Basic concepts on fuzzy sets and on measures of fuzzy information are given in section 1. The theory of possibility is related to the theory of fuzzy sets by defining the concepts of a possibility distribution as a fuzzy restriction. Possibility distributions of composite and qualified propositions are studied in section 2. We present a fuzzy model for the diagnosis of coronary artery stenosis using Shannon entropy in the third section. Risk factors and remedial measures play an important role in medical surgery. We arrive at, in the fourth section, the set of remedial measures which in the opinions of the group of surgeons gives the best remedy against the three different risk factors through fuzzy set theory.

4.1. FUZZY SETS, OPERATORS AND FUZZY MEASURE

Most of our traditional tools for formal modeling, reasoning and computing are crisp, deterministic and precise in character. Real situations are very often uncertain or vague in a number of ways. This type of uncertainty has been handled appropriately by probability theory and statistics. The vagueness concerning the description of the semantic meaning of the events, phenomena or statements themselves is called fuzziness. Fuzzy set theory provides a mathematical framework in which vague conceptual phenomena can be precisely and rigorously studied. Zadeh [114] and Goguen [40] show their intention to generalise the classical notion of a set and a proposition to accommodate fuzziness.

4.1.1. BASIC DEFINITIONS

Definition 4.1.1.

If X is a collection of objects denoted generically by x then a fuzzy set $\tilde{A}$ in X is defined as a set of ordered pairs:
\[ \tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}, \]

\(\mu_{\tilde{A}}(x)\) is called the membership function or grade of membership of \(x\) in \(\tilde{A}\) which maps \(X\) to the membership space \(M\).

A fuzzy set is obviously a generalization of a classical set and the membership function a generalization of the characteristic function. Since we are generally referring to a universal (crisp) set \(X\) some elements of a fuzzy set may have the degree of membership zero. Often it is appropriate to consider those elements of the universe which have a non-zero degree of membership in a fuzzy set.

**Definition 4.1.2.**

The support of a fuzzy set \(\tilde{A}\), denoted by \(S(\tilde{A})\), is defined as the crisp set of all \(x \in X\) such that \(\mu_{\tilde{A}}(x) > 0\).

**Definition 4.1.3.**

The (crisp) set of elements which belong to the fuzzy set \(\tilde{A}\) at least to the degree \(\alpha\) is defined as the \(\alpha\)-level-set:

\[ A_{\alpha} = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha \} \]

\(A'_{\alpha} = \{x \in X \mid \mu_{\tilde{A}}(x) > \alpha \}\) is called "strong \(\alpha\)-level-set" or "strong \(\alpha\)-cut".

Convexity also plays a role in fuzzy set theory. By contrast to classical set theory, however, convexity conditions are defined with reference to the membership function rather than the support of a fuzzy set.

**Definition 4.1.4.**

A fuzzy set \(A\) is convex if

\[ \mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda) x_2) \geq \min (\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)), x_1, x_2 \in X, \lambda \in [0, 1] \]

Alternatively, a fuzzy set is convex if all \(\alpha\)-level sets are convex.
Definition 4.1.5.
For a finite fuzzy set \( \tilde{A} \) the cardinality \( |\tilde{A}| \) is defined as
\[
|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x)
\]
Further \( ||\tilde{A}|| = \frac{|\tilde{A}|}{|X|} \) is called the relative cardinality of \( \tilde{A} \).

4.1.2. OPERATIONS ON FUZZY SETS

The membership function is obviously the crucial component of a fuzzy set. Hence the operations with fuzzy sets are defined via their membership functions.

Definition 4.1.6.
Let \( \tilde{A} \) and \( \tilde{B} \) be two fuzzy sets in \( X \). Then the union of \( \tilde{A} \) and \( \tilde{B} \), denoted by \( \tilde{A} \cup \tilde{B} \), is defined as a fuzzy set on \( X \) whose membership function is defined by
\[
\mu_{\tilde{A} \cup \tilde{B}}(x) = \max \{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x) \}, \quad x \in X.
\]

Definition 4.1.7.
Let \( \tilde{A} \) and \( \tilde{B} \) be two fuzzy sets in \( X \). Then the intersection of \( \tilde{A} \) and \( \tilde{B} \), denoted by \( \tilde{A} \cap \tilde{B} \), is defined as a fuzzy set on \( X \) whose membership function is defined by
\[
\mu_{\tilde{A} \cap \tilde{B}}(x) = \min \{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x) \}, \quad x \in X.
\]

Definition 4.1.8.
Let \( \tilde{A} \) be a fuzzy set in \( X \). Then the complement of \( \tilde{A} \), denoted by \( \tilde{C} \tilde{A} \), is defined as a fuzzy set on \( X \) whose membership function is defined by
\[
\mu_{\tilde{C} \tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x), \quad x \in X.
\]
If we call fuzzy sets, such as considered so far, as type 1 fuzzy sets, the type 2 fuzzy set can be defined as follows.

**Definition 4.1.9**

A type 2 fuzzy set is a fuzzy set whose membership values are type 1 fuzzy sets of \([0, 1]\).

**Definition 4.1.10**

A type \(m\) fuzzy set is a fuzzy set in \(X\) whose membership values are type \(m - 1\), \(m > 1\) fuzzy sets on \([0, 1]\).

There have been other attempts to include vagueness going beyond the fuzziness of ordinary type 1 fuzzy sets. One example is the “stochastic fuzzy model” by Norwich and Turksen [76]. Those authors were mainly concerned with the measurement and the scale level of membership functions. They view a fuzzy set as a family of random variables whose density functions are estimated by that stochasticity [77].

Hirota [48] also considers fuzzy sets for which the “value of membership functions is a random variable”.

**Definition 4.1.11**

The cartesian product of fuzzy sets is defined as follows:

Let \(\tilde{A}_1, \ldots, \tilde{A}_n\) be fuzzy sets in \(X_1, \ldots, X_n\). The cartesian product is then a fuzzy set in the product space \(X_1 \times \ldots \times X_n\) with the membership function

\[
\mu_{\tilde{A}_1 \times \tilde{A}_2 \times \ldots \times \tilde{A}_n}(x) = \min \{\mu_{\tilde{A}_i}(x) | x = (x_1, \ldots, x_n), x_i \in X_i\} \quad \ldots (4.1.6)
\]

**Definition 4.1.12**

The \(m\)th power of a fuzzy set \(A\) is a fuzzy set with the membership function
Additional algebraic operations are defined as follows:

**Definition 4.1.13**
Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in $X$. Then the algebraic sum (probabilistic sum) $C = \tilde{A} + \tilde{B}$ is defined as

$$ C = \{(x, \mu_{\tilde{A} + \tilde{B}}(x)) | x \in X \} $$

where

$$ \mu_{\tilde{A} + \tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x). \mu_{\tilde{B}}(x). $$

**Definition 4.1.14**
Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in $X$. Then the bounded sum $\tilde{C} = \tilde{A} \oplus \tilde{B}$ is defined as

$$ C = \{(x, \mu_{\tilde{A} \oplus \tilde{B}}(x)) | x \in X \} $$

where

$$ \mu_{\tilde{A} \oplus \tilde{B}}(x) = \min (1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)) $$

**Definition 4.1.15.**
Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in $X$. Then the bounded difference $\tilde{C} = \tilde{A} \ominus \tilde{B}$ is defined as

$$ \tilde{C} = \{(x, \mu_{\tilde{A} \ominus \tilde{B}}(x)) | x \in X \} $$

where

$$ \mu_{\tilde{A} \ominus \tilde{B}}(x) = \max (0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1) $$

**Definition 4.1.16**

The algebraic product of two fuzzy sets $\tilde{C} = \tilde{A} \tilde{B}$ is defined as

$$ \tilde{C} = \{(x, \mu_{\tilde{A}}(x). \mu_{\tilde{B}}(x)) | x \in X \} $$

Other operators have been suggested by Hamacher [45], Yager [111] and Zimmermann [120]. These suggestions vary with respect to the generality
or adaptability of the operators as well as to the degree to which and how they are justified. Justification ranges from intuitive argumentation to empirical or axiomatic justification. Adaptability ranges from uniquely defined, for example, non-adaptable, concepts via parametrized "families" of operators to general classes of operators which satisfy certain properties.

4.1.3. FUZZY MEASURE

Sugeno [97] defined a fuzzy measure as follows: \( \mathcal{B} \) is a Borel field of the arbitrary set (universe) \( X \).

**Definition 4.1.17**

A set function \( g \) defined on \( \mathcal{B} \) which has the following properties is called a fuzzy measure:

1. \( g(\emptyset) = 0, g(X) = 1 \)
2. If \( A, B \in \mathcal{B} \) and \( A \subseteq B \) then \( g(A) \leq g(B) \)
3. If \( A_n \in \mathcal{B}, A_1 \subseteq A_2 \subseteq ... \) then \( \lim_{n \to \infty} g(A_n) = g(\lim A_n) \)

Banon [6] shows that very many measures with finite universe such as probability measures, belief functions, plausibility measures, and so on are fuzzy measures in the sense of Sugeno.

**Measure of Fuzzy Information**

The main use of information is to remove uncertainty. Infact we measure information supplied by the amount of uncertainty removed so that the measure of information is essentially a measure of uncertainty. Uncertainty is of two types, namely, probabilistic and fuzzy. Probabilistic uncertainty arises when the outcomes depend on chance, while fuzzy uncertainty arises because of ambiguity about every outcome. The ambiguity is measured by a membership function \( \mu_A(x) \) which varies from
0 to 1. The uncertainty is maximum if \( \mu_A(x) = \frac{1}{2} \) and in other cases, the uncertainty lies between 0 and the maximum value. Thus uncertainty is a function \( f(\mu_A(x)) \) which satisfies the following conditions.

(i) \( f(\mu_A(x)) = 0 \) when \( \mu_A(x) = 0 \) or \( \mu_A(x) = 1 \)

(ii) \( f(\mu_A(x)) \) is maximum when \( \mu_A(x) = \frac{1}{2} \)

(iii) \( f(\mu_A(x)) \) increases from the value 0 to its maximum value as \( \mu_A(x) \) increases from 0 to \( \frac{1}{2} \) and then it decreases from its maximum value to 0 as \( \mu_A(x) \) goes from \( \frac{1}{2} \) to 1.

(iv) \( f(\mu_A(x)) = f(1-\mu_A(x)) \)

(v) \( f(\mu_A(x)) \) need not necessarily be convex or concave and it need not be differentiable at every print.

We shall call \( (\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)) \) as a fuzzy vector and its total uncertainty will be called fuzzy uncertainty or fuzzy entropy.

**Definition 4.1.18**

The measure of fuzziness based on Hamming distance is defined by

\[
f(\tilde{A}) = \sum_{x \in X} | \mu_A(x) - \mu_C(x) |,
\]

where \( C \) is the crisp set nearest to \( \tilde{A} \) for which,

\[
\mu_C(x) =
\begin{cases}
0 & \text{if } \mu_A(x) \leq \frac{1}{2} \\
1 & \text{if } \mu_A(x) > \frac{1}{2}
\end{cases}
\]

... (4.1.11)

**Definition 4.1.19**

The measure of fuzziness based on Euclidean distance is defined by

\[
f(\tilde{A}) = \left( \sum_{x \in X} (\mu_A(x) - \mu_C(x))^2 \right)^{1/2}
\]

... (4.1.12)

For a general metric distance \( D \), the measure of fuzziness has the form

\[
f_{C,D}(\tilde{A}) = D_C(Z, Z^C) - D_C(\tilde{A}, \tilde{A}^C)
\]

... (4.1.13)

where \( Z \) denotes any arbitrary crisp subset of \( X \) so that \( D_C(Z, Z^C) \) is the largest possible distance in the set \( \tilde{A}(X) \) of all fuzzy sets in \( X \).
The normalized version is
\[
\ell_{C,D}(\tilde{A}) = 1 - \frac{D_C(\tilde{A}, \tilde{A}^C)}{D_C(Z, Z^C)} \quad \ldots (4.1.14)
\]

Measures of fuzziness by contrast to fuzzy measures try to indicate the degree of fuzziness of a fuzzy set. A number of approaches to this end have become known. Some authors are strongly influenced by the Shannon entropy as a measure of information.

Using Shannon's function
\[
S(x) = x \ln(x) - (1-x) \ln(1-x) \quad \ldots (4.1.15)
\]
the Shannon entropy is defined as follows:

\textbf{Definition 4.1.20}

Let \( \tilde{A} \) be a fuzzy set in \( X = \{x_1, x_2, \ldots, x_n\} \). The Shannon entropy \( d \) as a measure of fuzziness of a fuzzy set \( \tilde{A} = \{(x, \mu_{\tilde{A}}(x))\} \) is defined as
\[
d(\tilde{A}) = K \sum_{i=1}^{n} S(\mu_{\tilde{A}}(x_i)) \quad \ldots (4.1.16)
\]
where \( n \) is the number of elements in the support of \( \tilde{A} \) and \( K \) is a positive constant.

\textbf{Classical Measures of Uncertainty}

Prior to the theory of fuzzy sets, two principal measures of uncertainty were recognized. One of them, proposed by Hartley [47], is based solely on the classical set theory. The other, introduced by Shannon [93], is formulated in terms of probability theory. Both of these measures pertain to some aspects of ambiguity, as opposed to vagueness or fuzziness. Hartley's measure pertains to non-specificity and Shannon's measure pertains to conflict or dissonance in evidence. These measures are often referred to as measures of information. It has been more
common to refer to the measure invented by Shannon as the Shannon entropy.

**Definition 4.1.21**

Let \((p(x), x \in X)\) be a probability distribution on a finite set \(X\). The Shannon entropy is defined as

\[
H(p(x) \mid x \in X) = - \sum_{x \in X} p(x) \log_2 p(x) \quad \ldots (4.1.17)
\]

Shannon entropy was considered for many years to be the only feasible basis for information theory. In fact, the Shannon entropy is considered as a significant measure of uncertainty and information. Shannon entropy has the branching property. In principle, the branching property allows us to calculate uncertainty either directly, in terms of the probability distribution on the universal set, or in stages, using probability distributions on various subsets of the universal set.

**Theorem 4.1.1. Uniqueness Theorem of Shannon Entropy [58]**

Let \(\mathcal{P}\) be the set of all probability distributions on a finite set \(X = \{x_1, x_2, \ldots, x_n\}\). If a function \(H: \mathcal{P} \to (0, \infty)\) satisfies the requirements of continuity, weak additivity, monotonicity, branching and normalization, then

\[
H(p_1, p_2, \ldots, p_n) = - \sum_{i=1}^{n} p_i \log_2 p_i \quad \ldots (4.1.18)
\]

all \(n \in \mathbb{N}\), the set of all natural numbers.

**Definition 4.1.22**

Let \(p(x)\) and \(p'(x)\) be two probability distributions defined on a finite set \(X\). Then the Shannon cross-entropy is defined as

\[
H(p(x), p'(x) \mid x \in X) = - \sum_{x \in X} p(x) \log p'(x)
+ \sum_{x \in X} p(x) \log p(x) \quad \ldots (4.1.19)
\]
The Shannon cross-entropy function is usually employed as a measure of the degree to which an estimated probability distribution \( p' \) approximates the distribution \( p \).

4.2. THEORY OF POSSIBILITY

When our main concern is with the meaning of information rather than with its measure the proper framework for information analysis is possibilistic rather than probabilistic in nature, thus implying that what is needed for such an analysis is not probability theory but the theory of possibility. The mathematical apparatus of the theory of fuzzy sets provides a natural basic for the theory of possibility [118], playing a role which is similar to that of measure theory in relation to the theory of probability. The importance of the theory of possibility stems from the fact that much of the information on which human decisions are based is possibilistic rather than probabilistic in nature. In particular, the intrinsic fuzziness of natural languages is possibilistic in origin. Based on this premise, it is possible to construct a universal language in which the translation of a proposition expressed in a natural language takes the form of a procedure for computing the possibility distribution of a set of fuzzy relations in a data base.

4.2.1. BASIC DEFINITIONS

Definition 4.2.1

Let \( \tilde{F} \) be a fuzzy set of the universe \( U \) characterized by a membership function \( \mu_F(u) \). \( \tilde{F} \) is a fuzzy relation on the variable \( X \) if \( \tilde{F} \) acts as an elastic constraint on the values that may be assigned to \( X \), in the sense that the assignment of the values \( u \) of \( X \) has the form: \( X = u: \mu_{\tilde{F}}(u) \). \( \mu_{\tilde{F}}(u) \) is the degree to which the constraint represented by \( \tilde{F} \) is satisfied when \( u \) is assigned to \( X \).
**Definition 4.2.2.**

Let \( \tilde{F} \) be a fuzzy subset of a universe of discourse \( U \) which is characterized by its membership function \( \mu_F \), with the grade of membership, \( \mu_F(u) \), interpreted as the compatibility of \( u \) with the concept labeled \( \tilde{F} \).

Let \( X \) be a variable taking values in \( U \), and let \( \tilde{F} \) act as a fuzzy restriction, \( R(X) \), associated with \( X \). Then the proposition "\( X \) is \( \tilde{F} \)”, which translates into

\[
R(X) = \tilde{F}, \quad \ldots \quad (4.2.1)
\]

associates a possibility distribution, \( \Pi_x \), with \( X \) which is postulated to be equal to \( R(X) \), i.e.,

\[
\Pi_x = R(X). \quad \ldots \quad (4.2.2)
\]

Correspondingly, the possibility distribution function associated with \( X \) (or the possibility distribution function of \( \Pi_x \)) is denoted by \( \pi_x \) and is defined to be numerically equal to the membership function of \( \tilde{F} \), i.e.,

\[
\pi_x = \mu_F. \quad \ldots \quad (4.2.3)
\]

Thus, \( \pi_x(u) \), the possibility that \( X = u \), is postulated to be equal to \( \mu_F(u) \).

**Remark 4.2.1.** In view of (4.2.2) the relational assignment equation (4.2.1) may be expressed equivalently in theorem

\[
\Pi_x = \tilde{F}, \quad \ldots \quad (4.2.4)
\]

placing in evidence that the proposition \( p = X \) is \( \tilde{F} \) has the effect of associating \( X \) with a possibility distribution \( \Pi_x \) which, by (4.2.2), is equal to \( \tilde{F} \). When expressed in the form of (4.2.4), a relational assignment equation will be referred to as a possibility assignment equation, with the understanding that \( \Pi_x \) is induced by \( p \).
Remark 4.2.2.

Equation (4.2.2) implies that the possibility distribution $\Pi_X$ may be regarded as an interpretation of the concept of a fuzzy restriction and, consequently, that the mathematical apparatus of the theory of fuzzy sets, especially the calculus of fuzzy restrictions, provides a basis for the manipulation of possibility distribution by the rules of this calculus. Further, the definition implies the assumption that our intuitive perception of the ways in which possibilities combine is in accord with the rules of combination of fuzzy restrictions.

The definition of $\pi_X(u)$ implies that the degree of possibility may be any number in the interval $[0, 1]$ rather than just 0 or 1. There is a fundamental difference between probability and possibility which will lead to a more careful differentiation between the characterizations of degrees of possibility vs. degrees of probability—especially in legal discourse, medical diagnosis and synthetic languages.

**Possibility/probability consistency principle:** If a variable $X$ can take the values $u_1 \ldots u_n$ with respective possibilities $\Pi = (\pi_1 \ldots \pi_n)$ and probabilities $P = (p_1 \ldots p_n)$, then the degree of consistency of the probability distribution $P$ with the possibility distribution $\Pi$ is expressed by

$$\gamma = \pi_1 p_1 + \ldots + \pi_n p_n.$$  

(4.2.5)

The possibility/probability consistency principle is not a precise law or a relationship that is intrinsic in the concepts of possibility and probability. Rather it is an approximate formalization of the heuristic of servation that a lessening of the possibility of an event tends to lessen its probability, but not vice-versa.
Definition 4.2.3.

Let $\tilde{A}$ be a fuzzy subset of $U$ and let $\Pi_x$ be a possibility distribution associated with a variable $X$ which takes values in $U$. The possibility measure, $\pi(\tilde{A})$, of $\tilde{A}$ is defined by

$$\text{Poss \{X is } \tilde{A}\} \triangleq \pi(\tilde{A})$$

$$= \sup_{u \in U} \mu_{\tilde{A}}(u) \wedge \pi_X(u). \quad \ldots (4.2.6)$$

where $\mu_{\tilde{A}}$ is the membership function of $\tilde{A}$ and $\wedge$ stands for min.

It should be noted that, the terms of the height of a fuzzy set, which is defined as the supremum of its membership function [115], (4.2.6) may be expressed compactly by the equation

$$\pi(\tilde{A}) \triangleq \text{Height} (\tilde{A} \cap \Pi_x) \quad \ldots (4.2.7)$$

Let $\tilde{A}$ and $\tilde{B}$ be arbitrary fuzzy subsets of $U$. Then, from the definition of the possibility measure of a fuzzy set (4.2.6), it follows that

$$\pi(\tilde{A} \cup \tilde{B}) = \pi(\tilde{A}) \vee \pi(\tilde{B}). \quad \ldots (4.2.8)$$

By comparison, the corresponding relation for probability measures of $\tilde{A}$, $\tilde{B}$ and $\tilde{A} \cup \tilde{B}$ (if they exist) is

$$P(\tilde{A} \cup \tilde{B}) \leq P(\tilde{A}) + P(\tilde{B}) \quad \ldots (4.2.9)$$

and, if $\tilde{A}$ and $\tilde{B}$ are disjoint (i.e., $\mu_{\tilde{A}}(u) \mu_{\tilde{B}}(u) = 0$).

$$P(\tilde{A} \cup \tilde{B}) = P(\tilde{A}) + P(\tilde{B}) \quad \ldots (4.2.10)$$

which expresses the basic additivity property of probability measures. Thus in contrast to probability measure, possibility measure is not additive. Instead, it has the property expressed by (4.2.8).

In a similar fashion the possibility measure of the intersection of $\tilde{A}$ and $\tilde{B}$ is related to those of $\tilde{A}$ and $\tilde{B}$ by

$$\pi(\tilde{A} \cap \tilde{B}) \leq \pi(\tilde{A}) \wedge \pi(\tilde{B}) \quad \ldots (4.2.11)$$
In particular, if $\tilde{A}$ and $\tilde{B}$ are non-interactive, we have

$$\pi(\tilde{A} \cap \tilde{B}) = \pi(\tilde{A}) \wedge \pi(\tilde{B}) \quad \ldots (4.2.12)$$

By comparison, in the case of probability measures, we have

$$P(\tilde{A} \cap \tilde{B}) \leq P(\tilde{A}) \wedge P(\tilde{B}) \quad \ldots (4.2.13)$$

and

$$P(\tilde{A} \cap \tilde{B}) = P(\tilde{A}) \wedge P(\tilde{B}) \quad \ldots (4.2.14)$$

if $\tilde{A}$ and $\tilde{B}$ are independent and non-fuzzy.

### 4.2.2. POSSIBILITY AND INFORMATION

If $p$ is a proposition of a form $p \triangleq X$ is $\tilde{F}$ which translates into the possibility assignment equation

$$\Pi_{\tilde{A}(x)} = \tilde{F} \quad \ldots (4.2.15)$$

where $\tilde{F}$ is a fuzzy subset of $U$ and $\tilde{A}(X)$ is an implied attribute of $X$ taking values in $U$, then the information conveyed by $p$, $I(p)$, may be identified with the possibility distribution, $\Pi_{\tilde{A}(x)}$, of the fuzzy variable $\tilde{A}(X)$. Thus, the connection between $I(p)$, $\Pi_{\tilde{A}(x)}$, $R(\tilde{A}(X))$ and $\tilde{F}$ is expressed by

$$I(p) \triangleq \Pi_{\tilde{A}(x)}, \quad \ldots (4.2.16)$$

where

$$\Pi_{\tilde{A}(x)} = R(\tilde{A}(X)) = \tilde{F}. \quad \ldots (4.2.17)$$

**N-ary possibility distributions**

In asserting that the translation of a proposition of the form $p \triangleq X$ is $\tilde{F}$ is expressed by

$$X \text{ is } \tilde{F} \rightarrow R(\tilde{A}(X)) = \tilde{F} \quad \ldots (4.2.18)$$

or, equivalently, $^{1}$

$$X \text{ is } \tilde{F} \rightarrow \Pi_{\tilde{A}(x)} = \tilde{F}. \quad \ldots (4.2.19)$$

we are assuming that $p$ contains a single implied attribute $\tilde{A}(X)$ whose possibility distribution is given by the right-hand member of (4.2.19).
More generally, \( p \) may contain \( n \) implied attributes \( \tilde{A}_1(X), \ldots, \tilde{A}_n(X) \), with \( \tilde{A}_i(X) \) taking values in \( U_i \), \( i=1, \ldots, n \). In this case, the translation of \( p \triangleq X \) is \( \tilde{F} \), where \( \tilde{F} \) is a fuzzy relation in the cartesian product \( U=U_1 \times \ldots \times U_n \) assumes the form

\[
X \text{ is } \tilde{F} \rightarrow R(\tilde{A}_1(X), \ldots, \tilde{A}_n(X)) = \tilde{F} \tag{4.2.20}
\]
or equivalently,

\[
X \text{ is } \tilde{F} \rightarrow \prod (\tilde{A}_1(x), \ldots, \tilde{A}_n(x)) = \tilde{F} \tag{4.2.21}
\]
where \( R(\tilde{A}_1(X), \ldots, \tilde{A}_n(X)) \) is an \( n \)-ary fuzzy restriction and \( \prod(\tilde{A}_1(x), \ldots, \tilde{A}_n(x)) \) is an \( n \)-ary possibility distribution which is induced by \( p \). Correspondingly, the \( n \)-ary possibility distribution function induced by \( p \) is given by

\[
\pi(\tilde{A}_1(x), \ldots, \tilde{A}_n(x)) (u_1, \ldots, u_n) = \mu_{\tilde{F}}(u_1, \ldots, u_n), \quad (u_1, \ldots, u_n) \in U, \tag{4.2.22}
\]
where \( \mu_{\tilde{F}} \) is the membership function of \( \tilde{F} \). In particular, if \( \tilde{F} \) is a cartesian product of \( n \) unary fuzzy relations \( \tilde{F}_1, \ldots, \tilde{F}_n \) then the right hand member of (4.2.20) decomposes into a system of \( n \) unary relational assignment equations, i.e.,

\[
X \text{ is } \tilde{F} \rightarrow \begin{align*}
R(\tilde{A}_1(X)) &= \tilde{F}_1 \\
F(\tilde{A}_2(X)) &= \tilde{F}_2 \\
& \vdots \\
R(\tilde{A}_n(X)) &= \tilde{F}_n
\end{align*} \tag{4.2.23}
\]

Correspondingly,

\[
\prod (\tilde{A}_1(x), \ldots, \tilde{A}_n(x)) = \prod (\tilde{A}_1(x) \times \ldots \times \tilde{A}_n(x)) \tag{4.2.24}
\]
and

\[
\pi (\tilde{A}_1(x), \ldots, \tilde{A}_n(x)) (u_1, \ldots, u_n) = \pi \tilde{A}_1(x)(u_1) \land \ldots \land \pi \tilde{A}_n(x)(u_n) \tag{4.2.25}
\]
where

\[
\pi \tilde{A}_i(x)(u_i) = \mu_{\tilde{F}_i}(u_i), \quad u_i \in U_i, \quad i = 1, \ldots, n \tag{4.2.26}
\]
and \( \land \) denotes min.
Marginal Possibility Distributions

The concept of a marginal possibility distribution bears a close relation to the concept of a marginal fuzzy restriction [116] which in turn is analogous to the concept of a marginal probability distribution.

More specifically, let $X = (X_1, ..., X_n)$ be an $n$-ary fuzzy variable taking values in $U = U_1 \times \ldots \times U_n$, and let $\Pi_X$ be a possibility distribution associated with $X$ with $\pi_X (u_1, u_2, ..., u_n)$ denoting the possibility distribution function of $\Pi_X$.

Let $q = (i_1, i_2, ... , i_n)$ be a subsequence of the index sequence $(1,2,...,n)$ and let $X_{(q)}$ be the $q$-ary fuzzy variables $X_{(q)} = (X_{i_1}, X_{i_2}, ..., X_{i_k})$. The marginal possibility distribution $\Pi_{X_{(q)}}$ is a possibility distribution associated with $X_{(q)}$ which is induced by $\Pi_X$ as the projection of $\Pi_X$ on $U_{(q)} = U_{i_1} \times \ldots \times U_{i_k}$. Thus, by definition,

$$\Pi_{X_{(q)}} \triangleq \text{Proj}_{U_{(q)}} \Pi_X, \quad ... \ (4.2.27)$$

which implies that the probability distribution function of $X_{(q)}$ is related to that of $X$ be

$$\pi_{X_{(q)}} (u_{(q)}) = \bigvee_{u_{(q')}} \pi_X (u) \quad ... \ (4.2.28)$$

where $u_{(q)} = (u_{i_1}, u_{i_2}, ..., u_{i_k})$, $q' = (j_1, ..., j_m)$ is a subsequence of $(1, ..., n)$ which is complementary to $q$ (e.g., if $n = 5$ and $q = (i_1, i_2) = (2,4)$, then $q' = (j_1, j_2, j_3) = (1,3,5)$, $(u_{q'}) = (u_{j_1}, ..., u_{j_m})$ and $\bigvee$ denotes the supremum over $(u_{q'})$.

$$(u_{j_1}, ..., u_{j_m}) \in U_{j_1} \times \ldots \times U_{j_m}.$$ 

By analogy with the concept of independence of random variables, the fuzzy variables
\[ X_{(q)} = (X_{i_1}, \ldots, X_{i_k}) \]

and

\[ X_{(q')} = (X_{j_1}, \ldots, X_{j_m}) \]

are noninteractive [117] if and only if the possibility distribution associated with \( X = (X_1, \ldots, X_n) \) is the cartesian product of the possibility distributions associated with \( X_{(q)} \) and \( X_{(q')} \) i.e.,

\[ \Pi x = \Pi x_{(q)} \times \Pi x_{(q')} \] \hspace{1cm} \ldots (4.2.29)

or, equivalently,

\[ \pi_x (u_1, \ldots, u_n) = \pi_{x_{(q)}} (u_{i_1}, \ldots, u_{i_k}) \wedge \pi_{x_{(q')}} (u_{j_1}, \ldots, u_{j_m}). \] \hspace{1cm} (4.2.30)

In particular, the variables \( X_1, \ldots, X_n \) are noninteractive if and only if

\[ \Pi x = \Pi x_1 \times \Pi x_2 \ldots \times \Pi x_n \] \hspace{1cm} \ldots (4.2.31)

The intuitive significance of noninteraction may be clarified by a simple example. Suppose that \( X \triangleq (X_1, X_2) \) and \( X_1 \) and \( X_2 \) are noninteractive, i.e.

\[ \pi_x (u_1, u_2) = \pi_{x_1} (u_1) \wedge \pi_{x_2} (u_2) \] \hspace{1cm} \ldots (4.2.32)

Furthermore, suppose that for some particular values of \( u_1 \) and \( u_2 \), \( \pi_{x_1} (u_1) = \alpha_1, \pi_{x_2} (u_2) = \alpha_2 < \alpha_1 \) and hence \( \pi_x (u_1, u_2) = \alpha_2 \). Now if the value of \( \pi_{x_1} (u_1) \) is increased to \( \alpha_1 + \delta_1, \delta_1 > 0 \), it is not possible to decrease the value of \( \pi_{x_2} (u_2) \) by a positive amount, say \( \delta_2 \), such that the value of \( \pi_x (u_1, u_2) \) remains unchanged. In this sense, an increase in the possibility of \( u_1 \) cannot be compensated by a decrease in the possibility of \( u_2 \) and vice versa. Thus, in essence, noninteraction may be viewed as a form of noncompensation in which a variation in one or more components of a possibility distribution cannot be compensated by variations in the complementary components.
In the manipulation of possibility distributions, it is convenient to employ a type of symbolic representation which is common used in the case of fuzzy sets. Specifically, assume, for simplicity, that \( U_1, ..., U_n \) are finite sets, and let \( r^i \equiv (r_{1i}, ..., r_{ni}) \) denote an \( n \)-tuple values drawn from \( U_1, U_2, ..., U_n \) respectively. Furthermore, let \( \pi_i \) denote the possibility of \( r^i \) and let the \( n \)-tuple \( (r_{1i}, r_{2i}, ..., r_{ni}) \) be written as a string \( r_{1i} r_{2i} ... r_{ni} \).

Using this notation, a possibility distribution \( \Pi_x \) may be expressed in the symbolic form

\[
\Pi_x = \sum_{i=1}^{N} \pi_i r_{1i} r_{2i} ... r_{ni} \quad \text{...(4.2.33)}
\]

or, in case a separator symbol is needed, as

\[
\Pi_x = \sum_{i=1}^{N} \pi_i / r_{1i} r_{2i} ... r_{ni}, \quad \text{...(4.2.34)}
\]

where \( N \) is the number of \( n \)-tuples in the table of \( \Pi_x \), and the summation should be interpreted as the union of the fuzzy singletons \( \pi_i(r_{1i}, ..., r_{ni}) \).

**Conditioned Possibility Distributions**

In the theory of possibilities, the concept of a conditioned possibility distribution plays a role that is analogous to that of a conditional probability distribution in the theory of probabilities.

More concretely, let a variable \( X = (X_1, ..., X_n) \) be associated with a possibility distribution \( \Pi_x \), with \( \Pi_x \) characterized by a possibility distribution function \( \pi_X(u_1, ..., u_n) \) which assigns to each \( n \)-tuple \( (u_1, ..., u_n) \) in \( U_1 x ... x U_n \) its possibility \( \pi_X(u_1, ..., u_n) \).

Let \( q = (i_1, ..., i_k) \) as \( s = (j_1, ..., j_m) \) be subsequences of the index sequence \( (1, ..., n) \) and let \( (a_{j_1}, ..., a_{j_m}) \) be an \( n \)-tuple of values assigned to
By definition, the conditioned possibility distribution of
\[ X(q) \equiv (X_{i1}, \ldots, X_{ik}) \]
given
\[ X(q') = (a_{j1}, \ldots, a_{jm}) \]
is a possibility distribution expressed as
\[ \prod_{X(q)} [X_{j1} = a_{j1}, \ldots, X_{jm} = a_{jm}] \]
whose possibility distribution function is given by
\[
\pi_{X(q)} (u_{i1}, \ldots, u_{ik} | X_{j1} = a_{j1}, \ldots, X_{jm} = a_{jm}) \\
= \pi_X (u_{i1}, u_{i2}, \ldots, u_n) | u_{i1} = a_{j1}, \ldots, u_{im} = a_{jm} \quad \ldots (4.2.35)
\]

In the foregoing discussion, we have assumed that the possibility distribution of \( X = (X_1, \ldots, X_n) \) is conditioned on the values assigned to a specified subset, \( X(s) \), of the constituent variables of \( X \). In a more general setting, what might be specified is a possibility distribution associated with \( X(s) \) rather than the values of \( X_{j1}, \ldots, X_{jm} \). In such cases, we shall say that \( \Pi_X \) is particularized by specifying that \( \Pi_{X(s)} = G \) where \( G \) is a given \( m \)-ary possibility distribution. It should be noted that in the present context \( \Pi_{X(s)} \) is a given possibility distribution rather than a marginal distribution that is induced by \( \Pi_X \).

To analyze this case, it is convenient to assume that \( X_{j1} = X_1, X_{j2} = X_2, \ldots, X_{jm} = X_m, m<n \). Let \( \tilde{G} \) denote the cylindrical extension of \( G \), that is, the possibility distribution defined by
\[ \tilde{G} = G X U_{m+1} X \ldots X U_n. \quad \ldots (4.2.36) \]
which implies that
\[
\mu_{\tilde{G}} (u_1, \ldots, u_n) = \mu_G (u_1, \ldots, u_m), u_j \in U_j, j = 1, \ldots, n. \quad \ldots (4.2.37)
\]
where \( \mu_G \) is the membership function of the fuzzy relation \( G \).
The assumption that we are given $\Pi_x$ and $G$ is equivalent to assuming that we are given the intersection $\Pi_x \cap \overline{G}$. From this intersection, we can deduce the particularized possibility distribution $\Pi x(q) [\Pi x(s) = G]$ by projection on $U(q)$. Thus

$$\Pi x(q) [\Pi x(s) = G] = \text{Proj}_{U(q)} \Pi_x \cap \overline{G}.$$ ... (4.2.38)

Equivalently, the left-hand member of (4.2.38) may be regarded as the composition of $\Pi_x$ and $G$.

4.2.3. POSSIBILITY DISTRIBUTIONS OF COMPOSITE AND QUALIFIED PROPOSITIONS

The concept of a possibility distribution provides a natural way for defining the meaning as well as the information content of a proposition in a natural language. Thus, if $p$ is a proposition in a natural language $NL$ and $M$ is its meaning, then $M$ may be viewed as a procedure which acts on a set of relations in a universe of discourse associated with $NL$ and yields the possibility distribution of a set of variables or relations which are explicit or implicit in $p$.

In constructing the meaning of a given proposition, it is convenient to have a collection of what might be called conditional translation rules [119] which relate the meaning of a proposition to the meaning of its modifications or combinations with other propositions.

Rules of type I

Let $p$ be a proposition of the form $X$ is $F$ and let $m$ be a modifier such as very, quite, rather, etc. The so-called modifier rule [10] which defines the modification in the possibility distribution induced by $p$ may be stated as follows.
If
\[ X \text{ is } F \rightarrow \Pi_{A(X)} = F \] ... (4.2.39)
then
\[ X \text{ is } mF \rightarrow \Pi_{A(X)} = F^+ \] ... (4.2.40)
where \( A(X) \) is an implied attribute of \( X \) and \( F^+ \) is the modification of \( F \) defined by \( m \). For example, if \( m = \text{very} \), then \( F^+ = F^2 \); if \( m = \text{more or less} \) than \( F^+ = V \) \( F \); and if \( m = \text{not less} \) then \( F^+ = F' = \text{complement of } F \).

**Rules of type II**

If \( p \) and \( q \) are propositions, then \( r = p \cdot q \) denotes a proposition which is a composition of \( p \) and \( q \). The three most commonly used modes of composition are (i) conjunctive, involving the connective "and"; (ii) disjunctive, involving the connective "or"; and (iii) conditional involving the connective "if...then." The conditional translation rules relating to these modes of composition are stated below:

**Conjunctive (noninteractive):** If
\[ X \text{ is } F \rightarrow \Pi_{A(X)} = F \] ... (4.2.41)
and
\[ Y \text{ is } G \rightarrow \Pi_{B(Y)} = G \] ... (4.2.42)
then
\[ X \text{ is } F \text{ and } Y \text{ is } G \rightarrow \Pi_{A(X), B(Y)} = F \times G \] ... (4.2.43)
where \( A(X) \) and \( B(Y) \) are the implied attributes of \( X \) and \( Y \), respectively, \( \Pi_{A(X), B(Y)} \) is the possibility distribution of the variables \( A(X) \) and \( B(Y) \), and \( F \times G \) is the cartesian product of \( F \) and \( G \). It should be noted that \( F \times G \) may be expressed equivalently as
\[ F \times G = \overline{F} \cap \overline{G} \] ... (4.2.44)
where \( \overline{F} \) and \( \overline{G} \) are the cylindrical extensions of \( F \) and \( G \), respectively.
Disjunctive (noninteractive): If (4.2.41) and (4.2.42) hold, then
\[ X \text{ is } F \text{ or } Y \text{ is } G \rightarrow \Pi_{(A(X), B(Y))} = F + G \]  \[ \text{... (4.2.45)} \]
where the symbols have the same meaning as in (4.2.41) and (4.2.42) and + denotes the unions.

Conditional (noninteractive): If (4.2.41) and (4.2.42) hold, then
\[ \text{If } X \text{ is } F \text{ then } Y \text{ is } G \rightarrow \Pi_{(A(X), B(Y))} = F' \oplus G \]  \[ \text{... (4.2.46)} \]
Where \( F' \) is the complement of \( F \) and \( \oplus \) is the bounded sum defined by
\[ \mu_{F' \oplus G} = 1 \land (1 - \mu_F + \mu_G) \]  \[ \text{... (4.2.47)} \]
in which + and \( \land \) denotes the arithmetic addition and subtraction and \( \mu_F \)
and \( \mu_G \) are the membership functions of \( F \) and \( G \), respectively.

4.3. FUZZY MODEL IN MEDICAL DIAGNOSIS

Imprecision and uncertainty play a dominant role in the field of medicine. Especially uncertainty always exists in the diagnosis of disease. The uncertainty that present in the medical diagnosis can be expressed in terms of fuzzy sets. Thus the field of medicine has rich potential for the applicability of fuzzy set theory and the theory of evidence.

4.3.1. DIAGNOSIS OF CORONARY ARTERY STENOSIS

Here we shall discuss the expression of uncertainty regarding the diagnosis of coronary artery stenosis (Fig.4.3.1, Fig.4.3.2) by using fuzzy sets. Stenosis of the main arteries supplying blood to the myocardium region of the heart is the most common underlying cause of heart attack. Thus it is important to determine how severely a coronary arteries blockage interferes with blood flow to the myocardium and which arterial abnormal. The regions of the heart with less than normal blood flow are said to have perfusion defect. In nuclear cardiology, stress thallium - 201 scintigraphy is used to detect the abnormalities in the distribution of blood flow to the myocardium. Scintigraphic images are recorded from the
anterior and left anterior oblique views with the patient supine and from the left lateral view with the patient in the right lateral decubitus position.

Counts are collected with a hi-tech crystal portable gamma camera. The scintigrams are processed by comparing the normalized count ratio for each pixel at a given anatomic location with that obtained previously from a group of subjects having normal myocardial perfusion. Pixels containing count ratios that fell more than 2.5 S.D. below normal were considered to be underperfused. Each of the three views was divided into 10 anatomic regions called segments (Fig.4.3.3). Segments 1 through 10 of the anterior view are referred to as regions 1 through 10, segments 1 through 10 of the left lateral view are referred to as regions 11 through 20 and segments 1 through 10 of the left anterior oblique view are referred to as regions 21 through 30. The percentage of underperfused pixels in each region represent the perfusion defect of that region. Thus 30 numbers ranging in size from 0 to 100, were generated from each patients scintigram.

4.3.2. REPRESENTATION THROUGH FUZZY SETS

Assuming a statistical relationship between a percentage of perfusion defect in a region and site of coronary artery stenosis, probability density functions are evaluated. If $X_i$ denote the percentage of perfusion defect of the $i^{th}$ region and if $Y$ denote the diagnosis of a patient ($Y = 0$ for patient with normal coronary stenosis and $Y=1$ for patient with stenosed coronary), then the p.d.f.'s are defined as

$$f_i(x) = P(X_i = x/Y=0), \quad 0 \leq x \leq 100 \quad \ldots (4.3.1)$$

$$g_i(x) = P(X_i = x/Y=1), \quad 0 \leq x \leq 100 \quad \ldots (4.3.2)$$

Now, the membership function $\mu_i(x)$ which represent the fuzzy set "stenosed" as defined as:

$$\mu_i(x) = \frac{\int_0^x g_i(x) \, dx}{\int_{100}^x f_i(x) \, dx + \int_0^x g_i(x) \, dx} \quad \ldots (4.3.3)$$
To construct a fuzzy set with discrete elements we digitize a perfusion defect, $x$, into eight ranges $L$:

$$
L = \begin{cases} 
  k & \text{if } 100 \frac{(k-1)}{8}^2 \leq x \leq 100 \frac{k}{8}^2, \\
  8 & \text{if } x = 100 
\end{cases} \quad (4.3.4)
$$

Our training data consists of 40 patients of which 11 with normal arteries and 29 with coronary arteries. Among the set of 29 patients with stenosis, 4 patients have stenosis in the circumflex artery (CCX), 16 patients have stenosis in the left anterior descending artery (LAD) and 9 patients have stenosis in the right coronary artery (RCA).

If $N_i(L)$ denote the number of normal patients and $M_i(L)$ denote the number of stenosed patients whose percentage of perfusion defect in the $i^{th}$ region lies in the range $L$, then the probability functions for the $i^{th}$ region can be evaluated as,

$$
f_i(L) = \frac{N_i(L)}{\sum_{j=1}^{8} N_i(j)} \quad (4.3.5)
$$

$$
g_i(L) = \frac{M_i(L)}{\sum_{j=1}^{8} M_i(j)} \quad (4.3.6)
$$

Also the distribution functions for the $i^{th}$ region are

$$
F_i(L) = \frac{\sum_{t=1}^{L} N_i(t)}{\sum_{j=1}^{8} N_i(j)} \quad (4.3.7)
$$

$$
G_i(L) = \frac{\sum_{t=1}^{L} M_i(t)}{\sum_{j=1}^{8} M_i(j)} \quad (4.3.8)
$$
After establishing the distribution functions for the $i^{th}$ region, we define the fuzzy set for the $i^{th}$ region by defining its corresponding membership function as

$$\mu_i(L) = \frac{G_i(L)}{G_i(L) + 1 - F_i(L)} \quad \ldots \ (4.3.9)$$

Thus we obtain fuzzy sets for all 30 regions. For instance, from our training data, the fuzzy set for region 14 as follows:

Table 4.3.1. Probability functions for left lateral view segment 4

<table>
<thead>
<tr>
<th>L</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{14}(L)$</td>
<td>7</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$M_{14}(L)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>$f_{14}(L)$</td>
<td>0.636</td>
<td>0.091</td>
<td>0.182</td>
<td>0.091</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$g_{14}(L)$</td>
<td>0.034</td>
<td>0.069</td>
<td>0.103</td>
<td>0.069</td>
<td>0.069</td>
<td>0.138</td>
<td>0.172</td>
<td>0.346</td>
</tr>
</tbody>
</table>

Table 4.3.2. Fuzzy set for left lateral view segment 14

<table>
<thead>
<tr>
<th>L</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{14}(L)$</td>
<td>0.636</td>
<td>0.727</td>
<td>0.909</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>$G_{14}(L)$</td>
<td>0.034</td>
<td>0.103</td>
<td>0.206</td>
<td>0.275</td>
<td>0.344</td>
<td>0.482</td>
<td>0.654</td>
<td>1.000</td>
</tr>
<tr>
<td>$M_{14}(L)$</td>
<td>0.085</td>
<td>0.274</td>
<td>0.694</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

4.3.3. DIAGNOSTIC RULES GENERATION

The training data set of patients is divided into 4 groups, namely, normal, LAD, RCA and CCX. Each iteration of the procedure produces both a rule and a reduced target set for the next iteration. Each rule may contain three features as its conditions. The following algorithm, similar to the algorithm described by Michalski [72], is used for the generation of the diagnostic rules.
Step 1. For all combination of 3 features, perform steps 2 and 3.

Step 2. For a choice of 3 features, test whether they group patients with the same kind of stenosis. For example, if an LAD group has stenosis in regions 15, 16 and 19 and none of other groups have the same features, then stenosis in region 15, 16 and 19 form a rule for LAD.

Step 3. For each choice of features, count the number of patients who satisfy as this rule and assign that number as an index associated with that rule.

Step 4. If there is more than one choice, find a rule with a corresponding maximum index. All patients' data covered by this rule are deleted from the target set. The reduced data set becomes the new target set for the next iteration.

Using the above algorithm for our training set the following diagnostic rules are generated:

Table 4.3.3.

<table>
<thead>
<tr>
<th>Stenosed Regions</th>
<th>Diagnosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>5, 6, 9</td>
<td>LAD</td>
</tr>
<tr>
<td>4, 7, 16</td>
<td>LAD</td>
</tr>
<tr>
<td>9, 10, 25</td>
<td>LAD</td>
</tr>
<tr>
<td>5, 6, 18</td>
<td>RCA</td>
</tr>
<tr>
<td>9, 25, 26</td>
<td>RCA</td>
</tr>
<tr>
<td>7, 14, 24</td>
<td>CCX</td>
</tr>
<tr>
<td>12, 18, 24</td>
<td>CCX</td>
</tr>
<tr>
<td>14, 18, 29</td>
<td>RCA</td>
</tr>
<tr>
<td>2, 3, 7</td>
<td>CCX</td>
</tr>
<tr>
<td>15, 16, 10</td>
<td>LAD</td>
</tr>
<tr>
<td>9, 14, 18</td>
<td>RCA</td>
</tr>
<tr>
<td>4, 11, 15</td>
<td>CCX</td>
</tr>
<tr>
<td>7, 10, 15</td>
<td>LAD</td>
</tr>
<tr>
<td>7, 15, 26</td>
<td>RCA</td>
</tr>
<tr>
<td>9, 10, 19</td>
<td>CCS</td>
</tr>
</tbody>
</table>
Applying these diagnostic rules to a group of 20 test patients we obtain the following percentage of sensitivity, specificity and accuracy results:

Table 4.3.4.
Performance of Diagnostic Rules

<table>
<thead>
<tr>
<th></th>
<th>Normal</th>
<th>LAD</th>
<th>RCA</th>
<th>CCX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity</td>
<td>100(5/5)</td>
<td>71(5/7)</td>
<td>75(3/4)</td>
<td>75(3/4)</td>
</tr>
<tr>
<td>Specificity</td>
<td>87(13/15)</td>
<td>92(12/13)</td>
<td>88(14/16)</td>
<td>94(15/16)</td>
</tr>
<tr>
<td>Accuracy</td>
<td>90(18/20)</td>
<td>85(17/20)</td>
<td>85(17/20)</td>
<td>90(18/20)</td>
</tr>
</tbody>
</table>

Thus, a set of inductive decision rules has been generated, by using fuzzy set theoretic technique, for the diagnosis of coronary artery stenosis. The rules have been tested with a group of test patients and its performance has been assessed.

The model described above for the diagnosis of coronary artery stenosis were aimed at reducing the number of decisive rules to the adopted by a cardiologist for the diagnosis of coronary artery stenosis. Subsequently Thiyagarajan and Ethirajalu [100] generalised the above diagnostic problem by taking into consideration the physiological conditions of the patient as well as the information derived by the physician from his preliminary investigations with the patient. The authors proposed a more general model for the diagnosis of coronary stenosis as mentioned below:

The percentage of perfusion defect in any region is signalised as in Table 4.3.5, with low value percentage of perfusion defect being signalised with high resolution and high value percentage of perfusion defect being signalised with low resolution, as the lower values of perfusion defect are significant in discriminating between normal and stenosed arteries.
Suppose that $X$ denote the universal set of all possible signals $x$ that may be communicated by the scintigraphic report. Let $\tilde{M}$ be the message received from the scintigraphic report. It is clear that $\tilde{M}$ is a fuzzy subset of $X$, in which $\mu_{\tilde{M}}(x)$ denote the degree of uncertainty of the receipt of the specific signal $x$. Usually the physician possesses some information about the patients in the form of possibilities and probabilities of the signals which can be expected, as he had already made some detailed preliminary investigations about the patient.

<table>
<thead>
<tr>
<th>Percentage of perfusion defect</th>
<th>0-1.5</th>
<th>1.6-6.2</th>
<th>6.3-13.5</th>
<th>14.0-24.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signals</td>
<td>$s_1$</td>
<td>$s_2$</td>
<td>$s_3$</td>
<td>$s_4$</td>
</tr>
<tr>
<td>Percentage of perfusion defect</td>
<td>25-38.9</td>
<td>39-55.9</td>
<td>56-76.9</td>
<td>77-100</td>
</tr>
<tr>
<td>Signals</td>
<td>$s_5$</td>
<td>$s_6$</td>
<td>$s_7$</td>
<td>$s_8$</td>
</tr>
</tbody>
</table>

4.3.4. TESTS FOR CONSISTENCY AND CONFLICT

Let $X = \{x_1, x_2, \ldots, x_n\}$ and let $p(x_i)$ denote the probability associated with the signal $x_i \in X$, $i = 1, 2, \ldots, n$, then the probability of the fuzzy event of the receipt of message $\tilde{M}$ is

$$p(\tilde{M}) = \sum_{i=1}^{n} \mu_{\tilde{M}}(x_i) p(x_i) \quad \ldots \quad (4.3.10)$$

The physician can make use of this information to assess the consistency of the received message. If this probability is high, then we can assume that little distortion was introduced by the distorting factors. In this case the message is very clear and unambiguous and hence appropriate diagnosis can be made at once. On the other hand, the expectation or background information may also be given in the form of a possibility.
distribution where \( q(x) \in [0,1] \) indicates the physician's belief in the possibility of signal \( x \) being the anticipated signal based on his preliminary investigation. The measure of conflict of the message \( \tilde{M} \) with the background information is

\[
\mathcal{G}(\tilde{M}, q) = -\log_2 \left[ 1 - \sum_{i,j=1}^{n} \frac{\mu_{\tilde{M}}(x_i) \cdot q(x_j)}{\sum_{i=1}^{n} \mu_{\tilde{M}}(x_i) \cdot \sum_{j=1}^{n} q(x_j)} \right] \quad (4.3.11)
\]

The choice of the logarithmic function in \( \mathcal{G}(\tilde{M}, q) \) is motivated by the arguments encountered in the discussion of Shannon entropy. This function \( \mathcal{G} \) takes values from 0 to \( \infty \), \( \mathcal{G}(\tilde{M}, q) \) takes values from 0 to \( \infty \), \( \mathcal{G}(\tilde{M}, Q) = 0 \) only if \( \tilde{M} \) and \( q \) do not conflict at all and \( \mathcal{G}(\tilde{M}, q) = \infty \) only if they conflict totally. If the received message \( \tilde{M} \) conflicts with the expectation, then the physician may ask for clarification by requesting a repetition of scintigraphy. Before the new scintigraphic message is available, the physician may probably have modified his expectations sensibly based on the previous message. If \( q_1 \) is the modified expectations of the physician, then \( q_1(x) \) is defined as follows:

\[
q_1(x) = \min (q^\alpha(x), \mu_{\tilde{M}}(x)) \quad (4.3.12)
\]

for all \( x \in X \) where \( \alpha \) indicates the degree to which past message \( \tilde{M} \) is considered relevant in the modification of expectations. Thus the physician's procedure for correct scintigraphic message detection consists of test of the consistency of \( \tilde{M} \) against his expectations and a test of the message \( \tilde{M} \) for clarity and dependability. If the test of consistency of \( \tilde{M} \) yield high value and \( \mathcal{G}(\tilde{M}, q) \) yield low value, then \( \tilde{M} \) can be assumed as the intended message. If the test of consistency of \( \tilde{M} \) yield low value and \( \mathcal{G}(\tilde{M}, q) \) yield high value, then the expectations should be modified and a repetition of scintigraphy is requested by the physician. In other cases either a repetition of scintigraphy is requested or the message \( \tilde{M} \) is accepted for diagnosis despite the presence of doubt, depending on the circumstances that warrant the physician. In the care of request for repetition of scintigraphy, if \( \tilde{M}_1 \) is the revised message received from the
repeated scintigraphic report, the test for consistency of $\tilde{M}_1$ and the test for conflict between $\tilde{M}_1$ and $q_1$ is conducted as before.

4.3.5. DIAGNOSIS OF STENOSIS

Let $\tilde{M}^*$ be the most reliable message received from the scintigraphic report by means of the procedure described above.

Let $Y$ be the universal set of all coronary stenosis. i.e. $Y=\{y_1, y_2, y_3, y_4\}$ where $y_1 = \text{normal}$, $y_2 = \text{LAD}$, $y_3 = \text{RCA}$, $y_4 = \text{CCX}$. Let $\tilde{R}$ be a fuzzy binary relation in $Y \times X$ where $\mu_{R}(y, x)$ represents the degree of appropriateness of diagnosing $y$ given signal $x$. A fuzzy diagnostic set $A$ in $Y$ can be generated as

$$\tilde{A} = \tilde{R} \circ \tilde{M}^*$$  

... (4.3.12)

where $\mu_{\tilde{A}}(y) = \max_{x \in X} [\min(\mu_{R}(y, x), \mu_{\tilde{M}^*}(x))]$

for each $y \in Y$. The membership grade of each diagnosis of $y$ in the fuzzy set $\tilde{A}$ thus corresponds to the degree to which the diagnosis is appropriate to the scintigraphic message.

Using this model to the same group of 20 testing patients we obtain the following sensitivity and specificity results:

<table>
<thead>
<tr>
<th></th>
<th>Normal</th>
<th>LAD</th>
<th>RCA</th>
<th>CCX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity</td>
<td>100(5/5)</td>
<td>86(6/7)</td>
<td>100(4/4)</td>
<td>75(3/4)</td>
</tr>
<tr>
<td>Specificity</td>
<td>93(14/15)</td>
<td>92(12/13)</td>
<td>94(15/16)</td>
<td>100(16/16)</td>
</tr>
<tr>
<td>Accuracy</td>
<td>95(19/20)</td>
<td>90(18/20)</td>
<td>95(19/20)</td>
<td>95(19/20)</td>
</tr>
</tbody>
</table>

The earlier investigations on diagnosing coronary artery stenosis were aimed at reducing the number of decisive rules to be adopted by a physician. Here, we have generalised the above problem taking into consideration the physiological conditions of the patient as well as the informations derived by the physician from his preliminary investigations.
Thus the problem is posed under a more general setup and is solved using information techniques through fuzzy set theory. A similar approach can be utilised for other serious common ailments like hypertension, viral infections and so on.

4.4. FUZZY RISK ANALYSIS MODEL

In this section we discuss a risk analysis model using fuzzy set theory. Thiyagarajan and Ethirajalu [103] apply this model to medical surgery and obtain interesting results.

BASIC DEFINITIONS

Definition 4.4.1.

An n-ary fuzzy relation is a fuzzy set $\tilde{A}$ in the product space $X \times X \times \ldots \times X$. The membership function of an n-ary fuzzy relation is of the form $\mu_{\tilde{A}}(x_1, \ldots, x_n)$, where $x_i \in X$, $1 \leq i \leq n$.

Definition 4.4.2.

A classical probability system is a triple $(\Omega', S, P)$ where $\Omega'$ is an arbitrary set which includes all possible outcomes of a situation, $S$ is a set of events and $P$ is a real valued function defined for each $A \subset S$ such that:

1) $0 \leq P(A) \leq 1$  \hspace{1cm} (4.4.1)

2) $P(\Omega') = 1$ \hspace{1cm} (4.4.2)

3) If $A_1, A_2, \ldots$ is any sequence of pairwise disjoint sets in $S$, then $P(\bigcup_{n} A_n) = \sum_{n} P(A_n)$ \hspace{1cm} (4.4.3)

A function $P$ which satisfies the three conditions above is called a probability measure.

Definition 4.4.3

Let $B$ be a Borel field of subsets of a sample space $\Omega$. A set function $\chi(.)$ defined on $B$ is called a fuzzy measure if it has the following properties:
1) \( \chi(\emptyset) = 0 \)

2) \( \chi(\Omega) = 1 \)

3) If \( \alpha, \beta \in B \) with \( \alpha \subset \beta \), then \( \chi(\alpha) \leq \chi(\beta) \)

4) If \( \{\alpha_j \mid 1 \leq j \leq \infty\} \) is a monotone sequence, then

\[
\lim_{j \to \infty} \chi(\alpha_j) = \chi[\lim_{j \to \infty} (\alpha_j)]
\]  

Then \((\Omega, B, \chi)\) is called a fuzzy measure space. The analog of a fuzzy measure space in probability is \((\Omega, S, P)\). The fuzzy measure of \((\Omega, B)\) is \(\chi(.)\).

**Definition 4.4.4.**

Let \( \mu \) be the membership function of a fuzzy set \( \tilde{A} \). Also let 
\( \mu : \Omega \to [0,1] \) and \( \xi_T = \{x \mid \mu(x) \geq T\} \). The function \( \mu \) is called a B-measurable function if \( \xi_T \in B \) for all \( T \in [0,1] \).

**Definition 4.4.5**

Let \( \mu \) be a B-measurable function. The fuzzy expected value (FEV) of \( \mu \) over a fuzzy set \( A \) with respect to the measure \( \chi(.) \) is defined to be

\[
\text{Sup} \{ \text{Min} [T, \chi(\xi_T)] \}
\]

\( T \to [0, 1] \)  

\( \)  

... (4.4.5)

where

\( \xi_T \in A, \xi_T = \{x \mid \chi(x) \geq T\} \)

We state without proof the following theorem.

**Theorem 4.4.1 (Kandel and Byatt [52])**

\[
\text{Sup} \{ \text{Min} [T, \chi(\xi_T)] \} = \text{Inf} \{\text{Max} [T, \chi(\xi_T)]\} = \text{FEV}(\mu).
\]

**Remark 4.4.1.**

The definition of the fuzzy expected value can be extended from the interval \([0,1]\) to any real interval \([a, b]\) by extending \( T, \chi \) and \( \xi_T \) to the interval \([a, b]\) under the same transformation which \( \mu \) undergoes.
Definition 4.4.6.

The fuzzy expected value of $\mu$ over a set $\tilde{A}$ with respect to the measure $\chi(.)$ when the membership function lies in the interval $[a,b]$ is

$$ \text{Sup}\{\text{Min}\left[ T^*, \chi^*(\xi_T^*)\right]\} $$

where $T^*$, $\mu^*$ and $\xi_T^*$ become $T^*$, $\mu^*$ and $\xi_T^*$ respectively by undergoing the same transformation from the interval $[0,1]$ as the function $\mu$.

Theorem 4.4.2

$\text{FEV}(a\mu + b) = b + a \cdot \text{FEV}(\mu)$.

Proof: Case 1. Let $a \geq 0$ and $b$ be constants and $\mu: \Omega \rightarrow [0, 1]$. Then

$$ \text{FEV}(a\mu + b) = \text{Sup}\{\text{Min}\left[ T^*, \chi^*(\xi_T^*)\right]\} $$

where

$T^* = aT + b,$

$\chi^* = a\chi(\xi_T^*) + b$

and

$$ \xi_T^* = \{ x \mid a\mu(x) + b > T^* \} $$

$$ = \{ x \mid a\mu(x) + b \geq aT + b \} $$

$$ = \{ x \mid a\mu(x) > aT \} $$

$$ = \{ x \mid \mu(x) > T \} $$

$$ = \xi_T. $$

Then

$$ \text{FEV}(a\mu + b) = \text{Sup}\{\text{Min}\left[ aT + b, a\chi(\xi_T) + b\right]\} $$

$$ = b + \text{Sup}\{\text{Min}\left[ aT, a\chi(\xi_T)\right]\} $$

$$ = b + a \cdot \text{FEV}(\mu). $$
Case 2. Let $a < 0$ and $b$ be constants and $\mu : \Omega \rightarrow [0,1]$. Then
\[
\text{FEV} (a\mu + b) = \sup_{T* \rightarrow [a+b, b]} \min \{ T* + \chi^* (\xi_{T*}) \}
\]
\[... (4.4.8)\]

where
\[
T* = aT + b,
\]
\[
\chi^* = a \chi (\xi_{T*}) + b
\]

and
\[
\xi_{T*} = \{ x \mid a\mu(x) + b \leq T* \}
\]
\[= \{ x \mid a\mu(x) + b \leq aT + b \}
\]
\[= \{ x \mid a\mu(x) \leq aT \}
\]
\[= \{ x \mid a\mu(x) \geq T \}
\]
\[= \xi_T.
\]

Thus
\[
\text{FEV} (a\mu + b) = \sup_{T \rightarrow [0, 1]} \{ \min \{ aT + b, a \chi (\xi_{T}) + b \} \}
\]
\[= b + \sup_{T \rightarrow [0, 1]} \{ \min \{ aT, a \chi (\xi_{T}) \} \}
\]
\[= b + a \cdot \sup_{T \rightarrow [0, 1]} \{ \max \{ T, \chi (\xi_{T}) \} \}
\]
\[= b + a \cdot \text{FEV} (\mu).
\]

Remark 4.4.2.

Assume there exists a finite set of data points in which there are $n+1$ distinct levels of compatibility such that $0 < a_1 < a_2 \ldots \leq a_{n+1} \leq 1$. This implies that there exist $n$ distinct levels of fuzzy measure $\chi (\xi_T)$, excluding $0$ and $1$. The median of the set of the $2n+1$ numbers obtained by combining the $n+1$ levels of compatibility and the $n$ levels of fuzzy measure and sorting them in increasing order of magnitude is the fuzzy expected value [52].

The basic concept of possibility theory is the possibility distribution. A possibility distribution arises from another closely related concept, the fuzzy restriction.
Let $\tilde{F}$ be a fuzzy set with a universe $U$. Let $X$ be a variable which takes values in universe $U$. If $\tilde{F}$ acts upon $X$ as a fuzzy restriction, then the assignment of $\tilde{F}$ as a fuzzy restriction upon $X$ associates with $X$ a possibility distribution, $\pi_X$, where $\pi_X = \frac{\mu_{\tilde{F}}(u_1)}{u_1} + \ldots + \frac{\mu_{\tilde{F}}(u_n)}{u_n}$ for all $u \in U$.

The negation of a restriction on a variable is similar to the concept of a complement of a fuzzy set. If we define a fuzzy set $\tilde{F}$, this type of restriction takes the form of a statement $X$ is not $\tilde{F}$. Thus the possibility distribution of $X$ is the membership function of the fuzzy set NOT $\tilde{F}$ which is in turn the complement of the fuzzy set $\tilde{F}$.

In terms of developing possibility distributions for variables or ordered $n$-tuples of variables, Zadeh [118] has shown that there are six obvious ways in which these conjunctions and disjunctions can occur.

First, two or more unrelated restrictions upon a single variable can be placed in conjunction. The method for this is to take the intersection of the fuzzy sets being used as restrictions upon the variables.

Second, two or more unrelated restrictions can be placed in disjunction upon a variable. In this case, the method is to take the union of the fuzzy sets being placed upon the variable as restrictions.

Third, $n$ unrelated restrictions upon $n$ variables can be joined in conjunction to create a possibility distribution for the $n$-tuples belonging to the cartesian product of the $n$ universes. There are two steps to the method in this case. First, each of the fuzzy sets being used as restrictions upon the variables are projected into the set of $n$-tuples obtained by
taking the cartesian product of the universes of the variables. Second, the projected fuzzy sets are joined in intersection to create a single fuzzy set.

Fourth, n unrelated restrictions upon n variables can be joined in disjunction to create a possibility distribution for the n-tuples belonging to the cartesian product of the n universes. This case is similar to the third. The first step is to project each of the fuzzy sets into the set of n-tuples belonging to the cartesian product of the universes. Then, the projected fuzzy sets are joined in union to create a single fuzzy set.

Fifth, n related restrictions upon n variables can be joined in conjunction to create a possibility distribution for the n-tuples belonging to the cartesian product of the universes. Finally, n related restrictions upon n variables can be joined in disjunction to create a possibility distribution for the n-tuples belonging to the cartesian product of the universes. In both of these cases, the method is to create a fuzzy relation between the variables to compute compatibility values for each n-tuple of variable values. The fuzzy set created by the application of this fuzzy relation over all the possible n-tuples in the cartesian product of the universes is the new restriction.

The consistency of probability and possibility distributions can be stated as follows:

1) An event which is impossible is bound to be improbable.
2) A high level of possibility does not guarantee a high level of probability.
3) The lessening of the possibility of an event tends to also lessen its probability, but not vice versa.

Consider the problem of making a projection of the event which a population feels is most likely to occur when the only information
available is a set of possibility distributions created by the subjective evaluation of this population. This problem can be broken into two parts. (1) given a single possibility distribution, finding a method to determine the most probable event in that distribution. (2) given that each of the possibility distributions for each member of the population have been analyzed to determine their most probable events, finding a method of combining these subjective evaluations of probability to choose a single event which is typical of the feelings of the group as a whole.

The problem of how to determine a typical event from a single possibility distribution is closely related to the probability-possibility consistency principle which was discussed earlier. As a general rule, when a situation occurs in which one desires to determine the probability of a set of events based only upon the possibility distribution of those events three guiding rules occur. First, an event which has no possibility or extremely low possibility can not have high probability. Second, when several events have the same possibility and nothing else is known about those events, it must be assumed that those events have the same probability. Finally, because the nature of the concept of a possibility value implies that if one possibility value is greater than another possibility value there is something inherent in the nature of the second event to make that event more difficult to occur than the first, an event with a higher possibility value than another event must be assigned a probability value greater than or equal to that of the other event.

There are several approaches which can be taken to question what is the most probable event in this universe.

One such approach which can be taken is based upon the concept of ease of occurrence. This approach argues that only those events which have the highest possibility will have the highest probability.
The second problem which must be solved in the search for a method to find a typical value given only the possibility distributions of a population is how to combine all the typical events derived from each single possibility distribution. Because the determination of the events regarded as typical is based upon the evaluation of a fuzzy or subjective distribution of the possibility of each single event for every person in the population being questioned, it would seem that the fuzzy expected value would be a proper tool.

The use of the fuzzy expected value in this situation proceeds as follows: First, each single distribution is analyzed to find its typical event(s). Then, once that event is found a counter containing the number of occurrences of the event as a typical event in the population is incremented. If there is more than one typical event in a single distribution, the amount by which the counter for each typical event is incremented is the probability value which has been determined for that event in that specific distribution. After all the single distributions have been evaluated in this fashion, a monotonic numeric series of values are assigned to the events in the universe if the events are not numeric. The next step is to compute the fuzzy expected value of the events using the numeric values assigned to the events as the compatibility values and the weights of occurrence as data for computing the measure ($\xi_T$). The fuzzy expected value in this case will be computed in the interval defined by the low and high numeric values assigned to the events. If the events are numeric events the fuzzy expected value gives a typical event which is produced by that population of possibility distributions. If the events are not numeric, the fuzzy expected value must be converted back to the original set of events. In either case, if the fuzzy expected value is a real number and the events are defined only in terms of integer numbers, the fuzzy expected value can be rounded to the closest integer to predict a typical event.
4.4.2. RISK ANALYSIS MODEL

The first step in the fuzzy risk analysis model is the identification of threats to the system. It is obvious that any and all models of security of any system must have some knowledge of what threats they are attempting to deter. Thus, the first step of the model will consist of the determination of a set of \( n \) threats \( T \) with a generic element of \( T \) being denoted by the symbol \( T_i, 1 \leq i \leq n \).

The second step in the model is the assessment of the risk which each threat induces. This step can be broken into four distinct parts.

The first part of the risk assessment step is the determination of the vulnerability of the system to the specific threat being considered. This can be done in two ways. First, if empirical evidence of the cost of recovering from the threat is available, it should be used. The second technique can be used if no empirical evidence of vulnerability is available. This technique is to take soft estimates of the vulnerability to the threat from the members of the group creating the security program. The estimates will fall between some upper and lower bound. A typical estimate of vulnerability can be computed by finding the fuzzy expected value of the estimates in the interval between the lower bound and the upper bound. The set of vulnerability estimates is called \( V \) with a generic element of \( V \) being denoted by the symbol \( V_i \).

The second part of the risk assessment step is the estimation of a probability of occurrence for each threat \( T_i \). The set of probability estimates is called \( P \) with a generic element of \( P \) being denoted by \( P_i \). Brock's states that the probability estimates which are used in his model are for the most part subjective evaluations. In the fuzzy model, there are two ways for estimating a member of \( P \). First, if empirical evidence is available, use it to get a hard probability estimate. Second, if a subjective
estimate of probability is required, have each member of the group of surgeons assign a membership value for the threat in the fuzzy set "EVENTS WHICH WILL OCCUR DURING THE SPECIFIED TIME PERIOD". The typical value of probability for the threat will be the fuzzy expected value of each of the membership values for the threat as given by the group.

The third part of the risk assessment step is the computation of the patient’s risk for each specific threat. The set of risks is called $R$ with a generic element of $R$ being denoted by $R_i$. The risk $R_i$ of the organization to a threat $T_i$ is computed by the formula $R_i = V_i \times P_i$. $R_i$ represents the amount which the patient could reasonably expect to lose if no protection is provided against threat $T_i$.

The final part of the risk assessment step is the computation of a set of priority percentages $PP_i$ corresponding to the set of threats $T$. The priority percentages are found according to the formula

$$PP_i = R_i / \sum_{j=1}^{n} R_j \quad \ldots \quad (4.4.9)$$

The priority percentages represent the relative amount of damage which can be expected from each threat when compared to the damage which can be expected from all the threats.

The third step in the fuzzy risk analysis model is the determination of a set of most effective protection mechanisms for the set of threats confronting the system. This step can be divided into five distinct parts.

The first part of the mechanism selection step is the setting of the maximum amount which the patient can afford to spend to provide security against all of the threats. This amount is called TOTAL.
The second part of the mechanism selection step is the creation of a list of possible protection mechanisms for each threat. This list should be created by consulting with eminent surgeons who have sound knowledge of the surgical system. The list of mechanisms created for any threat \( T \) is called \( M_i \). An element of \( M_i \) shall be denoted by \( M_{ij} \).

The set \( M_i \) should include as single elements each mechanism by itself, mechanisms which can be used in combination with each other and the null mechanism.

The third part of the mechanism selection step is to have each person in the security group estimate a compatibility value, \( \mu(M_{ij}) \), which represents the compatibility of mechanism \( M_{ij} \) to the security needs of the organization against threat \( T_i \). These compatibility values are to be assigned in the interval \([0, 1]\) with 1 representing the fact that mechanism \( M_{ij} \) will fully protect the organization against threat \( T_i \), 0 representing the fact that mechanism \( M_{ij} \) provides no protection against threat \( T_i \), and values between 0 and 1 representing partial protection against the threat. Each set of mechanisms \( M_i \) along with its corresponding set of compatibility values are a fuzzy set "MECHANISMS WHICH PROVIDE FULL PROTECTION AGAINST THE THREAT \( T_i \)." For each mechanism \( M_{ij} \), each of member of the security group should also estimate a cost \( c_{ij} \) for that mechanism. This estimate can either be hard or soft depending upon the availability of empirical evidence as to the cost. If we allow a variable \( X_i \) to take as its universe the mechanisms in set \( M_i \), the assignment \( X_i \) is restricted by the fuzzy set "MECHANISMS WHICH PROVIDE FULL PROTECTION AGAINST THE THREAT \( T_i \) create a possibility distribution for \( X_i \). This possibility distribution gives the possibility that each mechanism in \( M_i \) will fulfill the security requirements for the threat \( T_i \)."
The fourth part of the mechanism selection step is the conjunction of the possibility distributions for each $X_i$ to give a possibility distribution for the $n$-tuples of mechanisms created by taking the cartesian product of the sets $M_i$. This can be classified as the conjunction of several related restrictions upon $n$ variables. The restrictions are related for three reasons. First, the amount of security which each mechanism provides to the system as a whole is different. Second, the amount of security which the system as a whole receives is the sum of the security which each of the mechanisms provide. Finally there is a threshold cost which the $n$-tuple of mechanisms to be chosen can not exceed.

For each member of the group of eminent surgeons, there should be $n$ possibility distributions corresponding to the $n$ threats to the system. Each threat $T_i$ should have a priority percentage $PP_i$ which was computed earlier. Each mechanism $M_{ij}$ in the set of possibility distributions should have a cost $C_{ij}$ and a possibility value $Poss(M_{ij})$ which was created by the assignment of the fuzzy set "MECHANISMS WHICH PROVIDE FULL PROTECTION AGAINST THE THREAT $T_i$" as a restriction upon the variable $X_i$. The fuzzy relation which we will use in this model to give a possibility value for each ordered $n$-tuple of mechanisms which can be combined to provide security against the $n$ threats is the formula

$$Post(M_{1j}, \ldots M_{nj}) = \begin{cases} 
0 \text{ if } \sum_{i=1}^{n} C_{ij} > \text{TOTAL} \\
\sum_{i=1}^{n} PP_i \times Poss(M_{ij}) \text{ if otherwise}
\end{cases} \quad \ldots (4.4.10)$$

The application of this fuzzy relation to every $n$-tuple in the cartesian product of the sets $M_i$ creates a possibility distribution representing the possibility that those $n$-tuples will meet the system's protection requirements against the set of threats while not exceeding the
patients affordability. At this point in the model there should be one possibility distribution for each surgeon in the expert panel.

The final part of the mechanism selection step is the application of the typical value of a population of possibility distributions algorithm created previously to the possibility distributions of the groups of surgeons. The result of this algorithm will be the set of mechanisms which the group as a whole has chosen as both providing a maximum amount of protection while also staying within the patients budget.

The final step in the fuzzy risk analysis model is the implementation and monitoring of the security mechanisms chosen. This step is the same as in Brocks' model.

4.4.3. RISK ANALYSIS MODEL IN MEDICAL SURGERY [103]

During the modeling process, we very often see that some information inherent in the system is lost during modeling. The uncertainty which results from the loss of information in the model must be handled by the methods used in the model, otherwise the results which are obtained from the model will not be reliable. One method which has been developed for working with systems which have lost information in the modeling process is fuzzy set theory.

Here we shall discuss the fuzzy risk analysis model to medical surgery. We desire to provide remedial measures for the various risk factors so that the ultimate increase in expense for surgery may not exceed Rs.1000 per patient.

In order to develop a fuzzy risk analysis model for medical surgery we must identify the following:

i) the risk factors associated with the medical surgery
ii) the remedial measures that will provide protection against the different risk factors.

iii) the degrees of protection provided by the different remedial measures.

The first step in the fuzzy risk analysis model is the identification of risk factors pertaining to medical surgery. A group of three surgeons from the three different fields in medicine lists the following three major risk factors:

\[ T_1 : \text{Lapse in sterilization of operation theatre} \]
\[ T_2 : \text{Failure of monitors during surgery} \]
\[ T_3 : \text{Lapse in post surgical care} \]

The second step of the fuzzy risk analysis model is the assessment of risk. To do this, first, let us estimate the vulnerability \( V_i \). To obtain a vulnerability value \( V_i \) corresponding to the risk factor \( T_i \), each surgeon in the group estimates the vulnerability. The estimates are Rs.20000, Rs.15000, and Rs.12000. The group estimate for \( V_1 \) is the fuzzy expected value of the estimates in the interval \([12000, 20000]\). Thus \( V_1 = 15000 \). The estimates for \( V_2 \) from the surgeons are observed as Rs.2000, Rs.5000 and Rs.2500. Hence the group estimate for \( V_2 \) is Rs.2990. The estimates for \( V_3 \) are observed as Rs.500, Rs.600 and Rs.800. Hence the group estimated for \( V_3 \) is Rs.600. Thus we have \( V_1 = \text{Rs.}15000 \) \( V_2 = 2990 \) and \( V_3 = \text{Rs.}600 \).

Next we shall estimate the probability of occurrence \( P_i \) for each risk factor \( T_i \). The group of surgeons gave the estimates for the probability of occurrence \( P_i \) for \( T_1 \) as .05, .02 and .06. Their estimates for \( P_2 \) are .04, .02, and .01 and their estimates for \( P_3 \) are .25, .20 and .25. To obtain a typical estimate of the probability of the occurrence of risk factor, the fuzzy expected value of each set of estimates in the interval \([0, 1]\) is
computed. The group estimates for \( P_1, P_2, P_3 \) are .06, .04 and 25 respectively.

Next we compute the risk \( R_i \) for each factor \( T_i \). Using the formula \( R_i = V_i \times P_i \), the risks are computed to be \( R_1 = \text{Rs.}900 \), \( R_2 = \text{Rs.}119.60 \) and \( R_3 = \text{Rs.}150 \).

Now, we compute the priority percentages \( PP_i \) for each risk factor \( T_i \) using the formula

\[
PP_i = \frac{R_i}{\sum_{j=1}^{3} R_j} \quad \quad ... (4.4.11)
\]

In this case we have \( PP_1 = .77 \), \( PP_2 = .10 \) and \( PP_3 = .13 \).

The first part of the remedial measures selection step is the creation of lists of possible protection measures for each risk factor. The group of surgeons decides upon the protection measures given in Table 4.4.1

<table>
<thead>
<tr>
<th>Risk factors</th>
<th>Remedial measures</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_1 )</td>
<td>( M_{11} ) : Sterilizing the theatre before each surgery.</td>
</tr>
<tr>
<td></td>
<td>( M_{12} ) : Providing monitors to assess the standard of sterilization of theatre.</td>
</tr>
<tr>
<td></td>
<td>( M_{13} ) : Null</td>
</tr>
<tr>
<td>( T_2 )</td>
<td>( M_{21} ) : Providing service contracts for maintenance of monitors.</td>
</tr>
<tr>
<td></td>
<td>( M_{22} ) : Appointing Technician for the maintenance of monitors.</td>
</tr>
<tr>
<td></td>
<td>( M_{23} ) : Null</td>
</tr>
<tr>
<td>( T_3 )</td>
<td>( M_{31} ) : Providing IC (Intensive Care) facility for all post operative patients.</td>
</tr>
<tr>
<td></td>
<td>( M_{32} ) : Providing individual nursing assistants.</td>
</tr>
<tr>
<td></td>
<td>( M_{33} ) : Null</td>
</tr>
</tbody>
</table>
Now, we consider the assignments of grades of membership in the set of remedial measures which provide protection against the risk factor for each measure in $M_i$. The compatibility values as estimated by each member of the group of surgeons are given in Table 4.4.2.

<table>
<thead>
<tr>
<th>Surgeon</th>
<th>Compatibility of remedial measure to risk factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\mu (M_1) = .6/1 + .8/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$\mu (M_2) = .8/1 + .4/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$\mu (M_3) = .9/1 + .6/2 + 0/3$</td>
</tr>
<tr>
<td>II</td>
<td>$\mu (M_1) = .75/1 + .85/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$\mu (M_2) = .6/1 + .4/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$\mu (M_3) = .8/1 + .7/2 + 0/3$</td>
</tr>
<tr>
<td>III</td>
<td>$\mu (M_1) = .8/1 + .75/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$\mu (M_2) = .7/1 + .5/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$\mu (M_3) = .75/1 + .5/2 + 0/3$</td>
</tr>
</tbody>
</table>

Each surgeon of the group also estimates the cost of each of the remedial measures. The cost estimates are given in Table 4.4.3.

<table>
<thead>
<tr>
<th>Surgeon</th>
<th>Estimated cost in Rs. of remedial measure per patient</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$C_1 = 100/1 + 150/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$C_2 = 200/1 + 50/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$C_3 = 3000/1 + 600/2 + 0/3$</td>
</tr>
<tr>
<td>II</td>
<td>$C_1 = 120/1 + 170/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$C_2 = 250/1 + 50/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$C_3 = 4000/1 + 500/2 + 0/3$</td>
</tr>
<tr>
<td>III</td>
<td>$C_1 = 110/1 + 180/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$C_2 = 275/1 + 60/2 + 0/3$</td>
</tr>
<tr>
<td></td>
<td>$C_3 = 3600/1 + 750/2 + 0/3$</td>
</tr>
</tbody>
</table>
The next step is to derive the possibility distributions for the three surgeons of the group by using the formula

$$\text{Poss}(M_{ij}, \ldots M_{nj}) = \begin{cases} 0 & \text{if } \sum_{i=1}^{n} C_{ij} > \text{TOTAL} \\ \sum_{i=1}^{n} PP_i \times \text{Poss}(M_{ij}) & \text{otherwise} \end{cases}$$

The 3-tuples of remedial measures and their corresponding possibility values as estimated by each surgeon of the group are given in Table 4.4.4, 4.4.5 and 4.4.6.

Table 4.4.4.
Possibility Distribution For First Surgeon

<table>
<thead>
<tr>
<th>3-tuple</th>
<th>Possibility</th>
<th>3-tuple</th>
<th>Possibility</th>
<th>3-tuple</th>
<th>Possibility</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1,1)</td>
<td>0</td>
<td>(2,1,1)</td>
<td>0</td>
<td>(3,1,1)</td>
<td>0</td>
</tr>
<tr>
<td>(1,1,2)</td>
<td>.62</td>
<td>(2,1,2)</td>
<td>.774</td>
<td>(3,1,2)</td>
<td>.158</td>
</tr>
<tr>
<td>(1,1,3)</td>
<td>.542</td>
<td>(2,1,3)</td>
<td>.696</td>
<td>(3,1,3)</td>
<td>.08</td>
</tr>
<tr>
<td>(1,2,1)</td>
<td>0</td>
<td>(2,2,1)</td>
<td>0</td>
<td>(3,2,1)</td>
<td>0</td>
</tr>
<tr>
<td>(1,2,2)</td>
<td>.58</td>
<td>(2,2,2)</td>
<td>.734</td>
<td>(3,2,2)</td>
<td>.118</td>
</tr>
<tr>
<td>(1,2,3)</td>
<td>.502</td>
<td>(2,2,3)</td>
<td>.656</td>
<td>(3,2,3)</td>
<td>.04</td>
</tr>
<tr>
<td>(1,3,1)</td>
<td>0</td>
<td>(2,3,1)</td>
<td>0</td>
<td>(3,3,1)</td>
<td>0</td>
</tr>
<tr>
<td>(1,3,2)</td>
<td>.54</td>
<td>(2,3,2)</td>
<td>.694</td>
<td>(3,3,2)</td>
<td>.078</td>
</tr>
<tr>
<td>(1,3,3)</td>
<td>.462</td>
<td>(2,3,3)</td>
<td>.616</td>
<td>(3,3,3)</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.4.5.
Possibility Distribution For Second Surgeon

<table>
<thead>
<tr>
<th>3-tuple</th>
<th>Possibility</th>
<th>3-tuple</th>
<th>Possibility</th>
<th>3-tuple</th>
<th>Possibility</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1,1)</td>
<td>0</td>
<td>(2,1,1)</td>
<td>0</td>
<td>(3,1,1)</td>
<td>0</td>
</tr>
<tr>
<td>(1,1,2)</td>
<td>.7285</td>
<td>(2,1,2)</td>
<td>.8055</td>
<td>(3,1,2)</td>
<td>.151</td>
</tr>
<tr>
<td>(1,1,3)</td>
<td>.6375</td>
<td>(2,1,3)</td>
<td>.7145</td>
<td>(3,1,3)</td>
<td>.06</td>
</tr>
<tr>
<td>(1,2,1)</td>
<td>0</td>
<td>(2,2,1)</td>
<td>0</td>
<td>(3,2,1)</td>
<td>0</td>
</tr>
<tr>
<td>(1,2,2)</td>
<td>.7085</td>
<td>(2,2,2)</td>
<td>.7855</td>
<td>(3,2,2)</td>
<td>.131</td>
</tr>
<tr>
<td>(1,2,3)</td>
<td>.6175</td>
<td>(2,2,3)</td>
<td>.6945</td>
<td>(3,2,3)</td>
<td>.04</td>
</tr>
<tr>
<td>(1,3,1)</td>
<td>0</td>
<td>(2,3,1)</td>
<td>0</td>
<td>(3,3,1)</td>
<td>0</td>
</tr>
<tr>
<td>(1,3,2)</td>
<td>.6685</td>
<td>(2,3,2)</td>
<td>.7455</td>
<td>(3,3,2)</td>
<td>.091</td>
</tr>
<tr>
<td>(1,3,3)</td>
<td>.5775</td>
<td>(2,3,3)</td>
<td>.6545</td>
<td>(3,3,3)</td>
<td>0</td>
</tr>
</tbody>
</table>
The final part of the remedial measures selection step is the application of the typical value of a population of possibility distributions algorithm to the three possibility distributions given above. In this case the typical value of the distributions for surgeons I and II is (2, 1, 2) and the typical value of the distribution for surgeon III is (1,1,2). Thus the counter for the 3-tuple (2,1,2) is set to 2 and the counter for the 3-tuple (1,1,2) is set to 1. Counters for all other 3-tuples are set to 0. Now each 3-tuple is arbitrarily assigned a compatibility value which in this case will be an integer between 1 and 27. The arbitrary assignment in this case will be in reverse lexicographic order. After the scaled valued for $\chi^*(\xi_{T^*})$ are computed from the counters, the fuzzy expected value of the set of 3-tuples is computed in the interval [1,27]. The compatibility value and $\chi^*(\xi_{T^*})$ value for each 3-tuple is given Table 4.4.7.
<table>
<thead>
<tr>
<th>3-tuple</th>
<th>Compatibility value</th>
<th>$\chi^<em>(\xi_T^</em>)$</th>
<th>3-tuple</th>
<th>Compatibility value</th>
<th>$\chi^<em>(\xi_T^</em>)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1,1)</td>
<td>27</td>
<td>1</td>
<td>(2,2,2)</td>
<td>14</td>
<td>27</td>
</tr>
<tr>
<td>(1,1,2)</td>
<td>26</td>
<td>10</td>
<td>(2,2,3)</td>
<td>13</td>
<td>27</td>
</tr>
<tr>
<td>(1,1,3)</td>
<td>25</td>
<td>10</td>
<td>(2,3,1)</td>
<td>12</td>
<td>27</td>
</tr>
<tr>
<td>(1,2,1)</td>
<td>24</td>
<td>10</td>
<td>(2,3,2)</td>
<td>11</td>
<td>27</td>
</tr>
<tr>
<td>(1,2,2)</td>
<td>23</td>
<td>10</td>
<td>(2,3,3)</td>
<td>10</td>
<td>27</td>
</tr>
<tr>
<td>(1,2,3)</td>
<td>22</td>
<td>10</td>
<td>(3,1,1)</td>
<td>9</td>
<td>27</td>
</tr>
<tr>
<td>(1,3,1)</td>
<td>21</td>
<td>10</td>
<td>(3,1,2)</td>
<td>8</td>
<td>27</td>
</tr>
<tr>
<td>(1,3,2)</td>
<td>20</td>
<td>10</td>
<td>(3,1,3)</td>
<td>7</td>
<td>27</td>
</tr>
<tr>
<td>(1,3,3)</td>
<td>19</td>
<td>10</td>
<td>(3,2,1)</td>
<td>6</td>
<td>27</td>
</tr>
<tr>
<td>(2,1,1)</td>
<td>18</td>
<td>10</td>
<td>(3,2,2)</td>
<td>5</td>
<td>27</td>
</tr>
<tr>
<td>(2,1,2)</td>
<td>17</td>
<td>27</td>
<td>(3,2,3)</td>
<td>4</td>
<td>27</td>
</tr>
<tr>
<td>(2,1,3)</td>
<td>16</td>
<td>27</td>
<td>(3,3,1)</td>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>(2,2,1)</td>
<td>15</td>
<td>27</td>
<td>(3,3,2)</td>
<td>2</td>
<td>27</td>
</tr>
</tbody>
</table>

The fuzzy expected value of this set of 3-tuple is 17. Thus the set of remedial measures which in the opinion of the group of surgeons gives the best remedy against the three different risk factors at the same time meeting the budget of the patients is the set represented by the 3-tuple (2,1,2). In otherwords to provide remedy against the risk factors pertaining to the medical surgery at the same time meeting the budget of the patient, the hospital organisation may take the following steps:

1. Providing monitors to assess the standard of sterilization of the operation theatre.
2. Providing service contract for all monitors fixed in the operation theatre.
3. Providing individual qualified nursing assistants for each post-operative patient till his or her discharge from the hospital.
Coronary Artery Disease

Fig. 4.3.1. Coronary Artery Stenosis
Fig. 4.3.2. Longitudinal section of the Heart.
Fig. 4.3.3. Three Views of a Scan of the Heart and the Numbering Scheme for the Segments of each View.