + c_{13} \frac{\partial u_3}{\partial z}, \quad (4.38) \]

\[ p_{33} = c_{13} \frac{\partial u_2}{\partial y} + c_{33} \frac{\partial u_3}{\partial z}, \quad (4.39) \]

\[ p_{23} = c_{44} \left( \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} \right). \quad (4.40) \]

Solving (4.38) and (4.39), we obtain

\[ \frac{\partial u_2}{\partial y} = \Sigma^{-1} (c_{33} p_{22} - c_{13} p_{33}), \quad (4.41a) \]

\[ \frac{\partial u_2}{\partial z} = \Sigma^{-1} (c_{11} p_{33} - c_{13} p_{22}), \quad (4.41b) \]

where

\[ \Sigma = c_{11} c_{33} - c_{13}^2. \quad (4.42) \]

Integrating (4.41a, b), we find

\[ u_2 = \Sigma^{-1} \int (c_{33} p_{22} - c_{13} p_{33})dy + f(z), \quad (4.43a) \]
\[ u_3 = \mathbf{z}^{-1} \int (c_{11} p_{33} - c_{13} p_{22}) \, dz + g(y), \quad (4.43b) \]

where \( f \) and \( g \) are arbitrary functions. Equation (4.40) shows that \( f \) and \( g \) represent a rigid body displacement and can thus be disregarded in the analysis of deformation. Taking \( f = g = 0 \), (4.35), (4.36) and (4.43a, b) yield

\[
u_2 = \int_0^\infty \left[ p_1 (A e^{-\alpha k z} + B e^{\alpha k z}) + p_2 (C e^{-\beta k z} + D e^{\beta k z}) \right] (-\cos ky) \, dk,
\]

\[
u_3 = \int_0^\infty \left[ q_1 (A e^{-\alpha k z} - B e^{\alpha k z}) + q_2 (C e^{-\beta k z} - D e^{\beta k z}) \right] (-\sin ky) \, dk,
\]

where

\[
p_1 = \Sigma^{-1} (c_{33} \alpha^2 + c_{13}), \quad p_2 = \Sigma^{-1} (c_{33} \beta^2 + c_{13}), \quad (4.46)
\]

\[
q_1 = \Sigma^{-1} \left[ c_{13} \alpha + \frac{c_{11}}{\alpha} \right], \quad q_2 = \Sigma^{-1} \left[ c_{13} \beta + \frac{c_{11}}{\beta} \right]. \quad (4.47)
\]

We may write (4.36), (4.37), (4.44) and (4.45) in the form

\[
u = \int_0^\infty V (-\cos ky) \, dk,
\]

\[
u_3 = \int_0^\infty W (-\sin ky) \, dk,
\]

\[
p_{23} = \int_0^\infty S (-\sin ky) \, dk,
\]

\[
p_{23} = \int_0^\infty S (-\sin ky) \, dk,
\]

\[
\]
\[ p_{33} = \int_{0}^{\infty} \frac{\sin ky}{\cos ky} k^2 \, dk. \]  

(4.51)

The functions \( V, w, \sigma, N \) are given by the matrix relation

\[ [Y(z)] = [Z(z)] [K], \]  

(4.52)

where

\[ [Y(z)] = [V, w, \sigma, N]^T, \]  

(4.53)

\[ [K] = [A+B, A-B, C+D, C-D]^T, \]  

(4.54)

and \([\ldots]^T\) denotes the transpose of the matrix \([\ldots]\). The matrix \([Z(z)]\) is given below:

\[
[Z(z)] = \begin{bmatrix}
    p_1 \text{ch}(akz) & -p_1 \text{sh}(akz) & p_2 \text{ch}(\beta kz) & -p_2 \text{sh}(\beta kz) \\
    -q_1 \text{sh}(akz) & q_1 \text{ch}(akz) & -q_2 \text{sh}(\beta kz) & q_2 \text{ch}(\beta kz) \\
    -\alpha \text{sh}(akz) & \alpha \text{ch}(akz) & -\beta \text{sh}(\beta kz) & \beta \text{ch}(\beta kz) \\
    -\text{ch}(akz) & \text{sh}(akz) & -\text{ch}(\beta kz) & \text{sh}(\beta kz)
\end{bmatrix}
\]  

(4.55)

Results for the case when \( \alpha = \beta \) can be obtained by taking the limit \( \beta \rightarrow \alpha \) (Lekhnitskii, 1963).

4.5. Deformation of a Multilayered Half-Space by Surface Loads

We consider a semi-infinite medium made up of \( p-1 \) parallel, homogeneous, transversely isotropic layers lying over a
homogeneous, transversely isotropic half-space. The layers
are numbered serially, the layer at the top being layer 1 and
the half-space is designated as layer p. We place the origin
of the cartesian coordinate system (x, y, z) at the boundary of
the semi-infinite medium and the z-axis is drawn into the
medium. The nth layer is of thickness $d_n$ and is bounded by
interfaces $z = z_{n-1}$, $z_n$ so that $d_n = z_n - z_{n-1}$. Obviously
$z_0 = 0$ and $z_p = H$, where $H$ is the depth of the last interface.

4.5.1. ANTIPLANE STRAIN PROBLEM

Appending the subscript $n$ to the quantities pertaining
to the nth layer ($z_{n-1} \leq z \leq z_n$), (4.14) becomes

$$
\begin{bmatrix}
U_n \\
T_n
\end{bmatrix}
= \begin{bmatrix}
Z_n(z) \\
A_n + B_n
\end{bmatrix}
\begin{bmatrix}
A_n - B_n
\end{bmatrix},
$$

(4.56)

where the matrix $Z_n(z)$ is given by

$$
Z_n(z) = \begin{bmatrix}
ch(s_n k z) & -sh(s_n k z) \\
-s_n sh(s_n k z) & -s_n ch(s_n k z)
\end{bmatrix}
$$

(4.57)

From (2.95), we have
where the layer matrix \( [a_n] \), using (2.63), is given by

\[
[a_n] = \begin{bmatrix}
\text{ch}(s_n k d_n) & -s_n^{-1} \text{sh}(s_n k d_n) \\
-s_n \text{sh}(s_n k d_n) & \text{ch}(s_n k d_n)
\end{bmatrix}
\]  

(4.59)

Proceeding as in Section (2.3.2) and taking \( B_p = 0 \), we find

\[
\begin{bmatrix}
U_1 (0) \\
T_1 (0)
\end{bmatrix} = [E] \begin{bmatrix}
A_p \\
A_p
\end{bmatrix}
\]  

(4.60)

where

\[
[E] = \begin{bmatrix}
a_1 \\
\vdots \\
a_{p-1}
\end{bmatrix} \begin{bmatrix}
\mathbb{Z}_p (H)
\end{bmatrix}
\]  

(4.61)

Equation (4.60) gives the following equations

\[
U_1 (0) = (E_{11} + E_{12}) A_p ,
\]

(4.62)

\[
T_1 (0) = (E_{21} + E_{22}) A_p .
\]

(4.63)

From (4.62) or (4.63), \( A_p \) is known. Equation (4.62) is applicable when the surface displacement is prescribed and (4.63) is applicable when the surface load is prescribed. We shall confine our discussion to the latter case only.
When the surface load is prescribed the boundary condition is of the form

\[ P_{13} = f(y) \text{ at } z = 0. \] (4.64)

We shall write [cf. equation (4.13)]

\[ f(y) = \int_{0}^{\infty} \tilde{f}(k) \left( \frac{\sin ky}{\cos ky} \right) dk. \] (4.65)

Equations (4.13), (4.63) and (4.65) yield

\[ \frac{\tilde{f}(k)}{A_p \left( k(E_{21} + E_{22}) \right)} \] (4.66)

The field at any point of the medium can also be obtained. For \( z_{n-1} \leq z < z_n \), we find

\[
\begin{bmatrix}
U_n(z) \\
T_n(z)
\end{bmatrix} = \left[ G(z) \right] \begin{bmatrix} A_p \\ A_p \end{bmatrix},
\] (4.67)

where

\[
\left[ G(z) \right] = \left[ a_n(z_n-z) \right] \left[ a_{n+1} \right] \left[ a_{n+2} \right] \cdots \left[ a_{p-1} \right] \left[ z_p(H) \right],
\] (4.68)

and \( \left[ a_n(z_n-z) \right] \) is obtained from \( \left[ a_n \right] \) on replacing \( d_n \) by \( z_n - z \).

Inserting the value of \( A_p \) given in (4.66) into (4.67) and making use of (4.12) and (4.13), we find...
\[ u_1 = \int_0^\infty \tilde{f}(k) \left( \frac{G_{11} + G_{12}}{E_{21} + E_{22}} \right) (\sin ky)^{k-1} dk, \quad (4.69) \]

\[ p_{13} = \int_0^\infty \tilde{f}(k) \left( \frac{G_{21} + G_{22}}{E_{21} + E_{22}} \right) (\cos ky)^{k-1} dk. \quad (4.70) \]

### 4.5.2. PLANE STRAIN PROBLEM

Proceeding as in Section (2.3.1), we obtain, for the \( n \)th layer [cf. equation (2.62)]

\[ \begin{bmatrix} \gamma_{n}(y_{n-1}) \end{bmatrix} = \begin{bmatrix} a_n \end{bmatrix} \begin{bmatrix} \gamma_{n}(y_{n}) \end{bmatrix}, \quad (4.71) \]

where the layer matrix \( \begin{bmatrix} a_n \end{bmatrix} \) is given by (2.63):

\[ \begin{bmatrix} a_n \end{bmatrix} = \begin{bmatrix} z_n(-d_n) \end{bmatrix} \begin{bmatrix} z(0) \end{bmatrix}^{-1}. \quad (4.71a) \]

Using (4.55), we find

\[ \begin{bmatrix} -1 & 0 & 0 & -p_2 \\ \frac{\Sigma_1}{\Sigma_1} & 0 & 0 & 0 \\ 0 & \frac{\beta}{\Sigma_2} & -q_2 & 0 \\ 0 & 0 & \frac{\Sigma_2}{\Sigma_2} & 0 \end{bmatrix} \]

\[ \begin{bmatrix} -p_2 \\ \frac{\Sigma_1}{\Sigma_1} & 0 & 0 & 0 \\ 0 & \frac{\beta}{\Sigma_2} & -q_2 & 0 \\ 0 & 0 & \frac{\Sigma_2}{\Sigma_2} & 0 \end{bmatrix} \quad (4.72) \]
where

\[ \Sigma_1 = p_2 - p_1 , \]  
\[ \Sigma_2 = q_1 \beta = q_2 \alpha . \]  

From (4.55), (4.71a) and (4.72), the elements of the layer matrix are found to be:

\[
(11) = (-p_1 \text{ch } \theta + p_2 \text{ch } \phi)/\Sigma_1 , \\
(12) = (p_1 \beta \text{sh } \theta - q_2 \alpha \text{sh } \phi)/\Sigma_2 , \\
(13) = (-p_1 q_2 \text{sh } \theta + p_2 q_1 \text{sh } \phi)/\Sigma_2 , \\
(14) = p_1 p_2 (-\text{ch } \theta + \text{ch } \phi)/\Sigma_1 , \\
(21) = (q_1 \text{sh } \theta + q_2 \text{sh } \phi)/\Sigma_1 , \\
(22) = (q_1 \beta \text{ch } \theta - q_2 \alpha \text{ch } \phi)/\Sigma_2 , \\
(23) = q_1 q_2 (-\text{ch } \theta + \text{ch } \phi)/\Sigma_2 , \\
(24) = (-q_1 p_2 \text{sh } \theta + q_2 p_1 \text{sh } \phi)/\Sigma_1 , \\
(31) = (-\alpha \text{sh } \theta + \beta \text{sh } \phi)/\Sigma_1 , \\
(32) = \alpha \beta (\text{ch } \theta - \text{ch } \phi)/\Sigma_2 , \\
(33) = (-q_2 \text{ch } \theta + \beta q_1 \text{ch } \phi)/\Sigma_2 ,
\]
\( (34) = \left( -\alpha p_2 \sinh \theta + \beta p_1 \sinh \phi \right)/\Sigma_1 \),

\( (41) = \left( \cosh \theta - \cosh \phi \right)/\Sigma_1 \),

\( (42) = \left( -\beta \sinh \theta + \alpha \sinh \phi \right)/\Sigma_2 \),

\( (43) = \left( q_2 \sinh \theta - q_1 \sinh \phi \right)/\Sigma_2 \),

\( (44) = \left( p_2 \cosh \theta - p_1 \cosh \phi \right)/\Sigma_1 \),

with

\[ \theta = \alpha kd, \quad \phi = \beta kd. \quad (4.75) \]

Proceeding as in the Section (3.2.1), we find [cf. equation (3.16)]

\[ [Y_1(0)] = [E] \begin{bmatrix} A_p, A_p, C_p, C_p \end{bmatrix}^T, \quad (4.76) \]

with \([E]\) of (4.61). Equation (4.76) gives the following equations

\[ V_1(0) = (E_{11} + E_{12})A_p + (E_{13} + E_{14})C_p, \quad (4.77a) \]

\[ W_1(0) = (E_{21} + E_{22})A_p + (E_{23} + E_{24})C_p, \quad (4.77b) \]

\[ S_1(0) = (E_{31} + E_{32})A_p + (E_{33} + E_{34})C_p, \quad (4.78a) \]

\[ N_1(0) = (E_{41} + E_{42})A_p + (E_{43} + E_{44})C_p. \quad (4.78b) \]

For given displacements at the surface, \( A_p \) and \( C_p \) are
known from (4.77a, b). For given surface loads, $A_p$ and $C_p$ are known from (4.78a, b).

When the surface load is prescribed the boundary conditions are of the form

$$P_{23} = g(y), \quad P_{33} = h(y) \quad \text{at} \quad z = 0. \quad (4.79)$$

As before, we put [cf. equations (4.50) and (4.51)]

$$g(y) = \int_0^\infty \tilde{g}(k) \left( \cos ky - \sin ky \right) dk, \quad (4.80)$$

$$h(y) = \int_0^\infty \tilde{h}(k) \left( \sin ky + \cos ky \right) dk. \quad (4.81)$$

Equations (4.50), (4.51), (4.78a, b), (4.80) and (4.81) give the values of $A_p$ and $C_p$:

$$A_p = \frac{1}{\Sigma_3 k^2} \left[ (E_{43} + E_{44}) \tilde{g} - (E_{33} + E_{34}) \tilde{h} \right], \quad (4.82a)$$

$$C_p = \frac{1}{\Sigma_3 k^2} \left[ (E_{31} + E_{32}) \tilde{h} - (E_{41} + E_{42}) \tilde{g} \right], \quad (4.82b)$$

where

$$\Sigma_3 = (E_{31} + E_{32})(E_{43} + E_{44}) - (E_{33} + E_{34})(E_{41} + E_{42}). \quad (4.83)$$

The field at any point of the medium can be obtained from the relation ($z_{n-1} \leq z < z_n$)

$$\left[ v_n(z), w_n(z), s_n(z), n_n(z) \right]^T = [g(z)] \left[ A_p, A_p, C_p, C_p \right]^T, \quad (4.84)$$
where \( [G(z)] \) is defined in (4.68). From (4.48)-(4.51), (4.82a, b) and (4.84), we obtain

\[
u_2 = \int_0^\infty \left[ (G_{11} + G_{12}) \left\{ (E_{43} + E_{44}) \vec{g} - (E_{33} + E_{34}) \vec{h} \right\}
+ (G_{13} + G_{14}) \left\{ (E_{31} + E_{32}) \vec{h} - (E_{41} + E_{42}) \vec{g} \right\} \right] \Sigma_3^{-1} \frac{-1}{\cos ky} dk,
\]

(4.85)

\[
u_3 = \int_0^\infty \left[ (G_{21} + G_{22}) \left\{ (E_{43} + E_{44}) \vec{g} - (E_{33} + E_{34}) \vec{h} \right\}
+ (G_{23} + G_{24}) \left\{ (E_{31} + E_{32}) \vec{h} - (E_{41} + E_{42}) \vec{g} \right\} \right] \Sigma_3^{-1} \frac{-1}{\sin ky} dk,
\]

(4.86)

\[
p_{23} = \int_0^\infty \left[ (G_{31} + G_{32}) \left\{ (E_{43} + E_{44}) \vec{g} - (E_{33} + E_{34}) \vec{h} \right\}
+ (G_{33} + G_{34}) \left\{ (E_{31} + E_{32}) \vec{h} - (E_{41} + E_{42}) \vec{g} \right\} \right] \Sigma_3^{-1} \frac{-1}{\cos ky} dk,
\]

(4.87)

\[
p_{33} = \int_0^\infty \left[ (G_{41} + G_{42}) \left\{ (E_{43} + E_{44}) \vec{g} - (E_{33} + E_{34}) \vec{h} \right\}
+ (G_{43} + G_{44}) \left\{ (E_{31} + E_{32}) \vec{h} - (E_{41} + E_{42}) \vec{g} \right\} \right] \Sigma_3^{-1} \frac{1}{\sin ky} dk.
\]

(4.88)

4.6. Specified Surface Loads

The results obtained in Section (4.5) are of a general nature. We consider below a few particular cases in which the surface load is specified.
4.6.1. ANTIPLANE STRAIN PROBLEM

Suppose that a shear line load $h$ per unit length is applied at the origin to the surface $z = 0$ in the positive direction of $x -$ axis. Then

$$f(y) = -R \delta(y). \quad (4.89)$$

From (3.55), (4.65) and (4.89), we find that [cf. equation (3.75)]

$$\tilde{f}(k) = \frac{-R}{\pi}, \quad (4.90)$$

and we must choose the lower solution, $\cos ky$, in the expression (4.9) for $u_1$ and the corresponding solution in the succeeding equations. Putting this value of $\tilde{f}(k)$ in (4.69) and (4.70), we obtain

$$u_1 = \frac{-R}{\pi} \int_0^\infty \left( \frac{G_{11} + G_{12}}{E_{21} + E_{22}} \right) \cos ky \frac{1}{k} \, dk, \quad (4.91)$$

$$p_{13} = \frac{-R}{\pi} \int_0^\infty \left( \frac{G_{21} + G_{22}}{E_{21} + E_{22}} \right) \cos ky \, dk. \quad (4.92)$$

4.6.2. PLANE STRAIN PROBLEM

For the plane strain problem, we shall be considering the particular cases of a normal line load and a shear line load.
Let a normal line load \( P \) per unit length be applied at the origin to the surface \( z = 0 \) in the positive direction of the \( z \)-axis. Then

\[
g(y) = 0, \quad h(y) = -P\delta(y).
\]

From (3.55), (4.80), (4.81) and (4.93), we obtain [cf. equation (3.56)]

\[
\bar{g}(k) = 0, \quad \bar{h}(k) = \frac{-P}{\pi},
\]

and we must use the lower solution, \( \cos ky \), in the expression (4.34) for \( F \) and the corresponding solution of the succeeding equations. The displacements and stresses at any point of the medium are given by (4.85)–(4.88) and (4.94):

\[
u_2 = \frac{-P}{\pi} \int_0^\infty \left[ (G_{13} + G_{14})(E_{31} + E_{32}) - (G_{11} + G_{12})(E_{33} + E_{34}) \right] k^{-1} \Sigma^3 x \sin ky \, dk,
\]

\[
u_3 = \frac{-P}{\pi} \int_0^\infty \left[ (G_{23} + G_{24})(E_{31} + E_{32}) - (G_{21} + G_{22})(E_{33} + E_{34}) \right] k^{-1} \Sigma^3 \cos ky \, dk,
\]

\[
P_{23} = \frac{P}{\pi} \int_0^\infty \left[ (G_{33} + G_{34})(E_{31} + E_{32}) - (G_{31} + G_{32})(E_{33} + E_{34}) \right] \Sigma^3 \sin ky \, dk,
\]

\[
P_{33} = \frac{P}{\pi} \int_0^\infty \left[ (G_{43} + G_{44})(E_{31} + E_{32}) - (G_{41} + G_{42})(E_{33} + E_{34}) \right] \Sigma^3 \cos ky \, dk.
\]
SHEAR LIME LOAD

Suppose that a shear line load Q per unit length is applied at the origin to the surface \( z = 0 \) in the positive direction of the y-axis. Then

\[
g(y) = -Q \phi(y), \quad h(y) = 0. \quad (4.99)
\]

we then find [cf. equation (3.62)]

\[
\widetilde{g}(k) = -\frac{Q}{\pi}, \quad \widetilde{h}(k) = 0, \quad (4.100)
\]

and we should use the upper solution, \( \sin ky \), in the expression (4.34) for \( P \) and the corresponding solution in the succeeding equations. The displacements and stresses given by (4.85) - (4.88) and (4.100) are

\[
u_2 = \frac{Q}{\pi} \int_0^\infty \left[ (G_{11} + G_{12})(E_{43} + E_{44}) - (G_{13} + G_{14})(E_{41} + E_{42}) \right] k^{-1} \Sigma_3^{-1} \cos ky \, dk, \quad (4.101)
\]

\[
u_3 = \frac{Q}{\pi} \int_0^\infty \left[ (G_{21} + G_{22})(E_{43} + E_{44}) - (G_{23} + G_{24})(E_{41} + E_{42}) \right] k^{-1} \Sigma_3^{-1} \sin ky \, dk \quad (4.102)
\]

\[
p_{23} = \frac{Q}{\pi} \int_0^\infty \left[ (G_{31} + G_{32})(E_{43} + E_{44}) - (G_{33} + G_{34})(E_{41} + E_{42}) \right] \Sigma_3^{-1} \cos ky \, dk, \quad (4.103)
\]

\[
p_{33} = \frac{Q}{\pi} \int_0^\infty \left[ (G_{41} + G_{42})(E_{43} + E_{44}) - (G_{43} + G_{44})(E_{41} + E_{42}) \right] \Sigma_3^{-1} \sin ky \, dk. \quad (4.104)
\]
4.7. Uniform Half-Space

In Section (4.6), we have derived the displacements and stresses at any point of the medium caused by surface loads acting on the surface of a transversely isotropic multilayered half-space. These results are in the form of integrals over the variable k. These integrals can be evaluated numerically by using the method suggested by Jovanovich et al. (1974a,b). In the case of a transversely isotropic uniform half-space the integrals giving the stresses can be evaluated analytically. These closed form expressions can be used as a check over the numerical computation. The following basic transform integrals are used:

\[
\int_{0}^{\infty} e^{-ik} \sin ky \, dk = \frac{y}{y^2 + \zeta^2}, \quad (4.105)
\]

\[
\int_{0}^{\infty} e^{-ik} \cos ky \, dk = \frac{\zeta y}{y^2 + \zeta^2}. \quad (4.106)
\]

For a uniform half-space, \( p = 1 \) and

\[
|E| = [2 (0)], \quad (4.107)
\]

\[
[G] = [2(z)]. \quad (4.108)
\]

4.7.1. Antiplane Strain Problem

In this case,
\[
\begin{bmatrix}
1 & 0 \\
0 & -s
\end{bmatrix},
\tag{4.109}
\]

\[
\begin{bmatrix}
\text{ch}(skz) & -\text{sh}(skz) \\
\bar{s} & \text{sh}(skz)
\end{bmatrix},
\tag{4.110}
\]

where we have written \( s \) for \( s_1 \) and \( \bar{s} \) for \( \bar{s}_1 \). Using (4.92), (4.109) and (4.110), we find

\[
p_{13} = \frac{-R}{\pi} \left[ \frac{s z}{y^2 + s^2 z^2} \right].
\tag{4.111}
\]

In the case of an isotropic half-space \( s = 1 \) and (4.111) reduces to

\[
p_{13} = \frac{-R}{\pi} \left[ \frac{z}{y^2 + z^2} \right].
\tag{4.112}
\]

This result coincides with the corresponding equation (3.88) obtained in Chapter III.

4.7.2. PLANE STRAIN PROBLEM

Here,

\[
\begin{bmatrix}
p_1 & 0 & p_2 & 0 \\
0 & q_1 & 0 & q_2 \\
0 & \alpha & 0 & \beta \\
-1 & 0 & -1 & 0
\end{bmatrix},
\tag{4.113}
\]
and \([G]\) is defined in (4.55).

NORMAL LINE LOAD

Using (4.55), (4.83), (4.97), (4.98) and (4.113), the stresses are found to be:

\[
P_{23} = \frac{\alpha \beta p}{\pi (\beta - \alpha)} \left[ \frac{y}{y^2 + \beta^2 z^2} - \frac{y}{y^2 + \alpha^2 z^2} \right]
\]

\[
= -\frac{p}{\pi} \left[ \frac{\alpha \beta (\alpha + \beta) y z^2}{(y^2 + \beta^2 z^2)(y^2 + \alpha^2 z^2)} \right], \quad (4.114)
\]

\[
P_{33} = \frac{\alpha \beta p}{\pi (\beta - \alpha)} \left[ \frac{z}{y^2 + \beta^2 z^2} - \frac{z}{y^2 + \alpha^2 z^2} \right]
\]

\[
= -\frac{p}{\pi} \left[ \frac{\alpha \beta (\alpha + \beta) z^3}{(y^2 + \beta^2 z^2)(y^2 + \alpha^2 z^2)} \right]. \quad (4.115)
\]

The stresses for an isotropic uniform half-space can be deduced from (4.114) and (4.115). Putting \(\alpha = \beta = 1\), we find

\[
P_{23} = -\frac{p}{\pi} \left[ \frac{2 y z^2}{(y^2 + z^2)^2} \right], \quad (4.116)
\]

\[
P_{33} = -\frac{p}{\pi} \left[ \frac{2 z^3}{(y^2 + z^2)^2} \right]. \quad (4.117)
\]
These results coincide with the corresponding equations (3.81) and (3.82) obtained in chapter III.

**SHEAR LINE LOAD**

Using (4.53), (4.83), (4.103), (4.104) and (4.113) the stresses are found to be:

\[
P_{23} = \frac{-Q}{\pi(n-\alpha)} \left[ \frac{\alpha z}{y^2 + \alpha^2 z^2} - \frac{\beta z}{y^2 + \beta^2 z^2} \right] \tag{4.118}
\]

\[
= \frac{-Q}{\pi} \left[ \frac{(\alpha + \beta) y^2 z}{(y^2 + \alpha^2 z^2)(y^2 + \beta^2 z^2)} \right],
\]

\[
P_{33} = \frac{-Q}{\pi(n-\alpha)} \left[ \frac{y}{y^2 + \alpha^2 z^2} - \frac{y}{y^2 + \beta^2 z^2} \right] \tag{4.119}
\]

\[
= \frac{-Q}{\pi} \left[ \frac{(\alpha + \beta) y z^2}{(y^2 + \alpha^2 z^2)(y^2 + \beta^2 z^2)} \right].
\]

For an isotropic uniform half-space, (4.118) and (4.119) take the form

\[
P_{23} = \frac{-Q}{\pi} \left[ \frac{2y^2 z}{(y^2 + z^2)^2} \right], \tag{4.120}
\]

\[
P_{33} = \frac{-Q}{\pi} \left[ \frac{2yz^2}{(y^2 + z^2)^2} \right]. \tag{4.121}
\]

These results coincide with the corresponding equations (3.83) and (3.84) obtained in chapter III.
4.8. Numerical Results

In Section (4.7.2), we have derived analytical expressions for the stresses $p_{23}$ and $p_{33}$ caused by a normal line load acting at the surface of a transversely isotropic uniform half-space. We have performed numerical computations to study the variation of these stresses. For this purpose, we have defined the dimensionless quantity $Y$ through the relation

$$y = Yz. \quad (4.122)$$

The stresses $p_{23}$ and $p_{33}$, given by equations (4.114) and (4.115), can be put in the form

$$p_{23} = \frac{-P}{\pi z} \tau_{23}, \quad (4.123)$$

$$p_{33} = \frac{-P}{\pi z} \tau_{33}, \quad (4.124)$$

where

$$\tau_{23} = \frac{\alpha \beta (\alpha + \beta) Y}{Y^4 + Y^2 (\alpha^2 + \beta^2) + \alpha^2 \beta^2} \quad (4.125)$$

$$\tau_{33} = \frac{\alpha \beta (\alpha + \beta)}{Y^4 + Y^2 (\alpha^2 + \beta^2) + \alpha^2 \beta^2} \quad (4.126)$$

are dimensionless stresses. For an isotropic medium, we obtain, on putting $\alpha = \beta = 1$, 
Therefore, for an isotropic medium, $\tau_{23}$ and $\tau_{33}$ are independent of the elastic constants $\lambda$ and $\mu$ of the medium.

The numerical values of the elastic constants $c_{11}$, $c_{12}$, $c_{13}$, $c_{33}$ and $c_{44}$ for a number of transversely isotropic substances, extracted from Hearmon (1963), have been listed by Payton (1983, p. 3). Table (4.1) gives the numerical values (in the units of $10^{11}$ dyne/cm$^2$) of the elastic constants for Magnesium and Titanium. This table also gives the numerical values of the dimensionless quantities $\alpha$ and $\beta$ for these two substances. In the case of an isotropic medium, $\alpha = \beta = 1$ [see equation (4.33)]. The elastic constant $c_{12}$ does not occur in the expressions for $\alpha$ and $\beta$ [see equations (4.30) and (4.32)]. Therefore, $\tau_{23}$ and $\tau_{33}$ do not depend upon $c_{12}$. (The elastic constant $c_{12}$ occurs in the antiplane strain problem.)

![Table 4.1](image)

- **Substance** | $c_{11}$ | $c_{12}$ | $c_{13}$ | $c_{33}$ | $c_{44}$ | $\alpha$ | $\beta$
--- | --- | --- | --- | --- | --- | --- | ---
Magnesium | 5.92 | 2.57 | 2.14 | 6.14 | 1.64 | 1.403 | 0.699
Titanium | 16.2 | 9.2 | 6.9 | 18.1 | 4.67 | 1.255 | 0.754
Isotropic | $\lambda+2\mu$ | $\lambda$ | $\lambda$ | $\lambda+2\mu$ | $\mu$ | 1.000 | 1.000

\[
\tau_{23} = \frac{2Y}{(Y^2+1)^2} \quad (4.127)
\]

\[
\tau_{33} = \frac{2}{(Y^2+1)^2} \quad (4.128)
\]
The numerical values of the dimensionless stresses $\tau_{23}$ and $\tau_{33}$ for Magnesium, Titanium and an arbitrary isotropic medium are given in table (4.2) for various values of the dimensionless parameter $Y$. From table (4.1), we note that the numerical values of the elastic constants for Magnesium and Titanium differ considerably and that the numerical values of the dimensionless quantities $\alpha$ and $\beta$ for Magnesium and Titanium are significantly different from the numerical values for an isotropic medium. The numerical values for $\tau_{23}$ and $\tau_{33}$ for the three cases (Isotropic medium, Magnesium and Titanium) are almost the same [see table (4.2)]. Thus, we conclude that for realistic media, the anisotropy has no significant effect on the stresses caused by a normal line load.

Figure (4.1) shows the variation of $\tau_{23}$ with $Y$ for an isotropic medium. For an isotropic medium, $\tau_{23}$ has a maximum value of $3\sqrt{3}/8 = 0.6495$ for $Y = 1/\sqrt{3} = 0.577$. If the medium is Magnesium, $\tau_{23}$ has a maximum value of 0.6314 for $Y = 0.538$. In case the medium is Titanium, $\tau_{23}$ has a maximum value of 0.6394 for $Y = 0.544$. In each of the three cases, $\tau_{23}$ has a minimum value of 0 for $Y = 0$.

Figure (4.2) shows the variation of $\tau_{33}$ with $Y$ for an isotropic medium. At $Y = 0$, $\tau_{33}$ has maximum values of 2.0, 2.142 and 2.132 when the medium is isotropic, Magnesium and Titanium respectively.

The graphs for Magnesium and Titanium almost coincide with the corresponding graphs for the isotropic case.
FIG. 4.1. $\mathcal{Z}_{23}$ AS A FUNCTION OF $y$ FOR AN ISOTROPIC MEDIUM.
FIG. 4.2. $\tau_{33}$ AS A FUNCTION OF $Y$ FOR AN ISOTROPIC MEDIUM.
Table 4.2.

Numerical values of the dimensionless stresses $\tau_{23}$ and $\tau_{33}$ for various values of the dimensionless horizontal distance $Y$.

<table>
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<tr>
<th>$Y = y/z$</th>
<th>$\tau_{23}$</th>
<th></th>
<th>$\tau_{33}$</th>
<th></th>
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